Applications of Statistics and Probability, Lemaire, Favre & Mébarki (eds) © 1995 Balkema, Rotterdam. ISBN 90 5410 563 1

Melnikov necessary condition for noise-induced escapes

E. Simiu
National Institute of Standards and Technology, Gaithersburg, Md. & The Johns Hopkins University, Baltimore, Md., USA

M. Frey
Bucknell University, Lewisburg, Pa., USA

C. Hagwood
National Institute of Standards and Technology, Gaithersburg, Md., USA

ABSTRACT: For a wide class of nonlinear multistable deterministic systems, a necessary condition for the occurrence of chaos -- and jumps between phase space regions associated with potential wells -- is that the system's Melnikov function have simple zeros. The work presented in this paper is based on our extension of the Melnikov-based approach to a class of nonlinear stochastic differential equations with additive or multiplicative noise. The mean zero upcrossing rate for the stochastic system's Melnikov process is a weak upper bound for the system's mean escape rate. For systems excited by processes with tail-limited distributions the stochastic Melnikov approach yields a simple criterion guaranteeing the non-occurrence of chaos. This is illustrated for excitation by square wave, coin-toss dichotomous noise.

1 INTRODUCTION

The Melnikov approach is used to obtain necessary condition for the chaotic behavior of a wide class of multistable deterministic systems. For those systems no jumps are possible between regions associated with the potential wells unless the Melnikov-based necessary conditions for chaos are satisfied (Wiggins, 1992). The Melnikov approach has been applied to wind, ocean and seismic engineering, e.g., in studies of the behavior induced by wind in a quasi-geostrophic model of coastal currents over topography (Allen et al, 1991), ship capsizing due to wave forces (Thompson et al, 1990), and the rocking response of rigid objects to earthquakes (Yim and Lin, 1992). In these studies the excitation was assumed to be deterministic because Melnikov theory was originally developed for deterministic systems.

In fact the theory can be extended to a wide class of stochastic differential equations with various types of additive and multiplicative noise. This paper reviews and illustrates the recent development and application of stochastic Melnikov theory, a term we apply to our extension of Melnikov theory to stochastic systems. The extension is based on the observation that, for a wide class of dynamical systems, a stochastic additive or multiplicative excitation induces a stochastic Melnikov process (Frey and Simiu, 1993; Simiu and Frey, 1995; Frey and Simiu, 1995; Simiu and Hagwood, 1995). The Melnikov process has the property that its mean zero upcrossing rate, denoted by $1/\tau_u$, is a weak upper bound for the system's mean exit rate $1/\tau_e$: to within an approximation of order one, on average no transport can occur during a time interval smaller than τ_u across the system's pseudoseparatrix (for the definition of this term, see Wiggins, 1990, p. 528). Finally, for a system excited by noise with tail-limited distribution the stochastic Melnikov approach yields a remarkably simple criterion guaranteeing the non-occurrence of exits. We conclude that this approach can provide information on system behavior in a class of problems for which the Fokker-Planck equation -- otherwise a much more powerful approach -- is impractical or inapplicable.

Section 2 describes a class of systems to which the stochastic Melnikov approach is applicable, and briefly reviews basic material needed for our development of this approach. Section 3 applies the stochastic Melnikov approach to a typical system with colored or white Gaussian noise. It discusses the use of the mean zero upcrossing rate of the Melnikov process, $1/\tau_u$, as a weak upper bound for the system's mean exit rate $1/\tau_e$. Section 4 discusses the application of the stochastic Melnikov approach to systems excited by noise with tail-limited distributions. Conclusions are presented in Section 5.

