Twisted Mass Lattice QCD
$\Updownarrow$
Chirally twisted Wilson fermions

A formulation of lattice QCD $(N_f = 2, 4, \ldots)$ with

1. ultralocal action
2. no quark zero modes in the massive theory
3. easy/automatic O(a) improvement
4. ways of constructing composite operators
   with continuum-like mixing properties

"low" CPU cost

$1. \rightarrow$ wrt GW fermions

$2. \rightarrow$ wrt W fermions (no exceptionals,
no spikes in updating algorithms)

$3. \rightarrow$ cost $\sim a^{-k}$, $k \sim 5 \div 7$

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$N_f = 2$ twisted mass QCD

$S = S_g[U] + S_F[U, \psi, \bar{\psi}]$, mass degenerate $\psi = (q_1, q_2)$

$$S_F = a^4 \sum_x \bar{\psi}(x)[\gamma \cdot \widetilde{\nabla} + e^{-i\omega\gamma_5\tau_3} W_{\rm cr}(r) + m_q]\psi(x)$$

$$W_{\rm cr}(r) = -a\frac{r}{2}\nabla^* \cdot \nabla + M_{\rm cr}(r) \, , \qquad \widetilde{\nabla}_\mu = \frac{1}{2}(\nabla^*_\mu + \nabla_\mu)$$

[F.– Grassi – Sint – Weisz, '99-'01; F. – Rossi, '03]

- m_q : bare quark mass, renormalizes multiplicatively
  $m_\pi^2 \sim m_q$ as $a \to 0$
- ω : UV regularization label, unphysical, controls O(a),
  $\omega = 0, \pi \Leftrightarrow$ Wilson
- action contains $M_{\rm cr}(r)$: need know it

  all numerical and O(a) uncertainties on $M_{\rm cr}(r)$, ω exact
- protection against exceptional configurations from $m_q \neq 0$

  $\mathrm{Det}[D_F] = \det[Q^2 + m_q^2 \sin^2\omega]$ [Aoki – Gocksch '89]

  $Q \equiv \gamma_5[\gamma \cdot \widetilde{\nabla} + W_{\rm cr}(r) + m_q \cos\omega] = Q^\dagger$
- three partially conserved currents (@ $\omega = \pi/2$: $A_\mu^{1,2}$, V_μ^3)

$M_{\rm cr}$ and renormalization

often convenient to work in unphysical quark basis $(\chi, \bar\chi)$

$$S_F[\psi,\bar\psi] = a^4 \sum_x \bar\psi[\gamma\cdot\widetilde\nabla + e^{-i\omega\gamma_5\tau_3} W_{\rm cr}(r) + m_q]\psi$$

$$\Updownarrow$$

$$S_F[\chi,\bar\chi] = a^4 \sum_x \bar\chi[\gamma\cdot\widetilde\nabla - a\frac{r}{2}\nabla^*\cdot\nabla + m_0 + i\mu\gamma_5\tau_3]\chi$$

$$\chi = e^{-i\omega\gamma_5\tau_3/2}\psi \qquad \bar\chi = \bar\psi e^{-i\omega\gamma_5\tau_3/2}$$

$$\mu = m_q \sin\omega \qquad m_0 = m_q\cos\omega + M_{\rm cr}(r)$$

- info on $M_{\rm cr}$ from the renormalized WTI (at any μ)

$$\partial_\mu[\bar\chi\gamma_\mu\gamma_5\tau_1\chi](x) \propto (m_0 - M_{\rm cr}(r))[\bar\chi\gamma_5\tau_1\chi](x) + {\rm O}(a)$$

tested at $\mu\neq 0 \to m_\pi > 0$ [Della Morte – F. – Heitger – Sint, '01]

- $M_{\rm cr}(r) = -M_{\rm cr}(-r)$ [Aoki '84; F.– Rossi, '03]

- Mass-independent schemes $(m_0 = M_{\rm cr},\ \mu = 0)$:

Z_O of $O(\psi,\bar\psi)$? From $O(\psi,\bar\psi) = \sum_i c_i^O Q_i^O[\chi,\bar\chi]$

$$Z_O \Longleftrightarrow Z_{Q_i}(\omega = 0 \leftrightarrow \text{Wilson})$$

- Parity and isospin breaking: O(a) effect $\Rightarrow$

operator mixing; states in spectral representation of CF's

Mass non-degenerate flavours

- $N_f = 2$ with $\omega = \pi/2$ & $\gamma_5\tau_1$–twist [F.–Rossi, '03]

$$S_{Fnd} = a^4 \sum_x \bar{\psi}(x)[\gamma\cdot\widetilde{\nabla} - i\gamma_5\tau_1 W_{\rm cr}(r) + m_q - \epsilon_q\tau_3]\psi(x)$$

