MELNIKOV PROCESSES AND NOISE-INDUCED EXITS FROM A WELL

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ABSTRACT: For a wide class of near-integrable systems with additive or multiplicative noise the mean zero upcrossing rate for the stochastic system's Melnikov process τ_u^{-1}, provides an upper bound for the system's mean exit rate, τ_e^{-1}. Comparisons between τ_u^{-1} and τ_e^{-1} show that in the particular case of additive white noise this upper bound is weak. For systems excited by processes with tail-limited distributions, the stochastic Melnikov approach yields a simple criterion guaranteeing the nonoccurrence of chaos. This is illustrated for the case of excitation by square-wave, coin-toss dichotomous noise. Finally, we briefly review applications of the stochastic Melnikov approach to a study of the behavior of wind-induced fluctuating currents over a corrugated ocean floor; the snap-trough of buckled columns with continuous mass distribution and distributed random loading; and open-loop control of stochastically excited multistable systems.

INTRODUCTION

Harmonically forced deterministic multistable dynamical systems used in a variety of analytical studies represent idealized models of systems whose excitations are in fact stochastic. For example, wind, wave, and seismic excitations have been idealized as harmonic in studies of the behavior induced by wind in a quasi-geostrophic model of coastal currents over topography (Allen et al. 1991), a ship capsizing due to wave forces (Thompson et al. 1990), and the rocking response of rigid objects to earthquakes (Yim and Lin 1992). In these and other studies the idealized system's Melnikov function has been used for inferences on the possibility of chaotic behavior.

The validity of such inferences depends on whether an appropriate choice was made of the amplitude and frequency of the harmonic function used as an idealization of the actual stochastic excitation. This cannot be determined unless a stochastic Melnikov theory is used. Therein lies one drawback of applying Melnikov theory to a deterministic counterpart of the stochastic system.

There is also a second drawback. For a wide class of multistable systems, simple zeros of the Melnikov function entail chaotic phase space transport from preferred regions associated with the potential wells, that is, they entail the occurrence of exits from these regions (Wiggins 1992; Frey and Simiu 1993a). For stochastic systems with Gaussian excitation, or any other excitation with infinitely tailed marginal distribution, exits can occur no matter how small the noise. On the other hand, for the idealized system, if the excitation is smaller than the value that causes the Melnikov function to have simple zeros, exits cannot occur. The idealized system thus fails to reflect the full range of transport possibilities in the stochastic system it purports to model.

A more natural and effective approach is to apply Melnikov theory directly to the stochastic system. In this paper we review and illustrate our recent development and application of stochastic Melnikov theory, a term we apply to an extension of Melnikov theory to stochastic dynamical systems. The extension is based on the observation that, for a wide class of dynamical systems, a stochastic additive or multiplicative excitation induces a stochastic Melnikov process (Frey and Simiu 1993b; Simiu and Frey 1993; Frey and Simiu 1994; Simiu and Frey 1994). The Melnikov process has the property that its mean zero upcrossing rate, denoted by $1/\tau_u$, is an upper bound for the system's mean exit rate $1/\tau_e$: to within an approximation of order one, on average no transport can occur across the system's pseudoseparatrix (Wiggins 1990, p. 528) during a time interval smaller than τ_u. Finally, for a system excited by noise with tail-limited distribution the stochastic Melnikov approach yields a remarkably simple criterion guaranteeing the nonoccurrence of exits. We conclude that this approach can provide information on system behavior in a class of problems for which the Fokker-Planck equation—otherwise a more powerful approach—is impractical or inapplicable.

The following section describes a class of systems to which the stochastic Melnikov approach is applicable, and briefly reviews basic material needed for our development of this approach. The next section applies the stochastic Melnikov approach to a typical system with colored or white Gaussian noise. It discusses the use of the mean zero upcrossing rate of the Melnikov process $1/\tau_u$ as an upper bound for the system's mean exit rate $1/\tau_e$, and the problem of selecting the amplitude and frequency of the stochastic excitation's idealized harmonic counterpart. Next, comparisons between τ_u and τ_e for additive white noise excitation are given. Then the application of the stochastic Melnikov approach to systems excited by noise with tail-limited distributions is reviewed, followed by review of applications of this approach to a study of the behavior of wind-induced fluctuating currents over a corrugated ocean floor, noise-induced snap-through of a column with continuous mass distribution, and open-loop control of a class of stochastic multistable systems. Conclusions, including a brief discussion of limitations of the Melnikov approach, follow.

DYNAMICAL SYSTEMS, AND MELNIKOV FUNCTIONS AND PROCESSES

We consider systems of the form

$$\ddot{x} = -V'(x) + \varepsilon[g(t) + \gamma G(t) - f(x, \dot{x})] \quad (1)$$

where $\varepsilon \ll 1$; γ = a constant; $g(t)$ = a bounded, uniformly continuous function; and $V(x)$ = a potential function. The function $f(x, \dot{x})$ may, for example, take the form $\beta\dot{x}$, $\beta > 0$, in which case it represents viscous damping. For definiteness, in the remainder of this paper we consider this form. We assume that (1) the unperturbed system ($\varepsilon = 0$) is integrable; and (2) $V(x)$ has the shape of a multiple well so that the unperturbed system has a center at the bottom of each well and a saddle point at the top of the barrier between two adjacent wells. The stable and unstable manifolds emanating from the saddle point of the unperturbed system then coincide (i.e., the saddle point is connected to itself by homoclinic orbits). Finally, we assume

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$G(t)$ is a bounded, uniformly continuous function, or a random process with properties to be defined later. As a typical example belonging to the class of systems just described, unless otherwise shown we consider in this paper the Duffing-Holmes equation, which has potential

$$V(x) = x^4/4 - x^2/2 \tag{2}$$

homoclinic orbits with coordinates

$$x_s(t) = (2)^{1/2}\ \text{sech}(t);\quad \dot{x}(t) = (2)^{1/2}\ \text{sech}(t)\tanh(t) \tag{3a,b}$$

and a modulus of the Fourier transform of the function $h(t) \equiv \dot{x}_s(-t)$

$$S(\omega) = (2)^{1/2}\pi\omega\ \text{sech}(\pi\omega/2) \tag{4}$$

We also note for later use that

$$c \equiv \int_{-\infty}^{\infty} \dot{x}_s^2(\tau)\, d\tau = 4/3 \tag{5}$$

We now review briefly two cases.

