**2002**

**NASA SUMMER FACULTY FELLOWSHIP PROGRAM**

**MARSHALL SPACE FLIGHT CENTER**
**THE UNIVERSITY OF ALABAMA**

# AN IMPROVED RLV STABILITY ANALYSIS VIA A CONTINUATION APPROACH

| | |
|---|---|
| Prepared By | Mark Friedman |
| Academic Rank | Professor |
| Institution and Department | University of Alabama in Huntsville<br>Department of Mathematical Sciences |
| NASA/MSFC | Transportation |
| MSFC Colleague: | Mark Jackson |

**Introduction.**

The Continuation of Invariant Subspaces (CIS) algorithm [1] and [2] produces a smooth orthogonal similarity transformation to block triangular form of $A(s)$.

- Smooth dependence on $s$: $h_i$ decreases if $\mathcal{R}(s)$ changes fast, $h_i$ increases if $\mathcal{R}(s)$ changes slowly.
- Works if $\Sigma_1(s)$ and $\Sigma_2(s)$ do not come close.

1. Want to detect *bifurcations*: $\lambda(s)$ crosses $\operatorname{Re}\lambda = 0$.
2. Want to ensure: only $\lambda(s) \in \Sigma_1(s)$ can bifurcate.

- *An application:* stability analysis of a simulation model of the reusable launch vehicle X-33. *Problem*: choosing the points in flight that capture critical events affecting stability and performance.

**Background.**

**Theorem 1** *Given $A \in C^1([0,1],\ \mathbb{R}^{n\times n})$ with:*

$$\Sigma(s) = \Sigma_1(s) \cup \Sigma_2(s), \quad \Sigma_1(s) \cap \Sigma_2(s) = \emptyset, \tag{1}$$

$\mathcal{R}(s) \equiv \mathcal{R}(\Sigma_1(s))$, $\dim(\mathcal{R}) = m > 0$;

***(i)*** *There exist smooth $T$ and $Q$, $Q$ orthogonal:*

$$Q^T(s)A(s)Q(s) := T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ 0 & T_{22}(s) \end{bmatrix}, \tag{2}$$

$Q = [Q_1\ Q_2]$, $\mathcal{R} = \operatorname{span}(Q_1)$, $\mathcal{R}^\perp = \operatorname{span}(Q_2)$.

***(ii)*** *Let*

$$\widehat{T} = \begin{bmatrix} \widehat{T}_{11} & \widehat{T}_{12} \\ E_{21} & \widehat{T}_{22} \end{bmatrix} := Q^T(0)A(h)Q(0), \tag{3}$$

$Y \in \mathbb{R}^{(n-m)\times m}$ *solve algebraic Riccati equation*

$$F(Y) := \widehat{T}_{22}Y - Y\widehat{T}_{11} + E_{21} - Y\widehat{T}_{12}Y = 0. \tag{4}$$

*Then $Q(h)$ and $T(h)$ in (2) are given by*

$$\begin{aligned} Q_1(h) &= (Q_1(0) + Q_2(0)Y)(I + Y^TY)^{-1/2}, \\ Q_2(h) &= (Q_2(0) - Q_1(0)Y^T)(I + YY^T)^{-1/2}, \end{aligned} \tag{5}$$

$$T(h) = \begin{bmatrix} T_{11}(h) & T_{12}(h) \\ 0 & T_{22}(h) \end{bmatrix} := Q^T(h)A(h)Q(h). \tag{6}$$

***(iii)*** *Let $Y_0$ solve the Sylvester equation*

$$T_{22}(0)Y_0 - Y_0T_{11}(0) = -E_{21}, \tag{7}$$

and let $h$ be so small that

$$\tilde\kappa(\widehat{T}) := \frac{\|\widehat{T}_{12}\|_F \|F(Y_0)\|_F}{\mathrm{sep}^2(\widehat{T}_{11} + \widehat{T}_{12}Y_0,\ \widehat{T}_{22} - Y_0\widehat{T}_{12})} < \frac{1}{4}. \tag{8}$$