Proceedings of the ICASP 7th Conference, July 10-13, 1995,
Paris, France. 1995

2. DYNAMICAL SYSTEMS; MELNIKOV FUNCTIONS AND PROCESSES

2.1 *Systems Definition*

We consider systems of the form

$$\ddot{z} = -V'(z) + \epsilon[g(t) + \gamma G(t) - f(z,\dot{z})] \tag{2.1}$$

where $\epsilon < 1$, γ is a constant, g(t) is a bounded, uniformly continuous function, and V(z) is a potential function. The function $f(z,\dot{z})$ may, for example, take the form $\beta\dot{z}$, $\beta > 0$, in which case it represents viscous damping. For definiteness, in the remainder of this paper we will consider this form. We assume that: (i) the unperturbed system ($\epsilon = 0$) is integrable; (ii) V(z) has the shape of a multiple well so that the unperturbed system has a center at the bottom of each well and a saddle point at the top of the barrier between two adjacent wells. The stable and unstable manifolds emanating from the saddle point form homoclinic or heteroclinic orbits. Finally, we assume G(t) is a bounded, uniformly continuous function, or a random process with properties to be defined later. As a typical example we consider the Duffing-Holmes equation, which has potential

$$V(z) = z^4/4 - z^2/2, \tag{2.2}$$

homoclinic orbits with coordinates

$$z_s(t) = (2)^{1/2}\mathrm{sech}(t);\ \dot{z}_s(t) = (2)^{1/2}\mathrm{sech}(t)\tanh(t) \tag{2.3}$$

and a modulus of the Fourier transform of the function $h(t) \equiv \dot{z}_s(-t)$

$$S(\omega) = (2)^{1/2}\pi\omega\,\mathrm{sech}(\pi\omega/2). \tag{2.4}$$

We also note for later use that

$$c \equiv \int_{-\infty}^{\infty} \dot{z}_s^2(\tau)d\tau = 4/3 \tag{2.5}$$

We review briefly the following cases.

2.2 *Case 1. G(t) is a bounded and uniformly continuous function.*

In this case, for sufficiently small ϵ, the perturbed system possesses invariant stable and unstable manifolds; their intersection with an arbitrary plane of section ("time slice"), t=const, is a pair of curves approaching asymptotically a saddle point that is ϵ-close to the saddle point of the unperturbed system. The stable and unstable manifolds of the perturbed system no longer coincide, as they do in the unperturbed case. To first order, the distance between them, known as the Melnikov distance, is proportional to the Melnikov function. The Smale-Birkhoff theorem states that a necessary condition for chaos is that the Melnikov function of the system have simple zeros (Guckenheimer and Holmes, 1986). The following example, based on work by Beigie et al (1991), provides a stepping stone we use later in this section to deal with excitations by Gaussian random processes.

Example 1. Consider the bounded and uniformly continuous function

$$G(t) = \sum_{i=1}^{N} \cos[\omega_i(t+t_i)] \tag{2.6}$$

where $t_i = \phi_i/\omega_i$ and ϕ_i denote phase angles. The Melnikov function induced by G(t) and g(t) is

$$M(t,t_o,t_1,..,t_N) = -\beta\int_{-\infty}^{\infty}\dot{z}_s^2(\tau)d\tau + \int_{-\infty}^{\infty}h(\tau)g(t+t_o-\tau)d\tau$$

$$+\gamma\int_{-\infty}^{\infty}h(\tau)\sum_{n=1}^{N}\cos[\omega_i(t+t_i-\tau)]d\tau \tag{2.7}$$

where $h(\tau) = \dot{z}_s(-\tau)$ (Wiggins, 1992). Then

$$M(t,t_o,..,t_N) = -\beta c + x(t,t_o) + \gamma\sum_{i=1}^{N} S(\omega_i)\sin[\omega_i(t+t_i)] \tag{2.8}$$

where $x(t,t_o)$ denotes the second integral in the r.h.s. of Eq. 2.7. $S(\omega_i)$ are admittance functions, referred to in the context of Melnikov theory as scaling factors for the frequencies ω_i (Beigie et al., 1991). The necessary condition for chaos is that $M(t,t_o,t_1..,t_N)$ have simple zeros.