Case 1

In case 1 $G(t)$ is a bounded and uniformly continuous function. For sufficiently small ε, the perturbed system possesses invariant stable and unstable manifolds; their intersection with an arbitrary plane of section ("time slice"), t = const, is a pair of curves approaching asymptotically a saddle point that is ε-close to the saddle point of the unperturbed system. The stable and unstable manifolds of the perturbed system no longer coincide, as they do in the unperturbed case. To first order, the distance between them, known as the Melnikov distance, is proportional to the Melnikov function. The Smale-Birkhoff theorem states that a necessary condition for chaos is that the Melnikov function of the system have simple zeros (Guckenheimer and Holmes 1986; Wiggins 1992). The following example, based on work by Beigie et al. (1991), provides a stepping stone we use later in this section to deal with excitations by Gaussian random processes.

Example 1

Consider the bounded and uniformly continuous function

$$G(t) = \sum_{i=1}^{N} \cos[\omega_i(t + t_i)] \tag{6}$$

where $t_i = \phi_i/\omega_i$ and ϕ_i denote phase angles. The Melnikov function induced by $G(t)$ and $g(t)$ is

$$M(t, t_0, t_1, \ldots, t_N) = -\beta \int_{-\infty}^{\infty} \dot{x}_s^2(\tau)\, d\tau + \int_{-\infty}^{\infty} h(\tau) g(t + t_0 - \tau)\, d\tau + \gamma \int_{-\infty}^{\infty} h(\tau) \sum_{n=1}^{N} \cos[\omega_i(t + t_i - \tau)]\, d\tau \tag{7}$$

where $h(\tau) = \dot{x}_s(-\tau)$ (Wiggins 1992). Denoting the modulus of the Fourier transform of $h(t)$ by $S(\omega)$, we have

$$M(t, t_0, t_1, \ldots, t_N) = -\beta c + z(t, t_0) + \gamma \sum_{i=1}^{N} S(\omega_i)\sin[\omega_i(t + t_i)] \tag{8}$$

where c = a constant and $z(t, t_0)$ denotes the second integral on the right-hand side of (7); and $S(\omega_i)$ are admittance functions, referred to in the context of Melnikov theory as scaling factors for the frequencies ω_i (Beigie et al. 1991). The necessary condition for chaos is that $M(t, t_0, t_1 \ldots, t_N)$ have simple zeros.

Case 2

In case 2, $G(t)$ is a nearly Gaussian, ensemble-uniformly-continuous (EUC) random process with specified one-sided spectral density. A stochastic process $G(t)$ is EUC if, given any $\delta_1 > 0$, there exists $\delta_2 > 0$ such that, if $|t_2 - t_1| < \delta_2$, then $|G(t_2) - G(t_1)| < \delta_1$ for all times t_1 and t_2 and all realizations of $G(t)$ (Frey and Simiu 1993b). Each realization of $G(t)$ of a EUC process is bounded and uniformly continuous. A sufficiently small ε guarantees that to the random process $G(t)$ there corresponds an ensemble of stable and unstable manifolds such that their intersection with an arbitrary plane of section, t = const, is an ensemble of pairs of curves approaching asymptotically an ensemble of saddle points that are ε-close to the saddle point of the unperturbed system. To first order, the distance between the stable and unstable manifold corresponding to a realization of $G(t)$ is proportional to the Melnikov function induced by that realization. For any realization of $G(t)$, the necessary condition for chaos is that the corresponding Melnikov path have simple zeros. The two following examples are largely based on work by Frey and Simiu (1993b), and provide a method for dealing with Gaussian noise and Gaussian white noise in the context of Melnikov theory.

Example: Colored Gaussian Noise

We consider the bounded, EUC random process

$$G(t) \equiv G_N(t) = (2/N)^{1/2} \sum_{i=1}^{N} \cos(\omega_i t + \phi_i) \tag{9}$$

where the parameter N of the process is finite, and ϕ_i and ω_i ($i = 1, \ldots, N$) are independent, identically distributed random variables with, respectively, uniform distribution over the interval $[0, 2\pi]$, and probability density function $p(\omega_i) = 2\pi\Psi(\omega_i)$. The process $G_N(t)$, known as Shinozuka noise, has unit variance and spectral density $2\pi\Psi(\omega)$ (Shinozuka 1971). The Melnikov random process induced by $G_N(t)$ is

$$M_N(t) = -\beta c + z(t) + \gamma \int_{-\infty}^{\infty} h(\tau) G_N(t - \tau)\, d\tau \tag{10}$$

where notations of (7) and (8) are used and the parameters t_0, $t_1, \ldots, t_N$ may be omitted (Frey and Simiu 1993b). The expectation, spectral density, and variance of $M_N(t)$ are

$$E[M_N(t)] = -\beta c + z(t);\quad \Psi_{MN}(\omega) = 2\pi\gamma^2 S^2(\omega)\Psi(\omega);\quad \text{Var}[M_N] = \gamma^2 \int_0^{\infty} S^2(\omega)\Psi(\omega)\, d\omega \tag{11a,b,c}$$

