ROGERS' CONTINUED FRACTIONS: NEW PROPERTIES

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Abstract

Rogers gives three cases of infinite continued fractions which terminate for certain parameter values. We have analyzed the associated integrals and produced equivalent rational factor ratios.

Key words: Asymptotes; fixed points; Frullani integrals; terminated c.f.s.

1 Introduction

In 1907 Rogers introduced the three continued fractions (c.f.s),

$$R_1(a,x) = \exp\left\{\int_0^\infty \frac{1}{t}\left(1 - \frac{\cosh(at)}{\cosh(t)}\right) e^{-t/(2x)} dt\right\} \tag{1}$$
$$= 1 + \frac{2(1^2-a^2)x^2}{1+}\,\frac{(3^2-a^2)x^2}{1+}\,\frac{(5^2-a^2)x^2}{1+}\cdots. \qquad \left(\frac{1}{2x} > a-1\right)$$

$$R_2(a,x) = \tanh\left\{\frac{1}{2}\int_0^\infty \frac{\sinh(2at)}{t\cosh(t)} e^{-t/x} dt\right\} \tag{2}$$
$$= \frac{ax}{1+}\,\frac{(1^2-a^2)x^2}{1+}\,\frac{(2^2-a^2)x^2}{1+}\,\frac{(3^2-a^2)x^2}{1+}\,\frac{(4^2-a^2)x^2}{1+}\cdots \qquad \left(\frac{1}{x} > 2a-1\right)$$

and

$$R_3(a,x) = \tanh\left\{\int_0^\infty \frac{\sinh(at)}{t\cosh(t)} e^{-t/x} dt\right\} \tag{3}$$
$$= \frac{ax}{1+}\,\frac{1^2x^2}{1+}\,\frac{(2^2-a^2)x^2}{1+}\,\frac{3^2x^2}{1+}\,\frac{(4^2-a^2)x^2}{1+}\,\frac{5^2x^2}{1+}\cdots \qquad \left(\frac{1}{x} > 2a-1)\right)$$

Rogers in his first paper derived these by an intricate manipulation of power series; the forms (1), (2), and (3) are given in Rogers' second paper (Supplementary note). Although convergence questions are addressed there is no attempt to analyze the integrals involved.

Here we take the case when these c.f.s terminate, i.e. when the parameter a is an integer. Since there is little to be gained by considering negative values of a we assume that a is positive and an integer.

2 The case $R_1(a,x)$ and periodicity

Note first of all some elementary properties of the binomial

$$(X-Y)^n \qquad (X = e^t, Y = e^{-t}),$$

when n is a positive integer. If n is even, there is a middle term. Moreover this middle term may be positive or negative, positive for $4n$, negative for $4n+2$. Returning to

R_1, and $a = 4n+1$

$$\frac{X^{4n+1}+Y^{4n-1}}{X+Y} = \sum_{s=0}^{2n}(-1)^s(X^{4n+1-2s}+Y^{4n+1-2s})$$
$$= \sum_{s=0}^{2n}(-1)^s(X^{4n+1-2s}-1+Y^{4n+1-2s}-1)+1.$$

Looking at R_1 we find the unity above is removed.

Now Hardy (1904) introduced the Frullani integral, a simple example of which is

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt = \ln\left(\frac{b}{a}\right) \qquad (b > a > 0)$$

leading to the following value:

$a = 5$, $n = 1$:

$$\exp\{\ln(1-8x)+\ln(1+8x)-\ln(1-6x)+\ln(1+6x)\} = \frac{1-8^2x^2}{1-4^2x^2} \qquad (|8x| \leq 1)$$

$a = 9$, $n = 2$:

$$\exp\{\ln(1-16x)+\ln(1+16x)+\ln(1-8x)+\ln(1+8x)$$
$$-[\ln(1-12x)+\ln(1+12x)+\ln(1-6x)+\ln(1+6x)]\}$$
$$= \frac{(1-16^2x^2)(1-8^2x^2)}{(1-12^2x^2)(1-4^2x^2)} \qquad (4x \leq 1).$$

In general then for $a = 4n+1$,

$$R_1(n,x) = \frac{\prod_{s=0}^{n-1}\{1+[8(n-s)x]^2\}}{\prod_{s=0}^{n-1}\{1+[2(4n-4s-2)x]^2\}} \qquad (|8nx| \leq 1)$$

Similarly for $a = 4n-1$,

$$R_1(n,x) = \frac{\prod_{s=0}^{n-1}\{1+[2(4n-4s-2)x]^2\}}{\prod_{s=0}^{n-2}\{1+[2(4n-4s-4)x]^2\}} \qquad (n = 2, 3, \cdots)$$

for $|2(4n-1)x| < 1$.

