# Relativistic Models for a GAIA-Like Astrometry Mission

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## 1. Introduction

A non-perturbative general relativistic approach to global astrometry was developed by de Felice *et al.* (1998) to handle satellite astrometry data in a genuine relativistic framework. In this contribution, the framework above has been further exploited to account for stellar motions and parallax. Because of the relevance that accurate knowledge (to $10^{-5}$ or better) of the relativistic parameter $\gamma$ has to fundamental physics, a Parametrized Post-Newtonian (PPN) model has also been implemented, which allows the direct estimation of $\gamma$ along with the astrometric parameters. These models have been tested on end-to-end simulations of the mission GAIA. The results show that, within the limitation of the simulation and the assumptions of the adopted model, measurements accurate to 100 $\mu$arcsec of large arcs among stars repeated over a few years can be modelled to establish a dense reference frame with a precision of a few tens of $\mu$arcseconds. Moreover, our experiments indicate that $\gamma$ can be estimated to better than $10^{-6}$.

## 2. Relativistic effects in GAIA

In the data reduction of the Hipparcos astrometric mission, the observations were pre-corrected for relativistic effects to $(v/c)^2 \simeq GM_\odot/c^2R_\oplus \sim 2$ mas. The observation equations were then formulated in a three-dimensional Euclidean space. The expected astrometric precision for the GAIA mission is of the order of 100 $\mu$arcsec for stars of magnitude $V = 17$, and $\sim 10$ $\mu$arcsec for $V = 12$ (Lindegren and Perryman, 1996). To meet such goal, it is mandatory that the observations be modelled at the level of $\sim 1$ $\mu$arcsec, in a framework which utilizes a relativistic theory of gravitation.

Especially significant for the GAIA mission are metric perturbations due to the solar system planets which will affect the path of the photons reaching the satellite. The following table lists the light deflection angles due to the major planets at their closest approach to Earth, and for different elongations from the

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light source. We have not considered the internal planets Venus and Mercury because they would be too close to the Sun to be observed by GAIA. The mass of the planet is denoted by $M$, $r_{\oplus}$ is the minimum distance planet-Earth, $D$ the corresponding angular diameter of the planet and $\delta\alpha$ the light deflection angle. It can be seen that the mass of Jupiter produces significant effects up to angular distances of $\sim 15°$, while Saturn needs to be considered up to $\sim 2°$. The mass of Uranus and Neptune might be important only for the brightest stars.

Table 1. Deflections of a light ray due to solar system planets as seen from the Earth for a limb-grazing ray ($\delta\alpha_{lg}$), and for 2 and 15 degrees of elongation.

| Planet | $M \cdot 10^{-23}$ kg | $r_{\oplus} \cdot 10^{-6}$ km | $D$ ($''$) | $\delta\alpha_{\rm lg}$ mas | $\delta\alpha(2°)$ $\mu$arcsec | $\delta\alpha(15°)$ |
|---|---|---|---|---|---|---|
| Mars | 6.4191 | 78.3 | 8.94 | 0.116 | – | – |
| Jupiter | 18992 | 628.7 | 23.43 | 16.289 | 53.0 | 7.0 |
| Saturn | 5686.5 | 1277.4 | 9.76 | 5.765 | 7.8 | – |
| Uranus | 868.49 | 2720.0 | 1.80 | 2.235 | – | – |
| Neptune | 102.35 | 4347.0 | 1.06 | 2.800 | – | – |

## 3. Modelling the observations

The adopted mathematical model consists of a satellite (the observer orbiting on a spatially circular geodesic around the Sun. The latter is assumed non rotating, spherically symmetric and therefore generating a Schwarzschild space-time metric, the Sun conciding with the barycenter of the satellite-Sun system. In this context it is convenient to adopt polar coordinates: the colatitude $\theta \in [0, \pi]$, the azimuth $\phi \in [0, 2\pi]$ and the polar distance $r \in ]0, \infty[$. When the Schwarzschild metric is expressed in the Parametrized Post-Newtonian (PPN) formalism and isotropic coordinates, the two new parameters $\gamma$ and $\beta$ are explicitely introduced (Misner *et al.*, 1973). These parameters are equal to unity in the theory of General Relativity.

The observable quantity for a GAIA-like mission is the light coming from two objects in different directions across the sky reaching the satellite at the same proper time $\tau$. In particular, the cosine $\psi_{12}$ of the angle between a star pair is given by (Brumberg, 1991):

$$\cos\psi_{12} = \frac{h_{\alpha\beta} k_1^\alpha k_2^\beta}{\sqrt{h_{\iota\pi} k_1^\iota k_1^\pi}\sqrt{h_{\rho\sigma} k_2^\rho k_2^\sigma}}, \tag{1}$$

where $k_1$ and $k_2$ are the null-geodesics from the two stars, $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, is a tensor projecting in the rest frame of the observer, and $u^\alpha$ is the four-velocity vector representing the observer's trajectory. Our aim is to express the observable as function of same space coordinates, as well as of the PPN parameters in the post-Newtonian approximation. Equation 1 does not explicitely contain such parameters, and they are introduced by integrating the path of the photons from the light source to the observer (de Felice *et al.*, 2000). Finally, we obtain an equation of the form:

$$\cos\psi_{12}(\tau) = f\left(r_1(t), \theta_1(t), \phi_1(t), r_2(t), \theta_2(t), \phi_2(t); \gamma, \beta\right). \qquad (2)$$

We note that the time at which the angular separation of the star pair is observed is the proper time $\tau$ of the observer. Since the catalog values of the star's parameters are referred to coordinate time, we have to convert $\tau$ into $t$ in order to compute the correct stellar position.