The integral of (10) has the same form as the sum of (8). The marginal distribution of that integral, and hence the marginal distribution of the process $M_N(t)$, is Gaussian in the limit $N \to \infty$ (Simiu and Frey 1993). By choosing a sufficiently large N, that marginal distribution can be made as close to a Gaussian distribution as desired; that is, given any $M_{max} > 0$ and $\delta > 0$, there exists N such that $|P_N(M) - P(M)| < \delta$ uniformly for all $M < M_{max}$, $P_N(M)$ is the marginal distribution of $M_N(t)$, and the distribution $P(M) = \lim_{N\to\infty} P_N(M)$ is Gaussian. For sufficiently large N the distribution $P_N(M)$ will be an entirely adequate approximation to $P(M)$, however close the requisite approximation. Owing to the technical requirement of boundedness and uniform continuity needed to prove that the saddle point persists under perturbation, we do not use the limit $N \to$

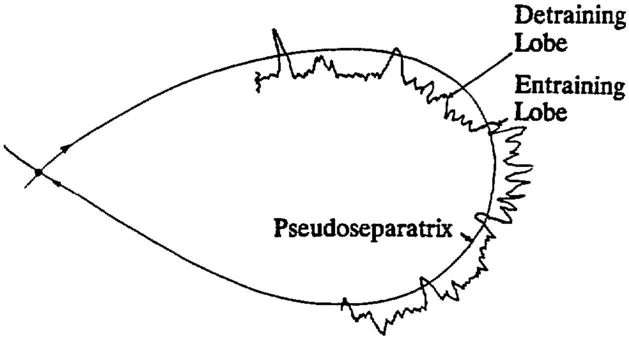

FIG. 1. Phase Plane Diagram Showing Intersecting Stable and Unstable Manifolds of Stochastically Excited System

∞, and use instead N finite but sufficiently large; that is, when we use the term Gaussian we refer to a process with distribution $P_N(M)$ that is arbitrarily close to the Gaussian distribution $P(M)$. Note that this approach in no way limits the usefulness or rigor of our results. It should be recalled that constructs such as the Gaussian distribution, the Dirac delta function, the Heaviside function, white noise, and so forth, are abstractions, rather than physical realities. If these abstractions are useful in certain mathematical contexts, they can be used because they are close approximations to the respective realistic representations of the actual physical processes. However, if there are mathematical contexts in which it is advantegeous to use representations other than the idealized constructs, this can again be done provided that those representations are physically as satisfactory as their idealized counterparts. This is indeed our case.

Example 2: Gaussian White Noise

We now consider the sequence of processes ($k = 1, 2, \ldots$)

$$G(t) \equiv G_{N,k}(t) = (2/N)^{1/2} \sum_{i=1}^{N} \cos(\omega_{i,k}t + \phi_{i,k}) \qquad (12)$$

with spectral densities

$$\Psi_k(\omega) = \begin{cases} 2\pi; & 0 < \omega \le k\omega_f \\ 0; & \omega > k\omega_f \end{cases} \qquad (13)$$

where ω_f = a constant frequency. The independent, identically distributed variates $\omega_{i,k}$ and $\phi_{i,k}$ have, respectively, probability density $\psi_k(\omega)/(k\omega_f)$ and uniform distribution over the interval $[0, 2\pi]$. The autocorrelation function of $G_{N,k}(t)$ is $\langle G_{N,k}(t)G_{N,k}(t + \tau)\rangle = [1/(\pi\omega_f\tau)\sin(k\omega_f\tau)$ (Papoulis 1962, p. 20). For any finite k, and for sufficiently large finite N, the process $G_{N,k}(t)$ approximates as closely as desired a Gaussian process $G_k(t)$ with spectral density $\Psi_k(\omega)$. If N and k are both sufficiently large, the process $G_{N,k}(t)$ approximates white noise as closely as desired, since the limit for $k \to \infty$ of the sequence of its autocorrelation functions is the delta function (Kanwal 1983, p. 5). The variance of $G_{N,k}(t)$ is $k\omega_f$. For the dimensional counterpart of the system, $G_{N,k}(t)$ and γ have dimension $[T^{-1/2}]$ and $[FT^{1/2}]$ (F = force), respectively, whereas for the dimensional counterpart of example 1 the excitation $G_N(t)$ is nondimensional and the dimension of γ is $[F]$. Comments similar to those made for example 1 on the use of the term "Gaussian" for a process that is as nearly Gaussian as desired are also applicable to the term "Gaussian white noise."

The Melnikov process $M_{N,k}(t)$ induced by $G_{N,k}(t)$ has expectation

$$E[M_{N,k}] = -\beta c + z(t) \qquad (14)$$

spectral density

$$\Psi_{M;N,k}(\omega) = \begin{cases} 2\pi S^2(\omega); & 0 < \omega \le k\omega_f \\ 0; & \omega > k\omega_f \end{cases} \qquad (15)$$

and variance

$$\mathrm{Var}[M_{N,k}] = \gamma^2 \int_0^{k\omega_f} S^2(\omega)\, d\omega \qquad (16)$$

It can be shown that since $S(\omega)$ is the modulus of the Fourier transform of $x_s(t)$, as $k \to \infty$ the integral in (16) converges to a limit denoted by σ_M^2. (In many cases of practical interest, e.g., the Duffing equation and the *rf* Josephson junction, closed-form expressions for $S(\omega)$ exist and the integrals can be calculated numerically.) The limit of the sequence $\mathrm{Var}[M_{N,k}]$ as $k \to \infty$ is then $(\gamma\sigma_M)^2$. For sufficiently large N and k, $M_{N,k}(t)$ approximates as closely as desired a Gaussian process with expectation $-\beta c + z(t)$ and standard deviation $\gamma\sigma_M$.