2.3 *Case 2. G(t) is a nearly Gaussian, ensemble-uniformly-continuous (EUC) random process with specified one-sided spectral density.*

A stochastic process G(t) is EUC if, given any

$\delta_1>0$, there exists $\delta_2>0$ such that, if $|t_2-t_1|<\delta_2$, then $|G(t_2)-G(t_1)|<\delta_1$ for all times t_1 and t_2 and all realizations of G(t) (Frey and Simiu, 1993). Each realization of G(t) of a EUC process is bounded and uniformly continuous. A sufficiently small ϵ guarantees that to the random process G(t) there corresponds an ensemble of stable and unstable manifolds such that their intersection with an arbitrary plane of section, t=const, is an ensemble of pairs of curves approaching asymptotically an ensemble of saddle points that are ϵ-close to the saddle point of the unperturbed system. To first order, the distance between the stable and unstable manifold for a realization of G(t) is proportional to the Melnikov function induced by that realization. For any realization of G(t), the necessary condition for chaos is that the corresponding Melnikov path have simple zeros. Examples 2a and 2b which follow provide a method for dealing with Gaussian colored and Gaussian white noise in the context of Melnikov theory.

Example 2a: Colored Gaussian Noise. We consider the bounded, EUC random process

$$G(t) \equiv G_N(t) = (2/N)^{1/2} \sum_{i=1}^{N} \cos(\omega_i t+\phi_i) \qquad (2.9)$$

where the parameter N of the process is finite, and ϕ_i and ω_i (i=1,..,N) are independent, identically distributed random variables with, respectively, uniform distribution over the interval $[0,2\pi]$, and probability density function $p(\omega_i)=2\pi\Psi(\omega_i)$. The process $G_N(t)$, known as Shinozuka noise, has unit variance and spectral density $2\pi\Psi(\omega)$ (Shinozuka, 1971). The Melnikov random process induced by $G_N(t)$ is

$$M_N(t) = -\beta c+ xz(t) + \gamma \int_{-\infty}^{\infty} h(\tau)G_N(t-\tau)d\tau \quad (2.10)$$

where notations of Eqs. 2.7 and 2.8 are used and the parameters $t_o,t_1,..,t_N$ may be omitted (Frey and Simiu, 1993). The expectation, spectral density and variance of $M_N(t)$ are

$$E[M_N(t)]=-\beta c+x(t);\ \Psi_{MN}(\omega)=2\pi\gamma^2 S^2(\omega)\Psi(\omega);$$

$$Var[M_N]=\gamma^2\int_0^{\infty} S^2(\omega)\Psi(\omega)d\omega \qquad (2.11a,b,c)$$

The integral of Eq. 2.10 has the same form as the sum of Eq. 2.8. The marginal distribution of that integral, and hence the marginal distribution of the process $M_N(t)$, is Gaussian in the limit $N\to\infty$ (Simiu and Frey, 1993). By choosing a sufficiently large N, that marginal distribution can be made as close to a Gaussian distribution as desired; that is, given any $M_{max}>0$ and $\delta>0$, there exists N such that $|P_N(M)-P(M)|<\delta$, where $M<M_{max}$, $P_N(M)$ is the marginal distribution of $M_N(t)$, and the distribution $P(M)=\lim_{N\to\infty}P_N(M)$ is Gaussian. For sufficiently large N the distribution $P_N(M)$ will be an entirely adequate approximation to P(M), however close the requisite approximation. Owing to the technical requirement of boundedness and uniform continuity needed to prove that the saddle point persists under perturbation, we do not use the limit $N\to\infty$, and resort instead to the technicality of using N finite but sufficiently large; that is, when we use the term Gaussian we refer to a process with distribution $P_N(M)$ that is arbitrarily close to the Gaussian distribution P(M).