The polar coordinates $(r, \theta, \phi)$ (the only ones appearing as arguments of the function $f$ in a non-perturbative approach) are functionally related to the classical quantities parallax and proper motions, as described in the following section.

## 4. The condition equation

The motion of a distant star at coordinate time $t$ can be formally described by using Taylor expansions with respect to an intial time $t_0$ as:

$$\begin{cases} r(t) &= r(t_0) + \dot{r}(t-t_0) + 0.5\,\ddot{r}(t-t_0)^2 + ... \\ \theta(t) &= \theta(t_0) + \dot{\theta}(t-t_0) + 0.5\,\ddot{\theta}(t-t_0)^2 + ... \\ \phi(t) &= \phi(t_0) + \dot{\phi}(t-t_0) + 0.5\,\ddot{\phi}(t-t_0)^2 + ... \end{cases} \qquad (3)$$

In principle, $r$, $\theta$, and $\phi$ are not directly observable; however, since $r \gg r_\oplus$, the above coordinates can be used to define the observable stellar *parallax* and *proper motion, i.e.*, $p \equiv r_\oplus/r, r_\oplus = 1AU$, $\mu_\theta \equiv \dot{\theta}$, $\mu_\phi \equiv \dot{\phi}$. To conveniently truncate the Taylor expansions, let us evaluate the rate of change of $p$, $\mu_\phi$ and $\mu_\theta$, *i.e.*, $\dot{p} = -\frac{r_\oplus}{r^2}\dot{r}$, $\dot{\mu_\theta} = \ddot{\theta}$, and $\dot{\mu_\phi} = \ddot{\phi}$. It can be seen that (Green, 1985) $\dot{p} = -0.2\ V_r p^2\ sin1''$, and $\dot{\mu} = -0.1\ V_r V_t p^2\ sin1''$, where $\mu = \sqrt{\mu_\theta + \mu_\phi}$ is expressed in arcseconds/yr, $V_r$ and $V_t$ are the star's barycentric radial and tangential velocities in km/sec, and $p$ is in arcseconds. Straightforward calculations show that, while $\dot{p}$ is negligible in virtually all cases, the effect of $\dot{\mu}$ will be detected by GAIA for high-velocity stars ($\approx 100$ Km/sec) in the vicinity of the Sun ($\approx 100$ pc), which represent a very low fraction of the total number of potential targets. Therefore, for the purpose of this experiment, the final model included the zero-th-order term in $r$ and up to the first-order terms in $\theta$ and $\phi$ of Equation 3. To estimate the astrometric parameters, along with the PPN parameters, from the measured arcs, we substitute Equation 3 in 2, linearize Equation 2 with respect to some *a priori* known values, then solve the linear system using a least-squares method. The resulting condition equation is the following:

$$-\sin\psi_{12}\delta\psi_{12} = \sum_{i=1}^{2}\left(A_i\delta p_i + B_i\delta\theta_i + C_i\delta\phi_i + D_i\delta\mu_{\theta i} + E_i\delta\mu_{\phi i}\right) + F\delta\gamma + G\delta\beta, \qquad (4)$$

where the coefficients $A_i$, $B_i$, $C_i$, $D_i$, $E_i$, $F$ and $G$ are partial derivatives of the function $f$ calculated at catalog values. The quantity $\delta\psi_{12}$ is the difference between the measured and the catalog value of the angular separation between the star pair. The small differences in the right-hand side of the condition equation represent corrections to the sought for parameters.

At present, we have only developed the calculations to estimate $\delta\gamma$, though the inclusion of $\delta\beta$ does not represent a major difficulty in principle.

## 5. Simulation of the observations

The end-to-end simulation of the GAIA observations follows the same three-step procedure used for the static case (de Felice *et al.*, 1998), with the obvious complications due to the inclusion of stellar proper motions and parallaxes, and the introduction of the relativistic parameter $\gamma$. Measurements (arcs between stars) are generated by a satellite that sweeps the sky according to a Hipparcos-like scanning law, and the option of three viewing directions (FOVs), studied for GAIA, is implemented.

Stars are simulated within a uniform density sphere of radius 500 pc (*i.e.* $p \geq 2$ mas) and proper motions are computed for each star on the basis of its radial velocities, randomly generated with $\langle v_r \rangle = 0$ km/s and $\sigma_{v_r} = 15$ km/s. Table 2 lists the numerical values adopted for the main simulation parameters.