Multiplicative Noise

We have so far assumed that the noise $G(t)$ is additive [see (1)]. If in (1) we consider multiplicative noise $F(x, \dot{x})G(t)$ instead of additive noise $G(t)$, then in the equations for the Melnikov process the function $h(\tau) = \dot{x}_s(-\tau)$ in the integral reflecting the contribution of the noise is simply replaced by the filter (Frey and Simiu 1994)

$$h_m(\tau) = \dot{x}_s(-\tau)F[x_s(-\tau), \dot{x}_s(-\tau)] \qquad (17)$$

STOCHASTIC VERSUS DETERMINISTIC MELNIKOV APPROACH FOR COLORED GAUSSIAN NOISE

Melnikov-Based Upper Bounds for Mean Exit Rate

Fig. 1 shows a hypothetical "time slice" through a realization of the stable and unstable manifolds of a stochastic dynamical system described by (1) and (9). The crossings of the pseudoseparatrix are assumed to be relatively rare events. They are associated with the formation of lobes. Chaotic transport across the pseudoseparatrix is carried out by the detraining and entraining turnstile lobes, (Beigie et al. 1991). On average, to within an approximation of order one, no transport across the pseudoseparatrix can occur during a time interval less than the mean zero upcrossing time τ_u of the Melnikov process. The mean zero upcrossing rate $1/\tau_u$ may therefore serve as an upper bound for the mean rate of exit from a well. We show later that, in the case of white noise, it is a weak upper bound.

Assume the stochastic excitation is Gaussian. The Melnikov process $M(t)$ is then Gaussian with mean $m(t) = -\beta c + z(t)$, standard deviation σ_M, and autocovariance function $\Gamma(\tau) = E\{(M(t) - m(t))(M(t + \tau) - m(t + \tau))\}$ [given by the Fourier transform of the Melnikov's process spectral density $\Psi_M(\omega)$]. The mean zero upcrossing rate for the Melnikov process is

$$\tau_u^{-1}(t) = \sigma_I\{\phi[m_I(t)/\sigma_I] + [m_I(t)/\sigma_I]\Phi[m_I(t)/\sigma_I]\}\phi[-m(t)/\sigma_M]/\sigma_M \qquad (18)$$

(Soong and Grigoriu 1992), where $\phi(\alpha) = (2\pi)^{-1/2}\exp(-\alpha^2/2)$, $\Phi(u) = \int_{-\infty}^{u} \phi(\alpha)\, d\alpha$

$$m_I(t) = m(t) - [\partial\Gamma(\tau)/\partial\tau|_{\tau=0}/\sigma_M^2]m(t) \qquad (19)$$

$$\sigma_I^2 = -\partial^2\Gamma(\tau)/\partial\tau^2|_{\tau=0} - \partial\Gamma(\tau)/\partial\tau|_{\tau=0}]^2/\sigma_M^2 \qquad (20)$$

For $g(t) \equiv 0$, $z(t) \equiv 0$, so that

$$\tau_u^{-1} = \nu\exp(-\kappa^2/2) \qquad (21)$$

$$\nu = (1/2\pi)\left\{\left[\int_0^{\infty} \omega^2\Psi_M(\omega)\, d\omega\right]\Big/\left[\int_0^{\infty} \Psi_M(\omega)\, d\omega\right]\right\}^{1/2};$$

$$\kappa = \beta c/\sigma_M \qquad (22a,b)$$

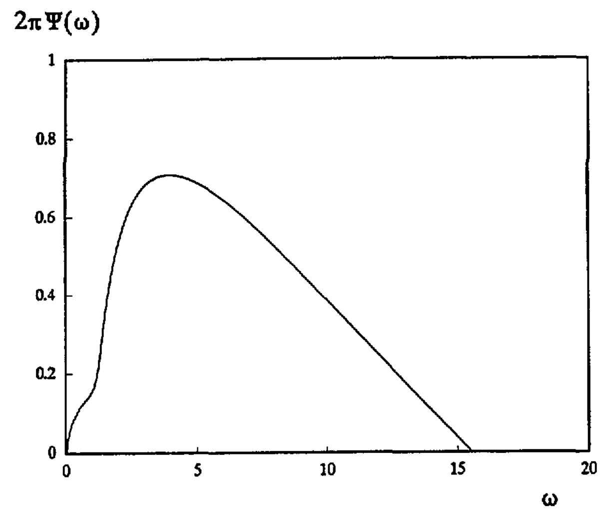

FIG. 2. Spectral Density of Excitation Corresponding to Unit Variance, $2\pi\Psi(\omega)$

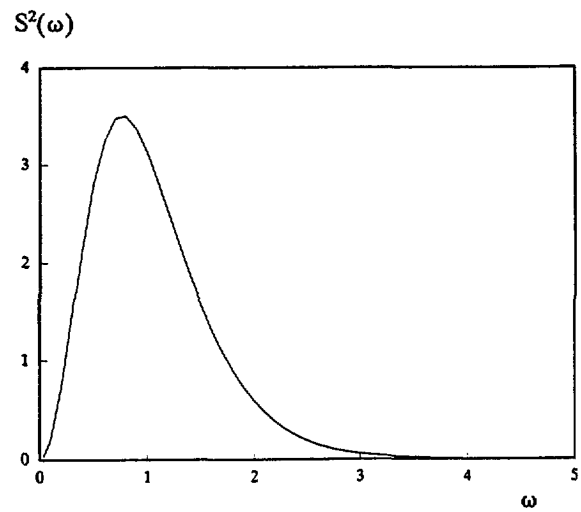

FIG. 3. Function $S^2(\omega)$

Melnikov-based Lower Bound for Probability of Nonoccurrence of Exits during Specified Time Interval

Let us again assume the Melnikov process is nearly Gaussian with expectation $m(t) = \beta c - z(t)$ and standard deviation σ_M. We define the ratio

$$\kappa = \{\beta c - \max[z(t)]\}/\sigma_M \tag{23}$$

which for $g(t) \equiv 0$ reduces to (22*b*). For κ sufficiently large (e.g., $\kappa > 2.5$, say), zero upcrossings are rare events, and the probability that there will be no upcrossings during a time interval $[T_1, T_2]$ can be closely approximated by a Poisson distribution. The probability that there will be at least one upcrossing during the interval $[T_1, T_2]$ is then

$$p_{T_1,T_2} = 1 - \exp\left(-\int_{T_1}^{T_2} dt/\tau_u(t)\right) \tag{24}$$

The probability p_{T_1,T_2} is an upper bound for the probability that exits from a well will occur during the interval $[T_1, T_2]$. If τ_u is a weak bound, so is p_{T_1,T_2}. If $g(t) \equiv 0$, the integral in (24) is $-(T_2 - T_1)/\tau_u = -T/\tau_u$, and we write

$$p_T = 1 - \exp(-T/\tau_u) \tag{25}$$

For details, see Rice (1954) or Simiu and Scanlan (1986, p. 549).