Example 2b: Gaussian White Noise. We now consider the sequence of processes (k=1,2,...)

$$G(t) \equiv G_{N,k}(t) = (2/N)^{1/2} \sum_{i=1}^{N} \cos(\omega_{i,k} t+\phi_{i,k}) \quad (2.12)$$

with spectral densities

$$\Psi_k(\omega) = \begin{cases} 2\pi & 0<\omega\le k\omega_f \\ 0 & \omega>k\omega_f, \end{cases} \qquad (2.13)$$

where ω_f is a constant frequency. The independent, identically distributed variates $\omega_{i,k}$ and $\phi_{i,k}$ have, respectively, distribution $1/(k\omega_f)$ and uniform distribution over the interval $[0, 2\pi]$. The autocorrelation function of $G_{N,k}(t)$ is $\langle G_{N,k}(t)G_{N,k}(t+\tau)\rangle=[1/(\pi\omega_f\tau)\sin(k\omega_f\tau)$ (Papoulis, 1962). For any finite k, and for sufficiently large finite N, the process $G_{N,k}(t)$ approximates as closely as desired a Gaussian process $G_k(t)$ with spectral density $\Psi_k(\omega)$. If N and k are both sufficiently large, the process $G_{N,k}(t)$ approximates white noise as closely as desired, since the limit for $k\to\infty$ of the sequence of its autocorrelation functions is the delta function. The variance of $G_{N,k}(t)$ is $k\omega_f$. For the dimensional counterpart of the system, $G_{N,k}(t)$ and γ have dimension $[T^{-1/2}]$ and $[FT^{1/2}]$ (F=force), respectively, whereas for the dimensional counterpart of example 2a the excitation $G_N(t)$ is nondimensional and the dimension of γ is [F]. Comments similar to those made in example 2a on our use of the term "Gaussian" for a process that is as nearly Gaussian as desired are also applicable to the term "Gaussian white noise."

The Melnikov process $M_{N,k}(t)$ induced by $G_{N,k}(t)$ has expectation, spectral density and variance

$$E[M_{N,k}]=-\beta c+x(t) \tag{2.14}$$

$$\Psi_{M;N,k}(\omega) = \begin{cases} 2\pi S^2(\omega) & 0<\omega\leq k\omega_f \\ 0 & \omega>k\omega_f, \end{cases} \tag{2.15}$$

$$\mathrm{Var}[M_{N,k}] = \gamma^2\int_0^{k\omega_f} S^2(\omega)d\omega \tag{2.16}$$

It can be shown that since $S(\omega)$ is the modulus of the Fourier transform of $\dot{z}_s(-t)$, as $k\to\infty$ the integral in Eq. 2.16 converges to a limit denoted by σ_M^2. The limit of the sequence $\mathrm{Var}[M_{N,k}]$ as $k\to\infty$ is then $(\gamma\sigma_M)^2$. For sufficiently large N and k, $M_{N,k}(t)$ approximates as closely as desired a Gaussian process with expectation $-\beta c+z(t)$ and standard deviation $\gamma\sigma_M$.

2.4 *Multiplicative Noise.*

We have so far assumed that the noise $G(t)$ is additive (see Eq. 1). If in Eq. 1 we consider multiplicative noise $F(z,\dot{z})G(t)$ instead of additive noise $G(t)$, then in the equations for the Melnikov process the function $h(\tau)=\dot{z}_s(-\tau)$ in the integral reflecting the contribution of the noise is simply replaced by the filter (Frey and Simiu, 1994)

$$h_m(\tau) = \dot{z}_s(-\tau)F[z_s(-\tau),\dot{z}_s(-\tau)]. \tag{2.17}$$

3. UPPER BOUND FOR MEAN EXIT RATE

3.1 *Melnikov-based Upper Bounds for Mean Exit Rate.*

We consider a "time slice" through a realization of the stable and unstable manifolds of a stochastic dynamical system described by Eqs. 2.1 and 2.9. The crossings of the pseudoseparatrix are assumed to be relatively rare events. They are associated with the formation of lobes. Chaotic transport across the pseudoseparatrix is carried out by the detraining and entraining turnstile lobes (Beigie et al, 1991). On average, to within an approximation of order one, no transport across the pseudoseparatrix can occur during a time interval less than the mean zero upcrossing time, τ_u, of the Melnikov process. The mean zero upcrossing rate $1/\tau_u$ may therefore serve as a weak upper bound for the mean rate of exit from a well.