fermionic determinant: real, positive for $|\epsilon_q| < |m_q|$

$$\det[Q_{\rm cr}^2 + m_q^2 - \epsilon_q^2]\ \det[1 + 2\epsilon_q B]\ \geq\ \det[Q_{\rm cr}^2 + m_q^2 - \epsilon_q^2]$$

$$Q_{\rm cr} \equiv \gamma_5[\gamma\cdot\widetilde{\nabla} + W_{\rm cr}(r)] = Q_{\rm cr}^\dagger$$

renormalized quark masses: $\hat{m}^\pm = Z_P^{-1}(r)m_q \pm Z_S^{-1}(r)\epsilon_q$

- $N_f = 1, 2, 3, \ldots$ with flavour diagonal action

$$S_F^{OS} = \sum_{f=1}^{N_f} \left\{ a^4 \sum_x \bar{q}_f(x)[\gamma\widetilde{\nabla} + e^{-i\gamma_5\theta_f} W_{\rm cr}(r_f) + m_f]q_f(x) \right\}$$

$\theta \equiv \sum_f \theta_f \neq 0 \;\Rightarrow\;$ N_f–QCD with a θ–term and

complex fermionic determinant

renormalized quark mass: $\hat{m}_f = Z_m(r_f)m_f$

[Österwalder–Seiler, '78; Kerler, '81; Seiler–Stamatescu, '82]

1. action for valence quarks
2. $[N_f = 4]$–QCD for $\theta_u = -\theta_d = \pi/2$, $\theta_s = \theta_c = 0$

   quark mass renormalization $\Leftarrow$ [Pena–Sint–Vladikas, '04]

O(a) improvement

- Symanzik's programme [many of you & co.]

$$-(ar/2)\nabla^*\cdot\nabla + M_{\rm cr}(r) \quad \longrightarrow$$
$$-(ar/2)\nabla^*\cdot\nabla + \overline{M}_{\rm cr}(r) + (ia/4)c_{SW}(r)\sigma\cdot F + \ldots$$

operators $\longrightarrow$ operators + O(a) counterterms

[extension to tmQCD: F.–Sint–Weisz, '01]

- Wilson Average method [F.–Rossi, '03]

1. multilocal, gauge-inv. , multiplicatively renormalizable O :

$$\mathrm{WA}: \quad \langle O(x_1,\ldots,x_n)\rangle^{(\omega)}_{(r,m_q)} + \langle O(x_1,\ldots,x_n)\rangle^{(\omega)}_{(-r,m_q)}$$
$$= \quad 2\zeta^O_O(r;\omega)\langle O(x_1,\ldots,x_n)\rangle^{\rm cont}_{(m_q)} + \mathrm{O}(a^2)$$

[based on spurionic ($r\to -r$ & $m_q\to -m_q$) and exact symmetries]

2. simplifies further for $\omega = \pm\pi/2$ (maximal twisting)

$$\mathrm{WA}: \quad \langle O(x_1,\ldots,x_n)\rangle^{(\pi/2)}_{(r,m_q)} + \eta_O\langle O(x_1^P,\ldots,x_n^P)\rangle^{(\pi/2)}_{(r,m_q)}$$

$$x^P \equiv (x_0, -\mathbf{x}), \quad \eta_O \equiv \text{parity of } O = O[\psi,\bar\psi,U]$$

due to symm. parity $\times$ ($r\to -r$) $\to$ WA from one simulation

$\Rightarrow$ carries over to mass non-degenerate quarks

$\Rightarrow$ by FT: WA's of energies and matrix elements

B_K without mixings and O(a)

QCD with $N_f = 4$ (qcd4): B_K extracted from (FT of)

$$C_{K\mathcal{O}K}^{(\mathrm{qcd4})} = \langle (\bar{d}\gamma_5 s)(x)[(\bar{s}\gamma_\mu d)^2 + (\bar{s}\gamma_\mu\gamma_5 d)^2](0)\bar{d}\gamma_5 s)(y)\rangle^{(\mathrm{qcd4})}$$

4s6v model: different regularizations for sea and valence quarks

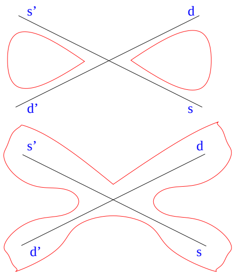
[F.–Rossi: '03 @ EU-IHP Orsay workshop, hep-lat/0406...]