Example

For definiteness we consider a Duffing oscillator excited only by the process $G(t)$ (i.e., $g(t) \equiv 0$), and assume that $G(t)$ has spectral density

$$2\pi\Psi(\omega) = \begin{cases} 0.03990\ln(\omega) + 0.12829; & 0.04 \le \omega \le 0.40 \\ 0.05755\ln(\omega) + 0.14493; & 0.40 \le \omega \le 1.20 \\ -0.38301[\ln(\omega)]^2 + 1.06192\ln(\omega) - 0.02941; & 1.20 \le \omega \le 15.40 \end{cases} \tag{26}$$

(Fig. 2). To a first approximation this spectrum is representative of low-frequency fluctuations of the horizontal wind speed (Van der Hoven 1957). It may be used as a model in applications where the length scale is sufficiently large that, owing to spatial correlation effects, higher frequency fluctuations are negligible (Simiu and Scanlan 1986, p. 169; Simiu 1994). In (19) $\omega = 4\Omega/\Omega_{pk}$, Ω is the dimensional frequency, $\Omega_{pk} \approx 2\pi/(4 \text{ days})$ is the dimensional frequency corresponding to the spectral peak, which occurs at $\omega = \omega_{pk} = 4$; $\Psi(\omega) = \Psi_u(\omega)/\sigma_u^2$, $\Psi_u(\omega)$ is the spectral density of the wind speed in m^2/s^2 (as a function of the nondimensional frequency ω), and the standard deviation of the dimensional wind speed fluctuations is $\sigma_u \approx 1.33$ m/s. The model implicit in our assumptions is Gaussian, although the physical reality is that wind speed fluctuations are bounded.

From (11*c*) and (22*a*), $\sigma_M^2 \equiv \text{Var}[M_N] = 0.14\gamma^2$ and $\nu = 0.24744$. Since $c = 4/3$ [(5)] and $g(t) \equiv 0$, $\kappa = 3.563\beta/\gamma$ [(22*b*)]. Let us assume $\beta/\gamma = 1$. Then $\tau_u = 2{,}312$ [(21)]. We consider the nondimensional time interval corresponding to 10 days. Since the dimensional time $T_d = 1$ day corresponds to a nondimensional frequency $\omega = 4$, that is, a nondimensional time $2\pi/4$, the nondimensional time corresponding to 10 days is $T = 15.71$, and the probability tht an exit will occur during a 10-day time interval has the upper bound $p_T \approx 0.007$ [(25)]. As we show in the next section, the actual exit probability is very much lower. However, knowledge of the upper bound p_T can be useful in some practical applications.

Amplitude and Frequency of Idealized, Harmonically Excited Counterpart of Stochastically Excited System

We now discuss the issue of the appropriate choice of excitation amplitude and frequency for the harmonically excited idealized system. Let the harmonic excitation be denoted by $\varepsilon(2)^{1/2}\gamma_H \cos(\omega_H t)$. Its root mean square (RMS) value is $\varepsilon\gamma_H$. It appears reasonable to choose γ_H such that $\varepsilon\gamma_H = \varepsilon\gamma$, where $\varepsilon\gamma$ is the RMS of the stochastic excitation [(1) and (9)].

We may then assume, for example, $\omega_H = \omega_{pk}$, where $\omega_{pk} = 4$ is the stochastic excitation's spectral peak (Fig. 2). The necessary condition for chaos would then be $\gamma/\beta > 4/3/[(2)^{1/2}\pi\omega_H \text{sech}(\pi\omega_H/2)] = 20.88$ [(4) and (8)]. Alternatively, we may assume $\omega_H = \omega_{m,S}$, where $\omega_{m,S} = 0.76$ is the frequency of the peak of the admittance function $S(\omega)$ (Fig. 3). We would then have $\gamma/\beta > 0.71$. A third choice, $\omega_H = \omega_{m,S\Psi}$ where $\omega_{m,S\Psi} = 1.5$ is the frequency of the peak of the spectral density of the Melnikov process (Fig. 4), would yield $\gamma/\beta > 1.06$. The ratio γ/β for which the necessary condition for chaos is satisfied is seen to depend on the choice of frequency ω_H. Given this dependence and the fact that a choice of ω_H would require knowledge of both the forcing spectrum and the function $S(\omega)$, it is more natural and advantageous to use the stochastic Melnikov approach, rather than the deterministic Melnikov approach applied to the idealized harmonically excited system. This conclusion is reinforced by the incorrect inference drawn

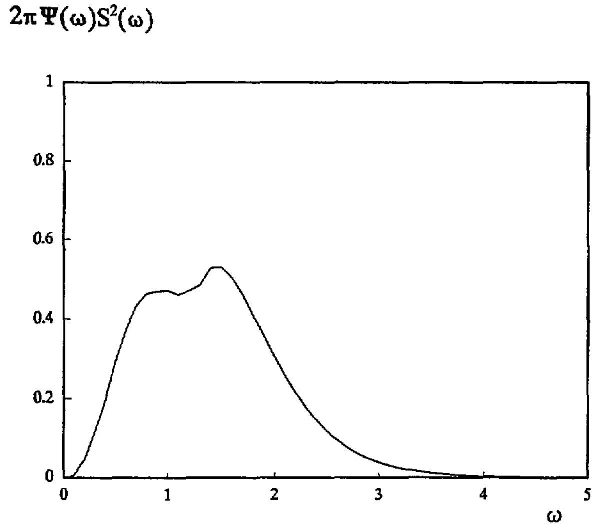

FIG. 4. Spectral Density of Melnikov Process, $2\pi\Psi(\omega)S^2(\omega)$

from the deterministic Melnikov approach that chaos is impossible for relatively small excitations, when in fact this is not true for the actual stochastic system.