Then (4) has unique solution $Y$ and Newton method with initial guess $Y_0$ converges for $\tilde\kappa(\widehat{T}) < 1/12$ .■

Here $\mathrm{sep}(A,B)$ is the smallest singular value of the Sylvester operator:

$$\mathrm{sep}(A,B) = \inf_{X\neq 0} \frac{\|AX - XB\|}{\|X\|}.$$

$\mathrm{sep}(A,B) = 0$ if and only if $A$ and $B$ have a common eigenvalue. It is small if there exists a small perturbation of either $A$ or $B$ that makes them have a common eigenvalue.

**A practical CIS algorithm.**

Replace the assumption (1) by

$$\Sigma_1(s) \equiv \{(\lambda_i(s))_{i=1}^m\}, \quad \Sigma_2(s) \equiv \left\{(\lambda_i(s))_{i=m+1}^n\right\},$$
$$\mathrm{Re}\,\lambda_1 \geq \ldots \geq \mathrm{Re}\,\lambda_{m_u} \geq 0 > \mathrm{Re}\,\lambda_{m_u+1} \geq \ldots \geq \mathrm{Re}\,\lambda_m,$$
$$\mathrm{Re}\,\lambda_m > \mathrm{Re}\,\lambda_{m+1} \geq \ldots \geq \mathrm{Re}\,\lambda_n,$$
$$m_s(s) \equiv m(s) - m_u(s) \geq m_s^{ref} > 0,$$

$m$, $m_u$, $m_s$ vary; $m_s^{ref}$ fixed, typically, $2 \leq m_s^{ref} \leq 4$.

- Algorithm, one step: given $Q(s)$, find $Q(s+h)$:
  - Can we actually compute $Q(s+h)$?
  - Once $Q(s+h)$ is computed, is it acceptable or it has to be modified (how?)?

1. Newton iteration for (4) fails to converge with $h_{\min}$.
   (a) Generically (Beyn et. al. 2000): real $\lambda_m$, $\lambda_{m+1}$ coalesce and become a complex conjugate pair.
   (b) Move $\lambda_{m+1}$ to $\Sigma_1(s)$ if $m_s \leq m_s^{ref}$, and $\lambda_m$ to $\Sigma_2(s)$ if $m_s > m_s^{ref}$. Recompute $Q(s)$ by (2).
2. Newton iteration for (4) converges.
   (a) Test for an *overlap*

   $$\psi_1 := \begin{cases} 0, & \mathrm{Re}\,\lambda_m > \mathrm{Re}\,\lambda_{m+1} \text{ (no overlap)}, \\ 1, & \mathrm{Re}\,\lambda_m < \mathrm{Re}\,\lambda_{m+1} \text{ (overlap)}. \end{cases}$$

   If $\psi_1 = 1$, decrease $h$ to ensure that *overlap* is only in one real eigenvalue or in one complex conjugate eigenpair.

(b) Test for a *bifurcation*

$$\psi_2 := \begin{cases} 0, & m_u(s) = m_u(s+h) \text{ (no bifurcation)}, \\ 1, & m_u(s) \neq m_u(s+h) \text{ (bifurcation)}. \end{cases}$$

If $\psi_2 = 1$, update $m$, $m_u$, and $m_s$.

(c) If $\psi_1 = \psi_2 = 1$, decrease $h$, if allowed, else *fail*.

(d) If $\psi_1 = \psi_2 = 0$, accept the point $Q(s+h)$.

(e) If $\psi_1 = 0$ & $\psi_2 = 1$, adapt $m$, $m_u$, and $m_s$, if needed, and then recompute $Q(s+h)$ by (2).

