Constraint stabilization for Linear Complementarity time-stepping methods for Multi-Rigid-Body Dynamics with Contact and Friction

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Model Requirements and Notations

- MBD system : generalized positions q and velocities v. Dynamic parameters: mass $M(q)$ (positive definite), external force $k(t, q, v)$.

- Non interpenetration constraints: $\Phi^{(j)}(q) \geq 0$, $1 \leq j \leq n_{total}$ and compressive contact forces at a contact.

- Joint (bilateral) constraints: $\Theta^{(i)}(q) = 0$, $1 \leq i \leq m$.

- Frictional Constraints: Coulomb friction, for friction coefficients $\mu^{(j)}$.

- Dynamical Constraints: Newton laws, conservation of impulse at collision.

Normal velocity: v_n

Normal impulse: c_n

d=0

c_n ↑ n1

c_n ↓ n2
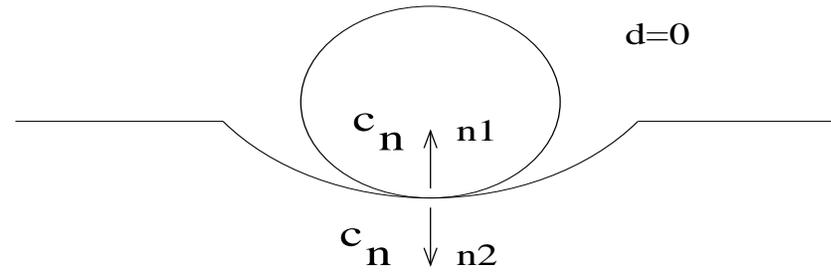
Contact Model

- Contact configuration described by the (generalized) distance function $d = \Phi(q)$, which is defined for some values of the interpenetration. Feasible set: $\Phi(q) \geq 0$.

- Contact forces are compressive, $c_n \geq 0$.

- Contact forces act only when the contact constraint is exactly satisfied, or

$$\Phi(q) \text{ is complementary to } c_n \text{ or } \Phi(q)c_n = 0, \text{ or } \Phi(q) \perp c_n.$$

Coulomb Friction Model

- Tangent space generators: $\widehat{D}(q) = \left[\widehat{d}_1(q), \widehat{d}_2(q) \right]$, tangent force multipliers: $\beta \in R^2$, tangent force $D(q)\beta$.

- Conic constraints: $||\beta|| \leq \mu c_n$, where μ is the friction coefficient.

- Max Dissipation Constraints: $\beta = \operatorname{argmin}_{||\widehat{\beta}|| \leq \mu c_n} v^T \widehat{D}(q)\widehat{\beta}$.

- v_T, the tangential velocity, satisfies $|v_T| = \lambda = -v^T \widehat{D}(q)\frac{\beta}{||\beta||}$. λ is the Lagrange multiplier of the conic constraint.

- Discretized Constraints: The set $\widehat{D}(q)\beta$ where $||\beta|| \leq \mu c_n$ is approximated by a polygonal convex subset: $D(q)\tilde{\beta}$, $\tilde{\beta} \geq 0$, $\left\|\tilde{\beta}\right\|_1 \leq \mu c_n$. Here $D(q) = [d_1(q), d_2(q), \ldots, d_m(q)]$.

For simplicity, we denote $\tilde{\beta}$ the vector of force multipliers by β.

Acceleration Formulation

$$M(q)\frac{d^2q}{dt^2} - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j=1}^{p} \left(n^{(j)}(q)c_n^{(j)} \quad + D^{(j)}(q)\beta^{(j)} \right) = k(t,q,\frac{dq}{dt})$$

$$\Theta^{(i)}(q) = 0, \qquad\qquad i = 1\ldots m$$

$$\Phi^{(j)}(q) \geq 0, \qquad \textbf{compl. to} \qquad c_n^{(j)} \geq 0, \quad j = 1\ldots p$$

$$\beta = \text{argmin}_{\widehat{\beta}(j)} v^T D(q)^{(j)}\widehat{\beta}^{(j)} \qquad \textbf{subject to} \qquad \left\|\widehat{\beta}^{(j)}\right\| \leq \mu^{(j)} c_n^{(j)}, \ j = 1\ldots p$$

Here $\nu^{(\mathbf{i})} = \nabla\mathbf{\Theta}^{(\mathbf{i})}$, $n^{(j)} = \nabla\Phi^{(j)}$.

It is known that these problems do not have a classical solution even in 2 dimensions, where the discretized cone coincides with the total cone.

Time-stepping scheme

Use Euler method, half-explicit in velocities, linearizing the geometrical constraints. Fundamental variables: velocities and impulses (h × force).