– sea quarks $(u_{\mathrm{sea}}, d_{\mathrm{sea}})$, $(s_{\mathrm{sea}}, c_{\mathrm{sea}})$ in pairs (FR action)

– valence species u, d, d', s, s', c with OS action

– B_K from the 4s6v–model correlator

$$C_{K'\mathcal{Q}K}^{(\mathrm{4s6v})} = \langle (\bar{d}'\gamma_5 s')(x) 2\mathcal{Q}_{VV+AA}^{\Delta S=2}(0)(\bar{d}\gamma_5 s)(y)\rangle^{(\mathrm{4s6v})}$$



$$\mathcal{Q} \ : \left\{ \begin{array}{l} (\bar{s}\gamma_\mu d)(\bar{s}'\gamma_\mu d') + (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}'\gamma_\mu\gamma_5 d') \\[2ex] (\bar{s}\gamma_\mu d')(\bar{s}'\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d')(\bar{s}'\gamma_\mu\gamma_5 d) \end{array} \right.$$

– $\underline{r_d = \pm r_{d'} = r_s = \mp r_{s'}}$ $\Rightarrow$ $\mathcal{Q}_{VV+AA}^{\Delta S=2}$ has no mixings

– $a \to 0$ @ $\hat{m}_d = \hat{m}_{d'} = \hat{m}_{d\mathrm{sea}}$, $\hat{m}_s = \hat{m}_{s'} = \hat{m}_{s\mathrm{sea}}$

– $\lim_{a\to 0}[C_{K'\mathcal{Q}K}^{(\mathrm{4s6v})}]_R \overset{\mathrm{Wick\ thm}}{=} \lim_{a\to 0}[C_{K\mathcal{O}K}^{(\mathrm{qcd4})}]_R$

provided $\hat{m}_f^{(\mathrm{4s6v})} = \hat{m}_f^{(\mathrm{qcd4})}$, $f = u, d, s, c$

- Extension to B_B^{stat}: d, d', $h^{\pm}$! [talk by Della Morte]

$K \to \pi(\pi)$ without wrong–χ mixings and O(a)

same strategy as for B_K [F.–Rossi, '04, hep-lat/0406...]

auxiliary 4s10v model

- valence OS u, u', u'', u''', d, s, c, c', c'', c'''
- two sea pairs $(u_{\rm sea}, d_{\rm sea})$, $(s_{\rm sea}, c_{\rm sea})$ (FR action)
- same renormalized mass for all quarks of a given phys. flavour

1. $K \to \pi\pi$ amplitudes from

$$C^{(4s10v)}_{\pm,\pi\pi} = \langle \Phi_\pi(x)\Phi_\pi(y)\mathcal{Q}^{\pm}_{VA+AV}(0)\Phi_{K^0}(y)\rangle^{(4s10v)}$$

$\Phi_{\pi^\pm}$, Φ_{π^0}, Φ_{K^0} with only valence u, d, s quarks

$$\mathcal{Q}^{\pm}_{VA+AV} = \mathcal{Q}^{\pm\,(u,c)}_{VA+AV} + \mathcal{Q}^{\pm\,(u',c')}_{VA+AV} - \frac{1}{2}\mathcal{Q}^{\pm\,(u'',c'')}_{VA+AV} - \frac{1}{2}\mathcal{Q}^{\pm\,(u''',c''')}_{VA+AV}$$

i) $\underline{r_d = r_s}$, $r_d = r_u$

ii) $\underline{r_u = r_c = r_{u''} = r_{c''} = -r_{u'} = -r_{c'} = -r_{u'''} = -r_{c'''}}$

$\Rightarrow$ $\mathcal{Q}^{\pm}_{VA+AV}$ mixes only with $(m_c^2 - m_u^2)(m_s - m_d)\,(\bar{s}\gamma_5 d)$

2. $K \to \pi$ amplitudes from matrix elements of

$$\mathcal{Q}^{\pm}_{VV+AA} = \mathcal{Q}^{\pm\,(u,c)}_{VV+AA} + \mathcal{Q}^{\pm\,(u',c')}_{VV+AA} - \frac{1}{2}\mathcal{Q}^{\pm\,(u'',c'')}_{VV+AA} - \frac{1}{2}\mathcal{Q}^{\pm\,(u''',c''')}_{VV+AA}$$

with i) $\underline{r_d = -r_s}$, $r_d = r_u$ and ii)

$K \to \pi(\pi)$ amplitudes with reduced mixings

[Pena–Sint–Vladikas, hep-lat/0405028]

$$\mathcal{L}_F^{OS} = \sum_{f=u,d,s,c} \bar{q}_f(x)[\gamma\cdot\widetilde{\nabla} + e^{-i\gamma_5\theta_f} W_{\rm cr}(r_f) + m_f] q_f(x)$$

$$\Leftrightarrow \sum_{f=u,d,s,c} \bar{q}'_f(x)[\gamma\cdot\widetilde{\nabla} + W_{\rm cr}(r_f) + e^{i\gamma_5\theta_f} m_f] q'_f(x)$$