Perhaps a more informative set of results is as follows, at least formally writing $y = 1/(4x)$:

			c.f. values	
a	c.f.s		$y=4$	$y=7$
3	$\frac{y^2-1}{y^2}$		0.937500	0.9795918
5	$\frac{y^2-2^2}{y^2-1}$	$(y > 2)$	0.800000	0.9375000
7	$\frac{(y^2-3^2)(y^2-1^2)}{(y^2-2^2)y^2}$	$(y > 3)$	0.546875	0.9375000
9	$\frac{(y^2-4^2)(y^2-2^2)}{(y^2-3^2)(y^2-1^2)}$	$(y > 4)$		0.7734375
11	$\frac{(y^2-5^2)(y^2-3^2)(y^2-1^2)}{(y^2-4^2)(y^2-2^2)y^2}$	$(y > 5)$		0.6332715
13	$\frac{(y^2-6^2)(y^2-4^2)(y^2-2^2)}{(y^2-5^2)(y^2-3^2)(y^2-1^2)}$	$(y > 6)$		0.4189453

3 The case $R_2(a,x)$

3.1 Basic formulas

$$R_2(a,x) = \tanh\left\{\frac{1}{2}\int_0^\infty \frac{\sinh(2at)}{t\cosh(t)} e^{-t/x} dt\right\} \qquad \left(\frac{1}{x} > 2a-1\right)$$
$$= \frac{ax}{1+}\,\frac{(1^2-a^2)x^2}{1+}\,\frac{(2^2-a^2)x^2}{1+}\,\frac{(3^2-a^2)x^2}{1+}\,\frac{(4^2-a^2)x^2}{1+}\cdots$$

When $a^2 = 1$, or 2^2 etc., the c.f. terminates. In the notation of §2 the hyperbolic component of the integral is

$$\frac{X^{2n}-Y^{2n}}{X+Y} = (X^{2n-1}-Y^{2n-1}) - (X^{2n-3}-Y^{2n-3}) + \cdots + (X-Y)(-1)^{n-1}$$

when $a = n$, a positive integer. Using Frullani integrals, the integrand leads to

$$R_2(n,x) = \frac{r_n^{(2)}(x) - r_n^{(2)}(-x)}{r_n^{(2)}(x) + r_n^{(2)}(-x)}, \qquad \left(\frac{1}{x} > 2n-1\right)$$

where

$$r_n^{(2)}(x) = \prod_{s=0}^{n-1}\{1+(-1)^s(2n-2s-1)x\} \quad (n = 1, 2, \cdots) \tag{4}$$

For examples,

$$R_2(1,x) = x, \quad R_2(2,x) = \frac{2x}{1-3x^2}, \quad R_2(3,x) = \frac{3x(1-5x^2)}{1-13x^2},$$
$$R_2(4,x) = \frac{4x(1-19x^2)}{1-34x^2+105x^4}, \quad R_2(5,x) = \frac{5x(1-46x^2+189x^4)}{1-70x62+789x^4}$$

The denominators are taken to be positive.