Assume the stochastic excitation is Gaussian. The Melnikov process $M(t)$ is then Gaussian with mean $m(t)=-\beta c+x(t)$, standard deviation σ_M, and autocovariance function $\Gamma(\tau)=E\{(M(t)-m(t))(M(t+\tau)-m(t+\tau))\}$ (given by the Fourier transform of the Melnikov process spectral density $\Psi_M(\omega)$). The mean zero upcrossing rate for the Melnikov process is

$$\tau_u^{-1}(t)=\sigma_I\{\phi[m_I(t)/\sigma_I]+[m_I(t)/\sigma_I]\Phi[m_I(t)/\sigma_I]\}$$
$$\phi[-m(t)/\sigma_M]/(2\pi\sigma_M) \tag{3.1}$$

(Soong and Grigoriu, 1992), where $\phi(\alpha)=(2\pi)^{-1/2}\exp(-\alpha^2/2)$,

$$\Phi(u)= \int_{-\infty}^{u}\phi(\alpha)d\alpha,$$

$$m_I(t)=m(t)-[\partial\Gamma(\tau)/\partial\tau\,|_{\tau=0}/\sigma_M^2]m(t), \tag{3.2}$$

$$\sigma_I^2=-\partial^2\Gamma(\tau)/\partial\tau^2|_{\tau=0} - \partial\Gamma(\tau)/\partial\tau\,|_{\tau=0}]^2/\sigma_M^2. \tag{3.3}$$

For $g(t)\equiv0$, $x(t)\equiv0$, so that

$$\tau_u^{-1}=\nu\exp(-\kappa^2/2) \tag{3.4}$$

$$\nu=\{[\int_0^\infty\omega^2\Psi_M(\omega)d\omega]/[\int_0^\infty\Psi_M(\omega)d\omega]\}^{1/2},\quad \kappa=\beta c/\sigma_M \tag{3.5a,b}$$

3.2 *Melnikov-based Lower Bound for Probability of Non-Occurrence of Exits During a Specified Time Interval.*

Let us again assume the Melnikov process is nearly Gaussian with expectation $m(t)=\beta c-x(t)$ and standard deviation σ_M. We define the ratio

$$\kappa=\{\beta c-\max[x(t)]\}/\sigma_M, \tag{3.6}$$

which for $g(t)\equiv0$ reduces to Eq. 3.5b. For κ sufficiently large (e.g., $\kappa>2$, say), zero upcrossings are rare events, and the probability that there will no upcrossings during a time interval $[T_1,T_2]$ can be closely approximated by a Poisson distribution. The probability that there will be at least one upcrossing during the interval $[T_1,T_2]$ is then

$$p_{T1,T2} = 1 - \exp(-\int_{T_1}^{T_2} dt/\tau_u(t)). \tag{3.7}$$

The probability $p_{T1,T2}$ is an upper bound for the probability that exits from a well will occur during the interval $[T_1,T_2]$. Like τ_u, it is a weak bound. If $g(t)\equiv 0$, the integral in Eq. 3.7 is $-(T_2-T_1)/\tau_u=-T/\tau_u$, and we write

$$p_T=1-\exp(-T/\tau_u) \tag{3.8}$$

For definiteness we consider a Duffing oscillator excited only by the process G(t) (i.e., $g(t)\equiv 0$), and assume that G(t) has spectral density

$$2\pi\Psi(\omega)=\begin{cases}0.03990\ell n(\omega)+0.12829 & 0.04\leq\omega\leq 0.40\\ 0.05755\ell n(\omega)+0.14493 & 0.40\leq\omega\leq 1.20\\ -0.383[\ell n(\omega)]^2+1.062\ell n(\omega)-0.0294 & 1.20\leq\omega\leq 15.4\end{cases} \tag{3.9}$$

To a first approximation this spectrum is representative of low-frequency fluctuations of the horizontal wind speed (Van der Hoven, 1957). In Eq. 3.9 $\omega = 4\Omega/\Omega_{pk}$, Ω is the dimensional frequency, $\Omega_{pk} \approx 2\pi/(4$ days) is the dimensional frequency corresponding to the spectral peak, which occurs at $\omega=\omega_{pk}=4$; $\Psi(\omega) = \Psi_u(\omega)/\sigma_u^2$, $\Psi_u(\omega)$ is the spectral density of the wind speed in m^2/s^2 (as a function of the nondimensional frequency ω), and the standard deviation of the dimensional wind speed fluctuations is $\sigma_u\approx 1.33$ m/s. The model implicit in our assumptions is Gaussian, although the physical reality is that wind speed fluctuations are bounded.