$i)$ $\theta_u = -\theta_d = \theta_s = -\theta_c = \pi/2$

$m_u = m_d, \quad m_s < m_c, \quad N_f = 0, 2$ (s and c quenched)

$ii)$ $\theta_u = -\theta_d = \pi/2,\ \theta_s = \theta_c = 0$

$m_u = m_d, \quad m_s < m_c, \quad N_f = 0, 2, 3, 4$

$i)$ no a^{-2} power divergency: by symmetries $\mathcal{O}^{\pm}_{VV+AA}$

$$\mathcal{O}^{\pm}_{VV+\ldots} = [(\bar{s}\gamma_\mu u)(\bar{u}\gamma_\mu d) \pm [\bar{s}\gamma_\mu d)(\bar{u}\gamma_\mu u) + \ldots] - [u \leftrightarrow c]$$

can mix only with

$$d_P^{\pm} a^{-1}(m_c - m_u)(m_s - m_d)(\bar{s}\gamma_5 d), \quad d_S^{\pm} \mathrm{O}(m^3)(\bar{s}d)$$

– $1/a$-divergent term can be subtracted by enforcing parity, affects $\langle \pi^+ | \mathcal{O}^{\pm}_{VV+AA} | K^+ \rangle_{\rm SW-improved\ action}$ only at O(a)

– term $\propto d_S^{\pm}$ can be fixed by imposing $\langle \pi, \mathbf{0} | \mathcal{Q} | K, \mathbf{0} \rangle = 0$

$ii)$ work with $\mathcal{Q} \equiv \mathcal{O}_{VA+AV} + i\mathcal{O}_{VV+AA}$

need: two O(a^{-1}) c.t.'s & imposing $\langle \pi, \mathbf{0} | \mathcal{Q} | K, \mathbf{0} \rangle = 0$

Continuum and Chiral Limits

- $a > 0$: in the action chiral symmetries broken by

  i) $\bar{\psi} m_q \psi$ ii) $\bar{\psi} e^{-i\omega\gamma_5\tau_3} W_{\rm cr}(r|U)\psi$

  physics $\sim m_q \Lambda^3_{QCD}$ lattice artifacts $\sim a\Lambda^5_{QCD}$

  $\omega \neq 0, \pi$: i) and ii) not parallel in chiral space

- physical world: chirality spontaneously broken

  as $a \to 0$ $\langle[\bar{\psi} i\gamma_5\tau_3\psi]_R\rangle \to 0$, $\langle[\bar{\psi}\psi]_R\rangle \to \Sigma \neq 0$

  for $a > 0$ O(a) due to ii) driven by m_q

Set $\omega = \pi/2$: if $m_q|_{\omega=\pi/2} = \mu$ is numerically $\leq a\Lambda^2_{QCD}$

$\Rightarrow$ impact on unquenched simulations ?

$\Rightarrow$ vacuum at $m_0 - M_{\rm cr} \leq \mathrm{O}(a)$ and $\mu \leq \mathrm{O}(a)$?

$\Rightarrow$ size of residual O(a^2) on improved quantities ?

To be on the safe side:

first continuum limit $(a \to 0)$, then chiral limit $(m_q \to 0)$

$a^2\Lambda^3_{QCD} \ll m_q$ $\leftarrow$ good scaling (after O(a) improv.)

$a\,\Lambda^2_{QCD} < m_q$ $\leftarrow$ unquenched simulations?

numerical test & analytic study (lattice ChPT, . . .)

Lattice ChPT for tmQCD

Talk by Münster [M.–Schmidt, '03 & M.–Schmidt–Scholz, '04]

Chiral Lagrangian including lattice artifacts linear in a:

$$L = \frac{F_0^2}{4}\left\{\mathrm{tr}[(\partial\Sigma)^\dagger\cdot\partial\Sigma] - \mathrm{tr}[(\chi+\rho)\Sigma^\dagger + \Sigma(\chi+\rho)^\dagger]\right\} + \mathrm{O}(p^4)$$

$$\chi = 2B_0\hat{m}_q\,, \qquad \rho = 2aW_0 e^{i\alpha\tau_3}\,, \qquad \mathrm{tg}\alpha = Z_{S^0}Z_P^{-1}\mathrm{tg}\omega$$

$\Rightarrow$ m_π^2, f_π, . . . to NLO: no flavour breaking for all α

$\Rightarrow$ O(a) artifacts vanish for $\alpha = \pi/2 \leftrightarrow \omega = \pi/2$