3.2 Fixed points and asymptotes

Consider the case $n = 4$. Here

$$r_4^{(2)}(x) = (1+7x)(1-5x)(1+3x)(1-x)$$

so that for example,

$$r_4^{(2)}\left(-\frac{1}{7}\right) = 0, \quad \text{and} \quad R_2\left(4, -\frac{1}{7}\right) = -1.$$

Similarly $R_2(4, \frac{1}{7}) = 1$ and we are lead to the fixed points located as follows:

$n=4$: Fixed points for x in the terminated c.f.

x	-1	-$\frac{1}{3}$	$-\frac{1}{5}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{3}$	1
$R_2(4,x)$	1	-1	1	-1	1	-1	1	-1

These exact fixed points are scarcely evident in the rational fraction in $R_2(5,x)$. In general these points can be easily identified. For example, for $R_2(5,x)$ we look at

$$(1+9x)(1-7x)(1+5x)(1-3x)(1+x),$$

and for example $x>0$, the points are

$$(\frac{1}{9},1),\ (\frac{1}{7},-1),\ (\frac{1}{5},1),\ (\frac{1}{3},-1), (\frac{1}{1},1).$$

Returning to $n=4$, zeros of $R_2(4,x)$ are solutions to

$$r_4^{(2)}(x)=r_4^{(2)}(-x),$$

and potential asymptotic solutions to

$$r_4^{(2)}(x)+r_4^{(2)}(-x)=0,$$

in both cases it appears that the factors in $R_2(4,x)$ are relative primes.

Figure 1 refer to the second Rogers c.f. $R_2(n,x)$, with $n=4$, $n=5$ and $n=8$. Note that $R_2(4,x)\to -\frac{76}{105x}$, whereas $R_2(5,x)\to\infty$, as $x\to\infty$. From Figure 1 ($n=8$) it is evident that the asymptotics are becoming dense. Overall the Figures make it clear that $R_2(a,x)$ when a is an integer can be zero, $\pm\infty$, and for a specified set of rational functions ($\frac{1}{15},\frac{1}{13},\cdots$, for $n=8$) take the values +1 or -1.

4 The case $R_3(a,x)$ and terminating c.f.

The c.f. terminates when $a^2=2^2,\ 4^2,\ \cdots$, but the integral comes out resembles that occurring in $R_2(a,x)$. Thus $a=2n$ in R_3 may be derive from R_2 taking $a=n$; note that R_3 involves $\tanh\phi$ whereas R_2 involves $\tanh\frac{1}{2}\phi$. We find

$$R_3(n,x)=\frac{\{r_n^{(2)}(x)\}^2-\{r_n^{(2)}(-x)\}^2}{\{r_n^{(2)}(x)\}^2+\{r_n^{(2)}(-x)\}^2}.$$

where $r_n^{(2)}(x)$ is given in (4). There are no asymptotes for real x; the zeros are from that of $r_n^{(2)}(x)-r_n^{(2)}(-x)$, and $r_n^{(2)}(x)+r_n^{(2)}(-x)$.
Examples:
$a=2$ in c.f., $n=1$ in R_3.