To recapitulate: (1) An attempt to assess the propensity for chaos of a stochastic system by applying the Melnikov approach to its harmonically forced idealization entails the difficulty of selecting an appropriate forcing frequency, for which there is no solution outside the stochastic Melnikov approach; (2) the deterministic Melnikov approach yields no information on chaos in the stochastic system for excitations with variance smaller than the variance of the minimum harmonic forcing for which deterministic chaos is possible; and (3) the direct application of the Melnikov approach to the stochastic system yields a lower (upper) bound for the probability that chaotic transport cannot (can) occur during a specified time interval. Simple approximate expressions for this probability are available for $\kappa > 2.5$ or so [(24) and (25)].

For a study of necessary conditions for escapes in systems excited by processes with continuous non-Gaussian distributions, see Simiu and Grigoriu (1995).

ADDITIVE WHITE NOISE: MEAN ZERO UPCROSSING TIME FOR MELNIKOV PROCESS VERSUS MEAN EXIT TIME

For additive white noise excitation and $g(t) \equiv 0$, we can compare the mean time $\tau_{u,w}$ between zero upcrossings of the Melnikov process, to the mean exit time from a well $\tau_{e,w}$. To within any desired approximation (that is, for sufficiently large N and k) $\tau_{u,w}$ is given by (21). Remembering that in (1) ε is asymptotically small, we have the result

$$\tau_{e,w}^{-1}(x) = E[X(t)^+]\alpha \exp[-2\beta V(x)/(\varepsilon\gamma^2)] \quad (27)$$

$$E[X(t)^+] = [\varepsilon/(4\pi\beta)]^{1/2}\gamma;$$

$$\alpha = 1\bigg/\left\{\int_{-\infty}^{\infty} \exp[-2\beta V(x)/(\varepsilon\gamma^2)]\,dx\right\} \quad (28a,b)$$

and $X(t)^+$ denotes positive value of $X(t)$ [see, e.g., Soong and Grigoriu (1992)]. We consider the Duffing oscillator, for which $V(x)$ is given by (2), and $\nu = 0.188$ [(22*a*), (15), and (4)]. For $\beta = 0.1$, $\gamma = 0.025$, $\varepsilon = 0.1$, $\tau_{e,w}(0) = 9.1 \times 10^{340}$ [(22*b*), (21) and (5)]; for $\beta = 0.01$, $\gamma = 0.0025$, $\varepsilon = 0.1$, $\tau_{e,w}(0) = 2.6 \times 10^{34{,}074}$. For both these cases $\tau_{u,w} = 160$.

For another illustration we now consider the case $V(x) = -\cos x$ (*rf*-driven Josephson equation with zero bias) and $g(t) \equiv 0$. For this system $S(\omega) = 2\pi\,\text{sech}(\pi\omega/2)$, and $c = 8$ (Genchev et al. 1983), $\nu = 0.0919$. For $\beta = 0.1$, $\gamma = 0.025$, $\varepsilon = 0.1$, $\tau_{e,w}(\pi) = 1.9 \times 10^{2{,}780}$; for $\beta = 0.01$, $\gamma = 0.0025$, $\varepsilon = 0.1$, $\tau_{e,w}(\pi) = 7.1 \times 10^{26{,}179}$. For both these cases $\tau_{u,w} = 7.66 \times 10^9$.

It is seen from these examples that, for the case of white noise, the performance of $\tau_{u,w}$ as a lower bound for $\tau_{e,w}$ is weak. It is shown by Sivathanu et al. (1995) that τ_u is also a weak lower bound for τ_e in the case of dichotomous noise. The weakness of such lower bounds is to be expected, since the Melnikov criterion provides only a necessary condition for chaos; that is, for any bounded excitation process the Melnikov criterion results in a lower excitation than the excitation that would actually induce chaos (escapes). It follows from (25) that the probability of exceedance of this lower excitation during a specified time interval is higher than (serves as an upper bound for) the corresponding probability for the excitation actually producing chaos.

SYSTEMS EXCITED BY NOISE WITH TAIL-LIMITED MARGINAL DISTRIBUTIONS

We now consider stochastic processes with tail-limited distributions whose paths may be approximated arbitrarily closely by uniformly continuous functions. For systems acted on by such noise the Melnikov approach yields a criterion guaranteeing that no exit from a well can occur, however long the waiting time. This is illustrated for the case of a Duffing oscillator excited by square-wave dichotomous noise. We assume in this section $g(t) \equiv 0$.

The expression for dichotomous coin-toss square-wave noise is

$$G(t) = a_n; \quad [\alpha + (n-1)]t_1 < t \leq (\alpha + n)t_1 \quad (29)$$

where n = the set of integers; α = a random variable uniformly distributed between 0 and 1; a_n = independent random variables that take on the values -1 and 1 with probabilities 1/2 and 1/2, respectively; and t_1 = a parameter of the process $G(t)$.

A rectangular pulse wave of amplitude a_n and length t_1 centered at coordinate $t_n = (\alpha + n - 1/2)t_1$ has Fourier transform $F_n(\omega)\ -a_n|(2/\omega)\sin(\omega t_1/2)\exp(-j\omega t_n)|$ (Papoulis 1962, p. 20). The pulse itself can then be expressed as a uniformly continuous sum of terms approximating as closely as desired the inverse Fourier transform of $F_n(\omega)$. Each realization of the coin-toss dichotomous square-wave can then be approximated as closely as desired by a superposition of such sums, which is itself a uniformly continuous function.