Talk by J. Wu [Sharpe–Wu] : inclusion of O(a^2) effects

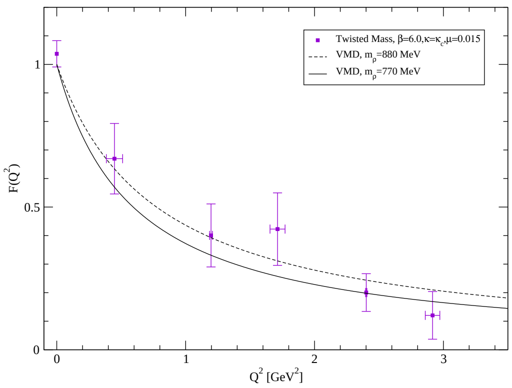
$$\Rightarrow\ m_{\pi^0}^2 - m_{\pi^\pm}^2 \sim a^2\Lambda_{QCD}^4 \frac{\hat{m}_q^2\sin^2\alpha}{\hat{m}_q^2\sin^2\alpha + (a\Lambda_{QCD}^2 + \hat{m}_q\cos\alpha)^2}$$

$\Rightarrow$ smoothing of Aoki phase or 1st order transition due to $\mu \neq 0$

Pion form factor (quenched)

$$\langle\pi^2, (\mathbf{p}+\mathbf{Q})|\ V_\mu^3\ |\pi^1, \mathbf{p}\rangle = (2p+Q)_\mu F(Q^2)$$

Talk by Abdel–Rehim [A.R. & Lewis]



Meson vector dominance:

$$F \sim [1 + Q^2/m_\rho^2]^{-1},\ Q^2 \ll m_\rho^2$$

$\beta = 6$, $\omega = \pi/2$, $am_q = 0.015$

WA by three-momentum averaging

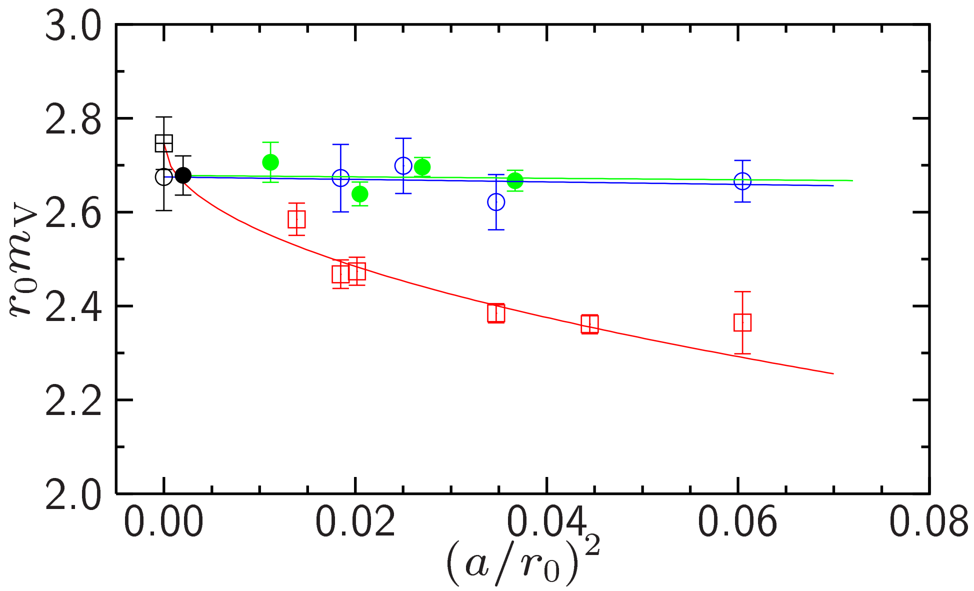
preliminary @ $(m_\rho, m_\pi) \sim (880, 470)$ Mev