(f) If $\psi_1 = 1$ & $\psi_2 = 0$, adapt $m$, $m_u$, and $m_s$ by swapping or moving some eigenvalues between $\Sigma_1$ and $\Sigma_2$ (depending on whether $m_s \leq m_s^{ref}$ or $m_s > m_s^{ref}$, and whether the eigenvalues in question are real or complex conjugate eigenpair(s)) and then recompute $Q(s+h)$ by (2).

**Stability analysis of the X-33.**

- A co-operative agreement between NASA and Lockheed Martin resulted in development of single stage to orbit reusable launch vehicle called the X-33.
- Simulation model of X-33 includes the 6 degree-of-freedom equations of motion and the engine, aerodynamic, sensor, actuator, wind disturbance, mass property, guidance and control system models.
- Flight control of the X-33 vehicle during assent mode involves attitude maneuvers through a wide range of flight conditions. The vehicle dynamics are rapidly changing in time and changing nonlinearly with respect to propellant usage, aerodynamics, engine thrust, control surface usage, and control command generation.
- Stability analysis during powered operation: linearizing the system at various operating points along its flight trajectory. It is not unusual for the system to be unstable (real part of eigenvalue is positive) for a portion of ascent flight; however, due to the high velocity nature of rocket flight, this instability might not adversely affect system performance because this phase of flight is passed so quickly.
- A problem in stability analysis: choosing the points in flight to capture all events that affect stability and performance of the vehicle.

**Discussion.**

*Continuation/Bifurcation Analysis Future Work.*

- Test approach with a sample RLV concept design.
  - Perform the bifurcation and stability analysis for models like X-33 and identify regions that need more analysis.

- Compare the computational results with experiments.

- Develop a method that relates migration of eigenvalues into the right half-plane to elements of A-matrix and parameters in the nonlinear system of equations.

- Develop an efficient method to determine effects of parameters variations on magnitude of eigenvalue $\lambda_1$ with the largest real part. Idea ($\lambda_1$ real), $A(s)q_1(s) = \lambda_1(s)q_1(s)$. From Eq. (2) with $m = 1$
$$\lambda_1(s) = q_1^{\mathrm{T}}(s)A(s)q_1(s). \qquad (9)$$
Hence the CIS algorithm produces $\lambda_1(s)$ as a *smooth function of parameters.* Hence can use minimization methods for $-\lambda_1(s)$ for smooth functions to determine $\max \lambda_1(s)$ rather than more expensive Monte Carlo.

- Addressing the issue of nonnormality. Recent results show that in the case of nonormal matrices (when the set of the eigenvectors does not form an orthogonal set) eigenvalues may not be a relevant tool to study stability. And one instead should use pseudospectra. Our preliminary results show that A-matrices can be highly nonnormal. This issue has to be investigated further.

*Benefits of Continuation/Bifurcation Analysis Approach.*

Continuation/bifurcation analysis addresses NASA's needs to reduce vehicle development cost and improve safety by aiding in the development of a more robust vehicle.

- *Need to develop safer vehicles*: identifying problematic operating conditions that would have been missed by a conventional stability analysis.

- *Need to reduce cost*: possibly reducing the time spent in sensitivity analyses. If this method of analysis can identify key parameters that affect vehicle stability, then a sensitivity analysis can focus on just these parameters, reducing number of iterations needed.

## Acknowledgements.

The author wishes to thank the NASA/Marshall Space Flight Center TD55/Vehicle Control Systems group: Mark Jackson and Charles Hall for their help in formulating the problem and Steve Ryan, Kerry Funston, and Don Krupp for helpful discussions.

## References.

[1] J.W Demmel, L. Dieci and M.J. Friedman, Computing connecting orbits via an improved algorithm for continuing invariant subspaces, *SIAM J. Sci. Comput.*, **22**, No. 1 (2001), 81–94.

[2] L.Dieci and M.J. Friedman, Continuation of Invariant Subspaces, *Numer. Lin. Alg. Appl.* **8** (2001), 317-327.