Uniformly continuous functions that would similarly approximate arbitrarily closely a process $G(t)$ with tail-limited marginal distributions would induce a Melnikov process approximating arbitrarily closely the process

$$M(t) = -\beta c + \gamma \int_{-\infty}^{\infty} h(\tau)G(t-\tau)\,d\tau \quad (30)$$

The necessary condition for chaos may therefore be developed simply by using (30); there is no need to carry out the approximation of the process $G(t)$ explicitly.

From (5) and (29)

$$M(t) = -4\beta/3 + (2)^{1/2}\gamma F(t) \quad (31)$$

$$F(t) \approx \sum_{n=-l}^{l} a_n\{-\text{sech}[(n+\alpha)t_1 - t]$$

$$+ \text{sech}[(n + \alpha - 1)t_1 - t]\} \quad (32)$$

where l is sufficiently large for the error due to the assumption that l is finite to be as small as desired.

The area under the curve $x_s(-t)$ (3*b*) in a half-plane is $(2)^{1/2}$. It follows immediately from the definition of $F(t)$ [see (31)

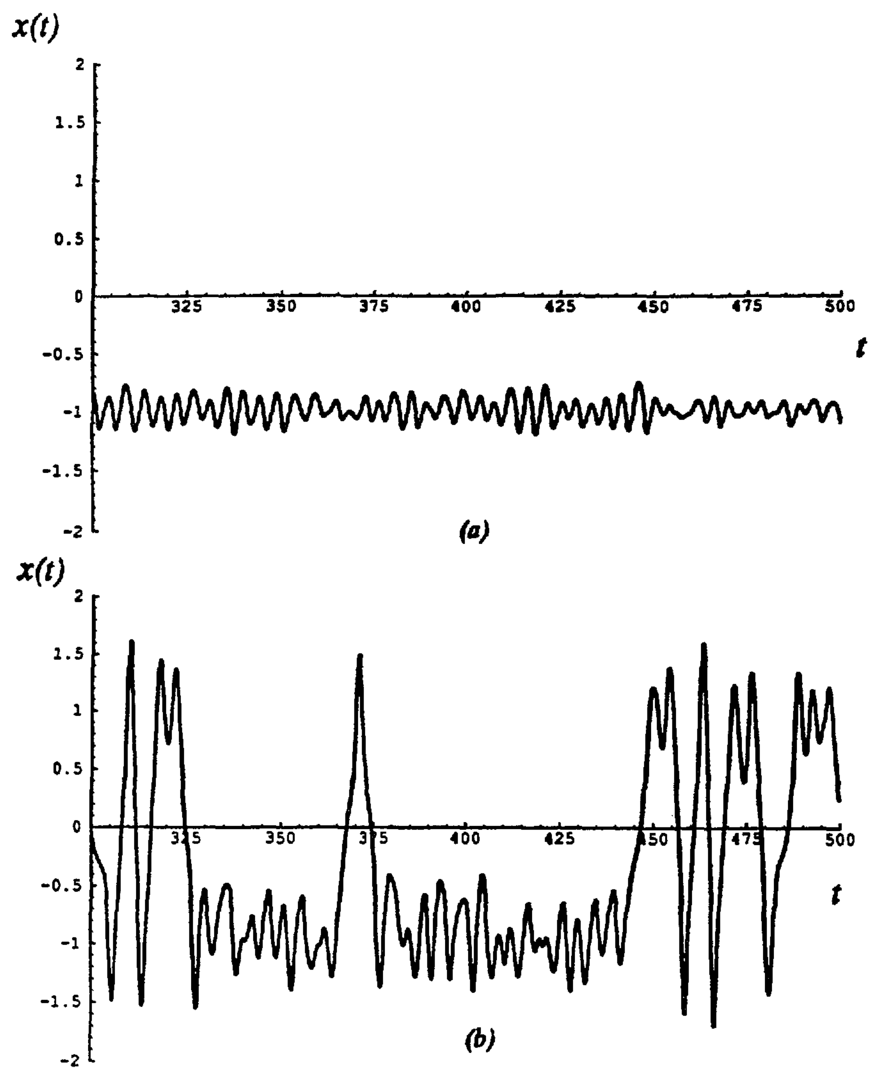

FIG. 5. Realizations of Stochastic Motions Induced by Square-Wave, Coin-Toss Dichotomous Noise: (a) Nonchaotic Motion; (b) Chaotic Motion

and (32)] that $-2 < F(t) < 2$. (For example, $F(t)$ would be equal to 2 if $\alpha = 0$, $a_n = 1$ for all n such that $t > 0$ and $a_n = -1$ for all n such that $t < 0$.) The necessary condition for chaos is that $M(t)$ have simple zeros. From (32) it follows therefore that if

$$\beta/\gamma > 2.121 \quad (33)$$

then this condition cannot be satisfied, and chaotic transport cannot occur. Eq. (33) is a remarkably simple criterion guaranteeing the nonoccurrence of exits. For additional details, see Simiu and Hagwood (1995).

From (30) it follows immediately that (33) is also valid for square wave dichotomous noise with random arrival times and, more generally, that criteria similar to (33) can be derived for any reasonable tail-limited random excitation. This can be seen by replacing in (30) $G(t)$ by its maximum possible value.

As was mentioned earlier, it was verified that if chaos is possible [i.e., if (33) is not satisfied], then the mean upcrossing rate of the Melnikov process is a weak lower bound for the mean exit rate (Sivathanu et al. 1995).