Scaling test of (quenched) tm-LQCD @ $\pi/2$

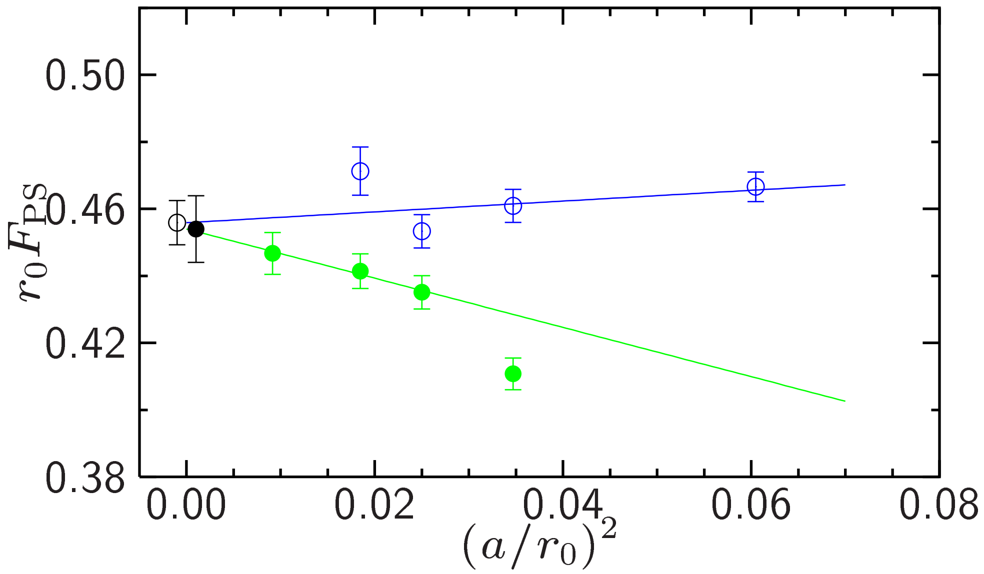
Keep fixed $r_0^2 F_{q\bar{q}}(r_0) = 1.65$ and $r_0 m_{PS} = 1.79$

[$r_0 \simeq 0.5$ fm and $m_{PS} \simeq 710$ Mev]

i) vector meson mass m_V vs. a^2



ii) pseudoscalar meson decay constant F_{PS} vs. a^2



Red : ordinary Wilson fermions (extrapol. linear in a) [Gockeler et al. '97]

Green: Symanzik (n.p.) improved Wilson fermions [ALPHA, Garden et al. '99]

Blue : tm-Wilson fermions [Jansen, Shindler, Urbach, Wetzorke '03]

More about quenched tmQCD

Precision computation of B_K [talk by C. Pena]

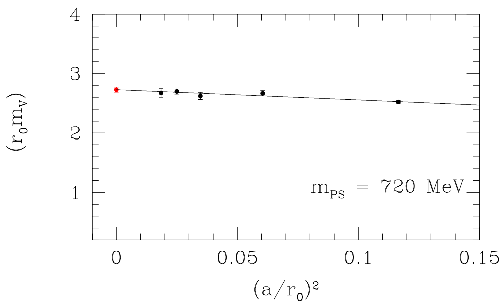
by Dimopoulos et al. (ALPHA Coll.) {Wingate's review}

$\Rightarrow \quad \hat{B}_K^{\mathrm{RGI}} = 0.817(22)$ (s untwisted)

n.p. renormalization, continuum limit; M_{PS} extrapolated to M_K

Extended scaling tests [talk by A. Shindler]

- several quark masses $\Leftrightarrow m_{PS} = 710, 515, 370$ MeV
- five β-values: [5.7,5.85,6,6.1,6.2] $\Leftrightarrow a \in [0.15, 0.075]$ fm



Still preliminary, unequal statistics

How does the size of O(a^2) effects

vary when decreasing m_q (m^2_{PS})?

tmQCD vs. overlap at $\beta = 5.85$:

[talks by K. Jansen & M. Papinutto]

- iterative methods for evaluating quark propagators
- comparison of CPU costs: $R = C_{\mathrm{ov}}/C_{\mathrm{tm}}$

  $V = 12^4, m_\pi = 720$Mev $\quad\Rightarrow\quad R \sim 22$

  $V = 12^4, m_\pi = 390$Mev $\quad\Rightarrow\quad R \sim 44$

  $V = 16^4, m_\pi = 720$Mev $\quad\Rightarrow\quad R \sim 30$

  $V = 16^4, m_\pi = 390$Mev $\quad\Rightarrow\quad R \sim 46$
- F_{PS}, M_V, octet & decuplet baryon masses (preliminary)

$N_f = 2$ simulations on large & coarse lattices

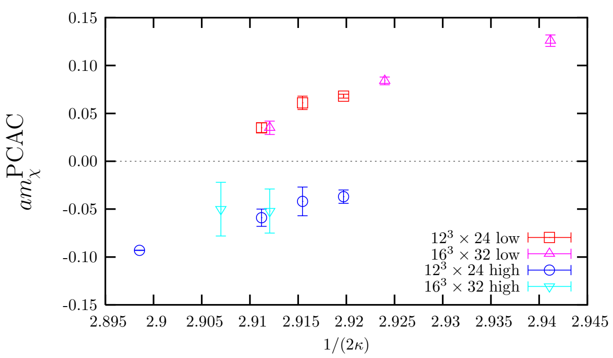
W-glue action: $\beta = 5.2$ ($a \sim 0.17$ fm), L=8,12,16 (T=2L)