$$R_3(1,x)=\frac{(1+x)^2-(1-x)^2}{(1+x)^2+(1-x)^2}=\frac{2x}{1+x^2}.$$

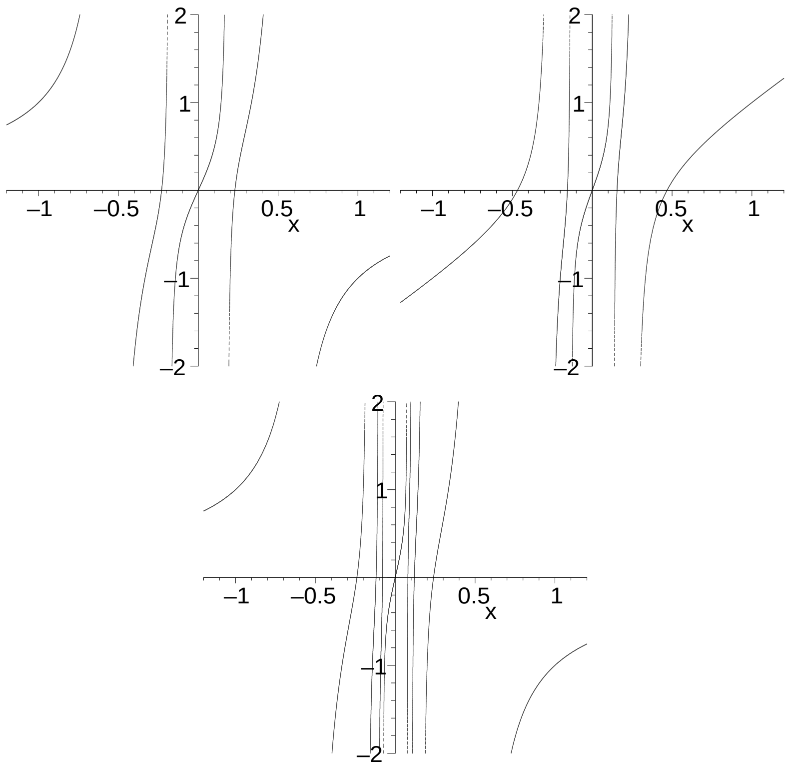

Figure 1: Rogers Case 2 with $n = 4$ (left), $n = 5$ (right), and $n = 8$ (center)

$a = 4$ in c.f., $n = 2$ in R_3.

$$r_2^{(2)}(x) = (1+3x)(1-x) = 1 + 2x - 3x^2$$
$$R_3(2,x) = \frac{(1+2x-3x^2)^2 - (1-2x-3x^2)^2}{(1+2x-3x^2)^2 + (1-2x-3x^2)^2} = \frac{4x(1-3x^2)}{1-2x^2+9x^4}.$$

5 Further examples of terminating c.f.

The graphics for $R_3(n,x)$ are similar to those of $R_2(n,x)$ except there are no real asymptotics.

It is appropriate to mention some examples of c.f.s which terminate, for a particular parameter values, given in Wall (1948, p346)

$$(a) \quad \frac{(1+z)^k - (1-z)^k}{(1+z)^k + (1-z)^k} = \frac{kz}{1+} \; \frac{(k^2-1)z^2}{3+} \; \frac{(k^2-2^2)z^2}{5+} \; \frac{(k^2-3^2)z^2}{7+} \cdots, \quad (\text{see } R_2(n,x))$$

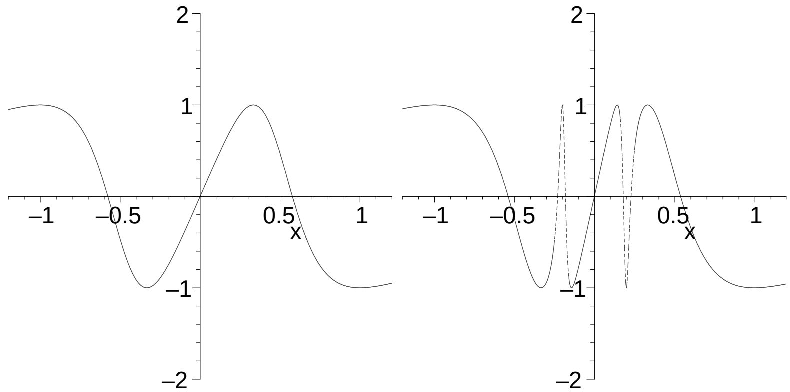

Figure 2: Rogers Case 3 with $n = 2$ (left), and $n = 4$ (right)

$$(b)\quad \left(\frac{z+1}{z-1}\right)^k = 1 + \frac{2k}{z-k-}\,\frac{(1-k^2)}{3z-}\,\frac{(2^2-k^2)}{5z-}\,\frac{(3^2-k^2)}{7z-}\cdots.$$

$$(c)\quad \exp\left(2k \arctan\frac{1}{z}\right) = 1 + \frac{2k}{z-k+}\,\frac{(1^2+k^2)}{3z+}\,\frac{(2^2+k^2)}{5z+}\,\frac{(3^2+k^2)}{7z+}\cdots$$

when $k = ni$, n a positive integer, c.f. reduces to $\dfrac{(z+i)^{2n}}{(z^2+1)^n}$.

$$(d)\quad \tan(k\phi) = \frac{k\tan\phi}{1-}\,\frac{(k^2-1)\tan^2\phi}{3-}\,\frac{(k^2-2^2)\tan^2\phi}{5-}\,\frac{(k^2-3^2)\tan^2\phi}{7-}\cdots$$