We show in Fig. 5 time history realizations corresponding to the dichotomous noise of (29), the parameters $\varepsilon = 0.1$, $\beta = 1.5$ and, respectively, $\beta/\gamma = 2.13 > 2.121$, $\beta/\gamma = 0.625$. The motion of Fig. 5(a) is confined to one well. Its irregularity is due to the stochastic nature of the excitation. The chaotic motion of Fig. 5(b) is similar to chaotic motions induced in the Duffing oscillator by harmonic or quasi-periodic excitation. Its irregularity is due to both the chaotic nature of the motion and the stochastic nature of the excitation. We note that, as is the case for equations with harmonic forcing (moon 1987), the necessary condition for the occurrence of chaos is helpful in the search for chaotic regions of parameter space even for relatively large ε.

APPLICATIONS

In this section we briefly review a few recent applications of the stochastic Melnikov approach.

Wind-Induced Current Flow over Corrugated Ocean Floor

We mentioned earlier the study by Allen et al. (1991) of a quasi-geostrophic model of along-coast ocean currents induced by fluctuating wind over bottom topography with periodic corrugations normal to the coastline. In the absence of friction and forcing this model exhibits homoclinic orbits due to the existence of the corrugations. The homoclinic orbits are associated with potential wells separated by a barrier. Under excitation by wind with low-frequency *harmonic* fluctuations, and in the presence of friction, if a particle starts its motion within a well with sufficiently small velocity, it will move within that well for all time—unless the system's Melnikov function has simple zeros, in which case the particle can behave chaotically, moving back and forth across the potential barrier in an apparently random fashion [see Allen et al. (1991), for details].

Consider now the more realistic case where the wind fluctuations are random. The excitation then induces a Melnikov process. Assume for simplicity that the excitation is Gaussian. Then, with probability one, the Melnikov process will have simple zeros, and exits are possible—provided that one waits long enough. However, the probability that exits will occur within a specified finite time interval is less than one. Using the approach described in this paper, this probability has been estimated in a specific case by Simiu (1995), to which the reader is referred for details.

Snap-through of Buckled Column with Distributed Mass and Distributed Random Loading

This system differs from (1) in that it is governed by a partial differential equation. For the deterministic, harmonic loading case the Melnikov function was obtained by Holmes and Marsden (1981). Melnikov-based criteria for snap-through in the stochastic case were obtained by Simiu and Frey (1996).

Open-Loop Control of Multistable Systems

The performance of certain nonlinear stochastic systems is deemed acceptable if, during a specified time interval, the systems have sufficiently low probabilities of escape from a preferred region of phase space. For example, the motion of a ship subjected to wave loading may be modeled by an equation of motion with a nonlinear restoring term [see, e.g., Hsieh et al. 1994). Given a design sea state with a specified mean return period, a coordinate defining the behavior of the ship (e.g., its roll angle) must have an acceptably small probability of exit from the ''safe'' region of phase space. One way to reduce the probability of escape during a specified time interval is to apply to the system, with a time lag that is relatively small in relation to a characteristic frequency of the excitation, a counterforce that reduces the effect of the excitation.

Eq. (11*b*), or Figs. 2–4, show that an open-loop control approach can be based on the observation that only part of the frequency components of the excitation $G(t)$ contribute significantly to the spectral density of the uncontrolled system's Melnikov process. These are determined by the portion of the function $S(\omega)$ [(4)] that is not negligibly small (for example, in Figs. 2–4, components with frequencies $0.25 < \omega < 2.5$). The following approach is therefore used. Instead of a control force proportional to the excitation $G(t - t_0)$, it would be more efficient to apply a control force obtained from the function $G(t - t_0)$ by filtering out from this function those frequency components that do not contribute significantly to the spectral density of the Melnikov process. Depending on the spectral density of the excitation and the characteristics of the system, this approach can reduce significantly the power needed for the system's control while achieving a comparable reduction of the ordinates—and the mean zero upcrossing time—of the controlled system's Melnikov process and, hence, a comparable reduction of the system's mean exit time. This was confirmed by studies reported by Franaszek and Simiu (1995).

CONCLUSIONS

In this paper we reviewed basic results on the application of the stochastic Melnikov approach to a class of near-integrable systems with additive or multiplicative noise. We compared this approach with the deterministic Melnikov approach applied to an idealization of the stochastic system wherein a harmonic excitation is substituted for the stochastic forcing. The stochastic Melnikov approach eliminates errors associated with the choice of amplitude and frequency of the stochastic excitation's harmonic counterpart. Unlike the approach based on a harmonic idealization of the stochastic forcing, for stochastic systems with noise having infinite-tailed marginal distributions the stochastic Melnikov approach reflects the systems' full range of chaotic transport possibilities.

For systems with additive or multiplicative noise, the mean zero upcrossing rate τ_u^{-1} for the stochastic system's Melnikov process provides an upper bound for the system's mean exit rate τ_e^{-1}. However, as shown by comparisons between τ_u^{-1} and τ_e^{-1} for the case of white noise, this upper bound can be very weak. This is explained, at least in part, by the fact that, for both stochastic and deterministic systems, the Melnikov condition for chaos is necessary but not sufficient.

For nonlinear systems excited by processes with tail-limited marginal distributions remarkably simple criteria can be derived that guarantee the nonoccurrence of exits. This was illustrated for square-wave, coin-toss dichotomous noise. We also reviewed briefly three applications of the stochastic Melnikov approach: (1) the estimation of upper bounds for probabilities that alongshore currents induced by wind with given mean speed over a corrugated ocean floor can experience exits from a preferred region of phase space during a specified time interval; (2) the snap-through of columns with distributed mass and distributed random loading; and (3) the efficient open-loop control of multistable systems.

The validity of the Melnikov approach can be proven rigorously for asymptotically small perturbations. Numerical experiments have shown that the approach can be useful also for systems with relatively large perturbations. It should be remembered, however, that beyond some perturbation level a system may not behave as predicted by the Melnikov approach. This is another limitation shared by the stochastic and deterministic Melnikov approaches.

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