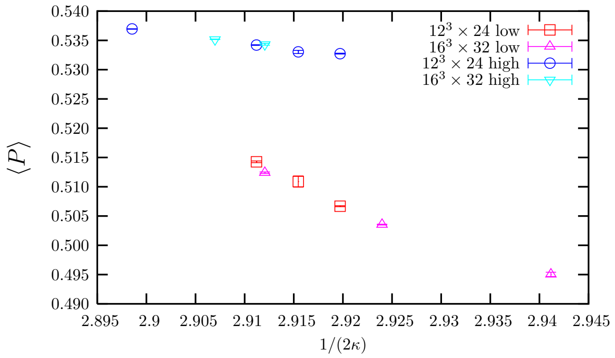
GHMC & TSMB algorithms (EO precond.): benefit from $\mu \neq 0$

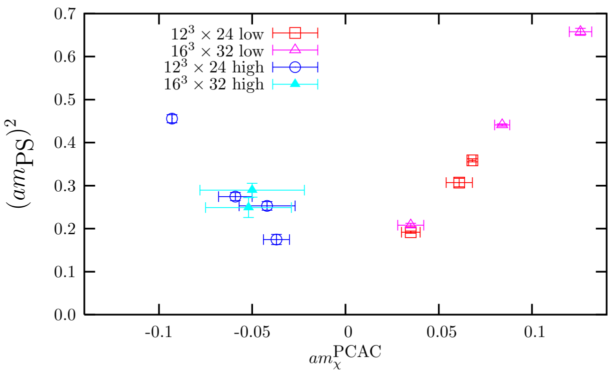
Talks by Urbach and Farchioni

plaquette, $m^2_{\pi^\pm}$, $m_\chi \equiv \dfrac{\partial_\mu \langle \bar\chi \gamma_\mu \gamma_5 \tau_1 \chi\, O^1 \rangle}{\langle \bar\chi \gamma_5 \tau_1 \chi\, O^1 \rangle}$ $\quad (m_\chi \propto m_0 - M_{\rm cr})$

show metastability signals for $a\mu$, $|am_\chi| < 0.1$!







$a\mu = 0.01$, # meas. O(100)

U-samples with opposite m_χ

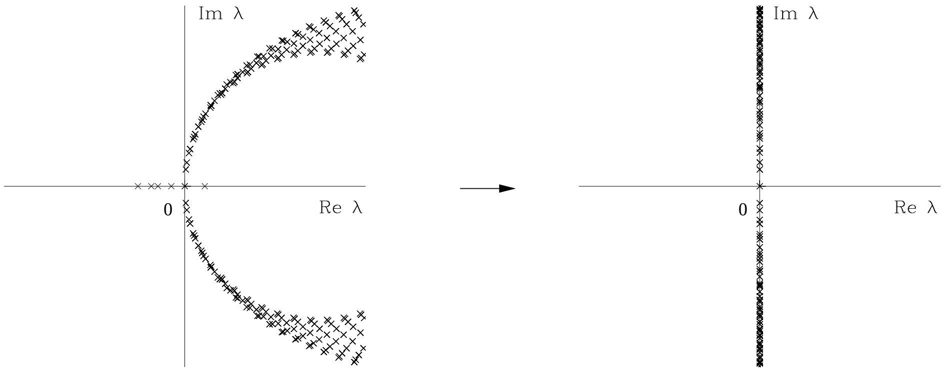
$m^2_{\pi^\pm}$ follows m_χ, not m_0!

[see JLQCD @ $N_f = 3$, Lat01]

- U-samples strongly reminiscent of hot or cold start
- $\langle \mathrm{Plaq} \rangle = \mathrm{O}(1) + b_3 a^3 \langle \bar\chi \chi \rangle + b_4 a\mu \langle i\bar\chi \gamma_5 \tau_3 \chi \rangle + \ldots$
  expect $\langle \bar\chi \chi \rangle$ driven by m_χ and $\langle i\bar\chi \gamma_5 \tau_3 \chi \rangle$ by μ
- phase transition(s) at small μ or at $\mu = 0$ or algorithmic problem? May ChPT to O(a^2) help understanding?

Exceptional configurations and tm-LQCD

Spectrum of the massless $(m_q = 0)$ Dirac matrix $D_{\rm cr}$ (sketch)