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j \in \mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)^T} v^{l+1} = 0, \qquad\qquad i = 1..m$$

$$\rho^{(j)} = n^{(j)^T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)^T} \beta^{(j)} \geq 0, \quad \textbf{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Here $\nu^{(\mathbf{i})} = \nabla \Theta^{(\mathbf{i})}$, $n^{(j)} = \nabla \Phi^{(j)}$. h is the time step. The set $\mathcal{A}$ consists of the active constraints. Stewart and Trinkle, 1996, MA and Potra,1997: The time-stepping scheme has a solution although the classical formulation doesn't!

Matrix Form of the Integration Step

$$
\begin{bmatrix}
M & -\tilde{\nu} & -\tilde{n} & -\tilde{D} & 0 \\
\tilde{\nu}^T & 0 & 0 & 0 & 0 \\
\tilde{n}^T & 0 & 0 & 0 & 0 \\
\tilde{D}^T & 0 & 0 & 0 & \tilde{E} \\
0 & 0 & \tilde{\mu} & -\tilde{E}^T & 0
\end{bmatrix}
\begin{bmatrix}
v^{(l+1)} \\
\tilde{c}_\nu \\
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}
+
\begin{bmatrix}
-Mv^{(l)} - hk \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}^T
\begin{bmatrix}
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
= 0,
\qquad
\begin{bmatrix}
\tilde{c}_n \\
\tilde{\beta} \\
\tilde{\lambda}
\end{bmatrix}
\geq 0,
\qquad
\begin{bmatrix}
\tilde{\rho} \\
\tilde{\sigma} \\
\tilde{\zeta}
\end{bmatrix}
\geq 0.
$$

Constraint Stabilization

The positions are updated by $q^{(l+1)} = q^{(l)} + h_l v^{(l+1)}$.

Due to the index reduction , the (geometrical) joint and non interpenetration constraints, which define the feasible set

$$\mathcal{F} = \left\{ q \,\middle|\, \Theta^{(i)}(q) = 0, \; 1 \leq i \leq m, \Phi^{(j)}(q) \geq 0, \; 1 \leq j \leq n_{total} \right\}$$

are replaced by constraints at the velocity level.

This may create constraint drift, in which the constraint infeasibility keeps growing. In interactive simulation this is particulatly annoying, since geometrical inconsistency is easy to notice.

Preventing constraint drift

- Change the approach to a nonlinear and potentially nonconvex optimization problem.

- Perform a nonlinear projection after each LCP (and eventually preserving the good energy properties (M)). If there is no friction the projection may be more costly than the LCP.

- Perform one step of an SQP applied to the nonlinear projection problem (Cline 2002). Extension of (Ascher, Chin, Reich 1994) from DAE. No analysis provided.

- Modify the right hand side of the LCP with an appropriate function of the infeasibility (this work, parameter-free, 02) and (Miller 02). This uses no additional subproblems, but may affect the energy balance.

Stabilized time-stepping scheme

$$M(v^{l+1} - v^{(l)}) - \sum_{i=1}^{m} \nu^{(i)} c_\nu^{(i)} - \sum_{j \in \mathcal{A}} (n^{(j)} c_n^{(j)} + D^{(j)} \beta^{(j)}) = hk$$

$$\nu^{(i)T} v^{l+1} + \frac{\Theta^{(i)}(q^{(l)})}{h_l} = 0, \qquad\qquad i = 1..m$$

$$\rho^{(j)} = n^{(j)T} v^{l+1} + \frac{\Phi^{(j)}(q^{(l)})}{h_l} \geq 0, \quad \textbf{compl. to} \quad c_n^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\sigma^{(j)} = \lambda^{(j)} e^{(j)} + D^{(j)T} v^{l+1} \geq 0, \quad \textbf{compl. to} \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}$$

$$\zeta^{(j)} = \mu^{(j)} c_n^{(j)} - e^{(j)T} \beta^{(j)} \geq 0, \quad \textbf{compl. to} \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A}.$$

Recall, $\nu^{(\mathbf{i})} = \nabla \mathbf{\Theta}^{(\mathbf{i})}$, $n^{(j)} = \nabla \Phi^{(j)}$, $q^{(l+1)} = q^{(l)} + hv^{(l+1)}$.

Note that, a large infeasibility may result in a large change in velocity!
We need to ensure that is not the case.

Assumption: Geometrical Regularity

We assume that the geometrical constraints satisfy the
Mangasarian-Fromovitz condition, or in dual form

$$\tilde{\nu}\tilde{c}_\nu + \tilde{n}\tilde{c}_n = 0, \quad \tilde{c}_n \geq 0 \Rightarrow \tilde{c}_\nu = 0, \ \tilde{c}_n = 0.$$

In the absence of an external force and friction, the internal forces are
nonzero.

In primal form, this is equivalent to $\tilde{\nu}$ has full rank and there exist an u
such that

$$\tilde{\nu}^T u = 0, \quad \tilde{n}^T u > 0.$$

This is equivalent to saying that the structure can be disassembled at zero
initial conditions with external action.