From Wall (1948, p347), there is a more complicated c.f. related to $\Phi(b+1, c+1; z)/\Phi(b, c; z)$, where

$$\Phi(b, c; z) = 1 + \frac{bz}{c} + \frac{b(b+1)z^2}{c(c+1)2!} + \frac{b(b+1)(b+2)z^3}{c(c+1)(c+2)z!} + \cdots$$

From Stieltjies (Wall, 1948, p359),

$$\int_0^\infty \left\{\frac{1-c}{e^{u(1-c)} - c^b}\right\}^a e^{zu} du = \frac{m^a}{z+}\,\frac{am}{1+}\,\frac{mc^b}{z+}\,\frac{(a+1)m}{1+}\,\frac{2mc^b}{z+}\,\frac{(a+2)m}{1+}\,\frac{3mc^b}{z+}\cdots$$

where $m = (1-c)/(1-c^b)$, $a > 0$, $b > 0$, $c > 0$, and z in the complex plane split from $-\infty$ to 0^+. But we may take a negative with an appropriate restriction on z; for example, when $a = -1$ the integral relates to

$$\frac{1}{1-c}\left\{\frac{1}{z+c-1} - \frac{c^b}{z}\right\},$$

agreeing with the terminated c.f.

From Wall (1948, P352),

$$\frac{\int_0^\infty \frac{e^{-u}u^{a-1}du}{(1+zu)^b}}{\int_0^\infty \frac{e^{-u}u^{a-1}du}{(1+zu)^{b-1}}} = \frac{1}{1+}\,\frac{az}{1+}\,\frac{bz}{1+}\,\frac{(a+1)z}{1+}\,\frac{(b+1)z}{1+}\,\frac{(a+2)z}{1+}\,\frac{(b+2)z}{1+}\cdots$$

provided the integrals exist, and for z in the split z-plane. The parameter b for negative integer values terminates the c.f..

From Perron (Vol 2, 1957, p36),

$$1+\frac{1^2+n^2}{2+}\,\frac{3^2+n^2}{2+}\,\frac{5^2+n^2}{2+}\cdots \equiv n\coth\left(\frac{n\pi}{4}\right) \qquad (n^2>-1).$$

Here we may set n to be Ni, N an integer, terminating the c.f.. We may also take $n=0$ and limit of the hyperbolic function, to obtain the known c.f.

$$\frac{4}{\pi}=1+\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots, \quad \text{or} \quad \frac{\pi}{4}=\frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots.$$

From Perron (1957, p33), for Psi function

$$\psi\left(\frac{x+3+n}{4}\right)+\psi\left(\frac{x+3-n}{4}\right)-\psi\left(\frac{x+1+n}{4}\right)-\psi\left(\frac{x+1-n}{4}\right)$$
$$=\frac{4}{x+}\,\frac{1^2-n^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2-n^2}{x+}\,\frac{4^2}{x+}\,\frac{5^2-n^2}{x+}\cdots$$

for $x>0$, $1>n^2>-\infty$. This expansion may be compared with Rogers' case $R_2(a,x)$ in (2). For $n=3$,

$$\psi\left(\frac{x}{4}+\frac{3}{2}\right)+\psi\left(\frac{x}{4}\right)-\psi\left(\frac{x}{4}+1\right)-\psi\left(\frac{x}{4}-\frac{1}{2}\right)=-\frac{1}{\frac{x}{4}+\frac{1}{2}}+\frac{1}{\frac{x}{4}-\frac{1}{2}}-\frac{4}{x}$$
$$=\frac{4}{x+2}+\frac{4}{x-2}-\frac{4}{x}=\frac{4(x^2+4)}{x(x^2-4)},$$

which agrees with the c.f. approach.

The hyper-geometric ratios

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)}$$

expressed as the Gauss c.f. provides several examples of c.f.s which terminates.

6 Concluding remarks

For a few moderate to small values of the parameter a in the three Rogers' c.f.s we have found exact rational function equivalents, featuring fixed points and for R_1 and

R_2 factoring asymptotes. The c.f. equivalents are possibly valid for all x; this might be considered using analytic continuation approaches .

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