From Eqs. 2.11c and 3.5a, $\sigma_M^2\equiv \mathrm{Var}[M_N]=0.14\gamma^2$ and $\nu=0.24744$. Since $c=4/3$ (Eq. 2.5) and $g(t)\equiv 0$, $\kappa=3.563\beta/\gamma$ (Eq. 3.5b).Let us assume $\beta/\gamma=1$. Then $\tau_u=2312$ (Eq. 3.4). We consider the nondimensional time interval corresponding to 10 days. Since the dimensional time $T_d=1$ day corresponds to a nondimensional frequency $\omega=4$, that is, a nondimensional time $2\pi/4$, the nondimensional time corresponding to 10 days is $T=15.71$, and the probability that an exit will occur during a 10-day time interval has the upper bound $p_T\approx 0.007$ (Eq. 3.8). It can be verified that the actual exit probability is very much lower. However, knowledge of the upper bound p_T can be useful in some practical applications.

4. SYSTEMS WITH NOISE HAVING TAIL-LIMITED MARGINAL DISTRIBUTION

We consider stochastic processes with tail-limited marginal distributions, whose paths may be approximated arbitrarily closely by uniformly continuous functions. For systems acted upon by such noise the Melnikov approach yields a criterion guaranteeing that no exit from a well can occur, however long the waiting time. This is illustrated for the case of a Duffing oscillator with $g(t)\equiv 0$, excited by square-wave dichotomous noise

$$G(t) = a_n, \qquad [\alpha+(n-1)]t_1<t\leq(\alpha+n)t_1 \tag{4.1}$$

where n is the set of integers, α is a random variable uniformly distributed between 0 and 1, a_n are independent random variables that take on the values -1 and 1 with probabilities 1/2 and 1/2, and t_1 is a parameter of the process G(t).

A rectangular pulse wave of amplitude a_n and length t_1 centered at coordinate $t_n=(\alpha+n-1/2)t_1$ has Fourier transform $F_n(\omega) = a_n|(2/\omega)\sin(\omega t_1/2)\exp(-j\omega t_n)|$ (Papoulis, 1962, p. 20). The pulse itself can then be expressed as a uniformly continuous sum of terms approximating as closely as desired the inverse Fourier transform of $F_n(\omega)$. Each realization of the coin-toss dichotomous square-wave can then be approximated as closely as desired by a superposition of such sums, which is itself a uniformly continuous function.

Uniformly continuous functions that would similarly approximate arbitrarily closely a process G(t) with tail-limited marginal distributions would induce a Melnikov process approximating arbitrarily closely the process

$$M(t) = -\beta c + \gamma\int_{-\infty}^{\infty} h(\tau)G(t-\tau)d\tau. \tag{4.2}$$

The necessary condition for chaos may thus be developed by using Eq. 4.2; the approximation of the process G(t) need not be carried out explicitly.

From Eqs. 2.5 and 4.2,

$$M(t) = -4\beta/3 + (2)^{1/2}\gamma F(t) \qquad (4.3a,b)$$

$$F(t) \approx \sum_{n=-\ell}^{\ell} a_n\{-\mathrm{sech}[(n+\alpha)t_1 - t] + \mathrm{sech}[(n+\alpha-1)t_1 - t]\}$$

where ℓ is sufficiently large so that the error due to the finiteness of ℓ be as small as desired.