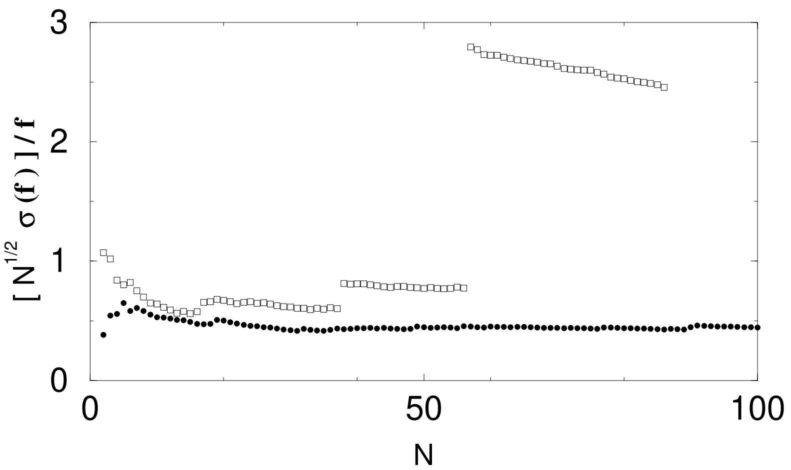


$\omega = 0$ (Wilson standard) $\qquad$ $\omega = \pi/2$ (maximal twist)

$\omega = 0$ and $r > 0$: assuming $m_q \geq 0$, negative real eigenvalues may yield spurious zero modes of $D_{\rm cr} + m_q$ $\Rightarrow$ spikes in fermionic observables

$\omega = 0$ and $r < 0$: spectrum reflected through the vertical axis, higher density of negative real eigenvalues $\Rightarrow$ "spikes" even more frequent

$\omega = \pm\pi/2$: no zero modes of $D_{\rm cr} + m_q$ as long as $m_q \neq 0$



clover, $\beta = 6$

$\frac{m_{\rm PS}}{m_{\rm V}} \simeq 0.47$

$L \simeq 2.2$ fm

Statistical test for N measurements: does $N^{1/2}\sigma \overset{N \to \infty}{\Longrightarrow}$ const.?

Göckeler, Hoferichter, Horsley, Pleiter, Rakow, Schierholz and Stephenson, '98

Della Morte, F., Heitger, '00: f = pion correl. at t=L, $\sigma^2(f)$ = its variance

QUENCHED : open symbols: $\omega = 0$, NO; filled symbols: $\omega = \pi/2$, YES

Key points for O(a) improvement via WA

Expectation values of multilocal gauge invariant operators

$$\langle O(x_1,\ldots,x_n)\rangle\Big|_{(r,m_q)}^{(\omega)} = Z_{\rm L}^{-1}\int_{\rm fields} e^{-[S_{\rm YM}^{\rm L}+S_{\rm F}^{\rm L,ph}]}O(x_1,\ldots,x_n)$$

$$S_{\rm F}^{\rm L,ph}[\psi,\bar\psi,U] = a^4\sum_x \bar\psi(x)\Big[\frac{1}{2}\gamma_\mu(\nabla_\mu^\star+\nabla_\mu)+$$

$$+\Big(-r\frac{a}{2}\nabla_\mu^\star\nabla_\mu + M_{\rm cr}(r)\Big)e^{-i\omega\gamma_5\tau_3}+m_q\Big]\psi(x)$$

- $M_{\rm cr}(-r) = -M_{\rm cr}(r)$ follows through the change of variables

$$\mathcal{R}_5 : \begin{cases} \psi & \to \psi' = \gamma_5\psi \\ \bar\psi & \to \bar\psi' = -\bar\psi\gamma_5 \end{cases}, \qquad \mathcal{R}_5^2 = 1 \qquad (1)$$

- spurionic symmetry $\mathcal{R}_5^{\rm sp} \equiv \mathcal{R}_5\times(m_q\to -m_q)\times(r\to -r)$

$$\mathcal{R}_5^{\rm sp}: \quad \langle O\rangle\Big|_{(r,m_q)}^{(\omega)} = (-1)^{P_{\mathcal{R}_5}[O]}\langle O\rangle\Big|_{(-r,-m_q)}^{(\omega)} \qquad (2)$$

$P_{\mathcal{R}_5}[O] = \gamma_5$-chirality of O

- symmetry $\mathcal{R}_5\times\mathcal{D}_d$, i.e.

$$\langle O(x_1,\ldots,x_n)\rangle\Big|_{(r,m_q)}^{(\omega)} = (-1)^{(P_{\mathcal{R}_5}+d)[O]}\langle O(-x_1,\ldots,-x_n)\rangle\Big|_{(r,m_q)}^{(\omega)}$$

- Symanzik expansion of $\langle O\rangle\Big|_{(r,m_q)}^{(\omega)}$ (O multipl. renormalizable)

$$\langle O\rangle\Big|_{(r,m_q)}^{(\omega)} = [\zeta_O^O(\omega,r)+am_q\xi_O^O(\omega,r)]\langle O\rangle\Big|_{(m_q)}^{\rm cont}+$$

$$+a\,\sum_\ell(m_q)^{n_\ell}\eta_{O_\ell}^O(\omega,r)\langle O_\ell\rangle\Big|_{(m_q)}^{\rm cont}+{\rm O}(a^2) \qquad (3)$$