Table 2. Relevant parameters of the GAIA dynamic simulation.

| *parameter* | *numerical value* |
|---|---|
| satellite orbital radius | $1.496 \cdot 10^{11}$ m |
| satellite precession angle | 43° |
| satellite spin period | 128 min |
| angles between FOVs | 54°, 78°.5 |
| amplitude of each FOV | 1°.6 |
| radius of the simulated sphere | 2 mas |
| mean stars transverse velocity | 0 km/s |
| $\sigma$ of the transverse velocity distribution | 15 km/s |
| catalog error for parallaxes | 2 mas |
| catalog error for angular coordinates | 2 mas |
| catalog error for proper motions | 2 mas |

## 6. Results and discussion

Table 3 shows the final astrometric errors for seven different simulations of the non-perturbative dynamical model. The mission length varies from one to five years. The catalog reference (mean) time is always at half the mission lifetime ($t_0 = \Delta T/2$), which provides the most accurate positions, while the correlation between position and proper motion is minimized (Eichhorn and Googe, 1969). We note that the true errors of the five stellar parameters scale well with the square root of the number of observations ($n_{obs}$). It is also evident that the model becomes inadequate below $\Delta T = 2$ yr, as the relatively short time baseline makes it difficult to disentangle the proper motion components of stellar motion from the parallax.

Table 4 reports the astrometric results along with the estimation of $\gamma$ in the perturbative (PPN) case for a static sphere (only star positions are considered), and a mission duration of 1 year. With these runs we have tested the influence of the number of stars ($N_*$) on the estimation of $\gamma$. The results show that with a measurement error of 10 $\mu$arcsec, an increase of $N_*$ from 2000 to 5000 gives $\delta\gamma$ in the range $5 \cdot 10^{-5} - 6 \cdot 10^{-6}$. These results made us confident that even better results will be achieved by GAIA; the inclusion of the PPN parameter $\beta$ can be done under the same framework with a little increase of mathematical complexity. Further experiments in this direction are underway.

Table 3. Astrometric errors for the non-perturbative dynamic case. Additional input parameters are: $\sigma_{oss}$ (single-measurement error) = 100 $\mu$arcsec and $N_*$ (number of simulated stars) = 2000.

| $\Delta T$ (yr) | $n_{obs}$ | $\sigma_{\delta p}$ ($\mu$as) | $\sigma_{\delta\theta}$ ($\mu$as) | $\sigma_{\sin\theta\,\delta\phi}$ ($\mu$as) | $\sigma_{\delta\mu_\theta}$ ($\mu$as/yr) | $\sigma_{\sin\theta\,\delta\mu_\phi}$ ($\mu$as/yr) |
|---|---|---|---|---|---|---|
| 5 | 286431 | 15.79 | 9.49 | 13.54 | 6.58 | 7.84 |
| 4 | 227869 | 17.88 | 10.67 | 15.28 | 9.60 | 11.31 |
| 3 | 172150 | 20.73 | 12.97 | 17.88 | 15.65 | 19.34 |
| 2.5 | 144063 | 22.98 | 15.56 | 19.91 | 21.52 | 25.32 |
| 2 | 115218 | 29.75 | 21.06 | 26.64 | 67.32 | 52.69 |
| 1.5 | 86028 | 262.92 | 200.75 | 136.83 | 679.07 | 751.66 |
| 1 | 57892 | 3691.33 | 1154.09 | 1927.44 | 5930.67 | 6827.99 |

Table 4. Final errors of astrometric parameters and the PPN-$\gamma$ parameter for the perturbative (PPN) static case. Additional input parameters are: $\sigma_{oss} = 10$ $\mu$arcsec and $\Delta T = 1$ yr.

| $N_*$ | $n_{obs}$ | $\langle\delta\theta\rangle$ ($\mu$as/yr) | $\sigma_{\delta\theta}$ ($\mu$as/yr) | $\langle\sin\theta\,\delta\phi\rangle$ ($\mu$as/yr) | $\sigma_{\sin\theta\,\delta\phi}$ ($\mu$as/yr) | $\delta\gamma \cdot 10^3$ ($\gamma_t - \gamma_c$) |
|---|---|---|---|---|---|---|
| 2000 | 58987 | -0.08 | 4.69 | -1.52 | 10.95 | 0.0532 |
| 2500 | 92061 | -0.01 | 2.09 | -1.87 | 2.72 | 0.0090 |
| 3000 | 132402 | -0.01 | 2.37 | -2.08 | 2.51 | 0.0364 |
| 3500 | 177642 | 0.06 | 2.40 | 2.26 | 2.34 | -0.0269 |
| 4000 | 230039 | 0.25 | 1.59 | 0.59 | 1.90 | -0.0071 |
| 4500 | 289399 | 0.03 | 1.63 | -2.03 | 1.78 | -0.0097 |
| 5000 | 358002 | -0.07 | 2.95 | 3.90 | 2.46 | -0.0066 |

**Acknowledgments.** This work has been partially supported by the Italian Space Agency (ASI).

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