The area under the curve $x_s(-t)$ (Eq. 2.3) in a half-plane is $(2)^{1/2}$. It follows immediately from the definition of F(t) (see Eqs. 4.3) that $-2<F(t)<2$. (For example, F(t) would be equal to 2 if $\alpha=0$, $a_n = 1$ for all n such that $t>0$ and $a_n = -1$ for all n such that $t<0$.) The necessary condition for chaos is that M(t) have simple zeros. From Eq. 4.3a, if

$$\beta/\gamma > 2.121 \qquad (4.4)$$

then this condition cannot be satisfied, and chaotic transport cannot occur. Eq. 4.4 is a simple, though generally weak, criterion guaranteeing that exits do not occur (Simiu and Hagwood, 1994).

5. CONCLUSIONS

For systems with additive or multiplicative noise, the mean zero upcrossing rate, τ_u^{-1}, for the stochastic system's Melnikov process is a weak upper bound for the system's mean exit rate, τ_e^{-1}.

For nonlinear systems excited by processes with tail-limited marginal distributions remarkably simple criteria can be derived that guarantee the non-occurrence of exits. This was illustrated for square-wave, coin-toss dichotomous noise.

The validity of the Melnikov approach can be proven rigorously for asymptotically small perturbations. However, numerical experiments have shown that the approach can be useful also for systems with relatively large perturbations.

ACKNOWLEDGMENTS

This work was supported in part by the Office of Naval Research, Ocean Engineering Division, Grants N00014-94-0028 and N00014-94-0284. Dr. Tom Swean served as project monitor.

REFERENCES

Allen, J.S., Samelson, R.M. and Newberger, P.A. (1991). Chaos in a Model of Forced Quasi-Geostrophic Flow Over Topography...*J. Fluid Mech.*, 226: 511-547.

Beigie, D., Leonard, A. and Wiggins, S. (1991). Chaotic Transport in the Homoclinic and Heteroclinic Tangle Regions of Quasi-periodically Forced Two-dimensional Dynamical Systems. *Nonlinearity* 4:775-819.

Frey, M. and E. Simiu (1993). Noise-induced chaos and phase space flux. *Phys. D* 63:321-340.

Frey, M. & E. Simiu (1995). Noise-induced Transitions to Chaos. *Proc., NATO Adv. Res. Workshop Spatio-Temp. Patterns in Nonequilibrium Complex Systems*. P.E. Cladis & P. Pallfy-Muhoray (eds.), Addison-Wesley.

Guckenheimer, J., and Holmes, P. (1986). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag.

Papoulis, A. (1962). *The Fourier Integral and Its Applications*. New York: McGraw-Hill.

Shinozuka, M. (1971). Simulation of Multivariate and Multidimensional Random Processes. *J. Acoust. Soc. of America,* 49:357-368.

Simiu, E., and Hagwood, C. (1995). Exits in Second-Order Nonlinear Systems Driven by Dichotomous Noise. *Proceedings, 2nd Intern. Conf. Comput. Stoch. Mech.*, (M. Shinozuka and P. Spanos, eds.), Athens. Elsevier.

Simiu, E. (1994). Melnikov Process for Stochastically Perturbed Slowly Varying Oscillators: Application to Model of Wind-Driven Current. *J. Appl. Mech.* (in press).

Simiu, E. & M. Frey (1995). Noise-induced Sensitivity to Initial Conditions. *Proceedings, Workshop on Fluctuations and Order*, (M. Millonas, ed.). Springer-Verlag (in press).

Soong, T.T. and Grigoriu, M. (1992). *Random Vibrations of Mechanical and Structural Systems*, New York: Prentice-Hall.

Thompson, J.M.T., Rainey, R.C.T., and Soliman, M.S. (1990). Ship stability criteria based on chaotic transients from incursive fractals. *Phil. Trans. Royal Soc. Lond. A* , 332: 149-167.

Van der Hoven, I. (1957). Power Spectrum of Horizontal Wind Velocity Fluctuations ... *J. Met.*, 14:1254-1255.

Wiggins, S. (1990). *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. New York: Springer-Verlag.

Wiggins, S. (1992). *Chaotic Transport in Dynamical Systems*. Springer-Verlag.

Yim, S.C.S. and Lin, H. (1992). Probabilistic analysis of a chaotic dynamical system, *Applied Chaos*. J.H. Kim and J. Stinger (eds.). Wiley.