Note that this assumption is weaker than linear independence of the
constraints! which in many cases does not hold in three dimensions

Assumption: pointed friction cone assumption

Friction cone: the convex subset of all possible constraint reaction impulses (for the appropriate external force).

$$FC(q) \quad = \quad \left\{ t = \tilde{\nu} c_\nu + \tilde{n} c_n + \tilde{D}\tilde{\beta} \,\middle|\, c_n \geq 0,\ \tilde{\beta} \geq 0, \right.$$
$$\left. \left\|\beta^{(j)}\right\|_1 \leq \mu^{(j)} c_n^{(j)},\ \forall j \in \mathcal{A} \right\}$$

- We say that the friction cone $FC(q)$ is pointed if it does not contain any proper linear subspace.

- This assumption is essential in ensuring convergence to a differential inclusions as $h_l \to 0$.

- This assumption implies the geometrical regularity assumption

Main Result: assumptions

- Over the course of the simulation the friction cone is uniformly pointed.

- The geometrical data of the problem, $\Phi^{(j)}(q)$ $1 \le j \le n_{total}$, the signed distance functions and $\Theta^{(i)}(q)$, $i = 1, 2, \ldots, m$ are twice continously differentiable in a neighborhood of the feasible set F. This holds if all bodies are smooth-shaped convex except bodies which are infinite and flat (walls) (M,96)

- The ratio between succesive time steps is uniformly lower bounded
$$\frac{h_l}{h_{l-1}} \ge \frac{1}{c_h}$$

- The mass matrix is constant (Newton-Euler formulation in body coordinates). This can be relaxed.

- There is no initial infeasibility and the coefficient of restitution is 0.

- Definition: The active set $\mathcal{A} = \left\{ j \mid 1 \le j \le n, \quad \Phi^{(j)}(q^{(l)}) \le \hat{\epsilon} \right\}$.

Main result, continued

Then for any fixed time interval T, there exists an $H > 0$, and a $V > 0$ and $C > 0$ such that if $h_l < H$, $\forall l$, we have that
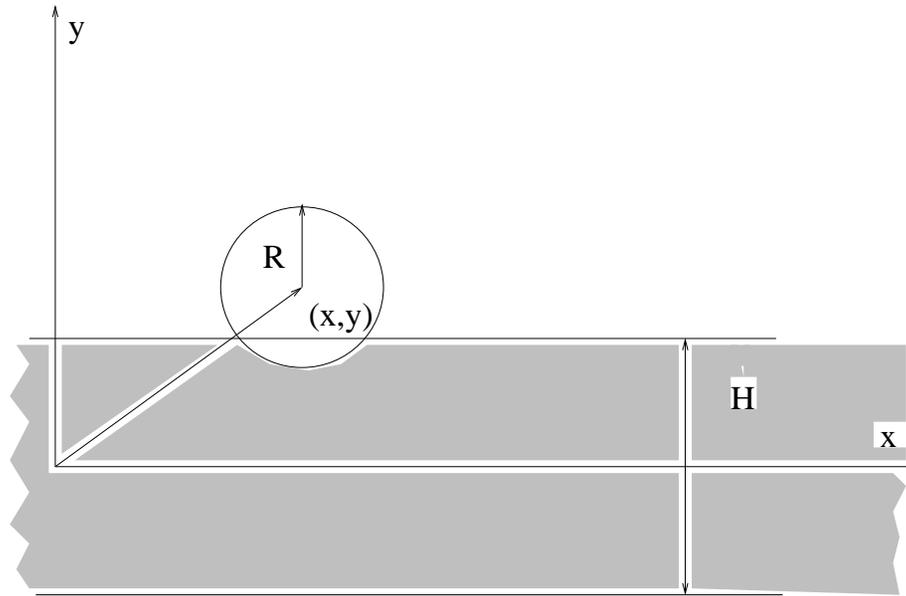
1. $v^{(l)} \leq V$, $\forall l$.

2. If $j \notin \mathcal{A}$ then $\Phi^{(j)}(q^{(l+1)}) \geq 0$.

3. The infeasibility, defined as

$$I(q) = \max_{1 \leq i \leq m, \ 1 \leq j \leq n_{total}} \left\{ 0, \left| \Theta^{(i)}(q) \right|, \Phi^{(j)}(q) \right\}$$

   satisfies $I(q^{(l+1)}) \leq C \left\| h_l v^{(l+1)} \right\|^2$, $\forall l$.

The last conclusion demonstrates constraint stabilization. The main tool in the proof is the stability of the solution of a strictly convex quadratic program with respect to its right hand side.

## Example of non differentiability of the signed distance

- Signed distance: $d_{12}(q) = |y| - R - \frac{H}{2}$ is not differentiable everywhere!.

- It is, however, differentiable over the set $d_{12}(q) \geq -\epsilon$ for any $\epsilon < R + \frac{H}{2}$. . We are OK if the infeasibility is not too large.

Elliptic body simulation

Ellipse Simulation
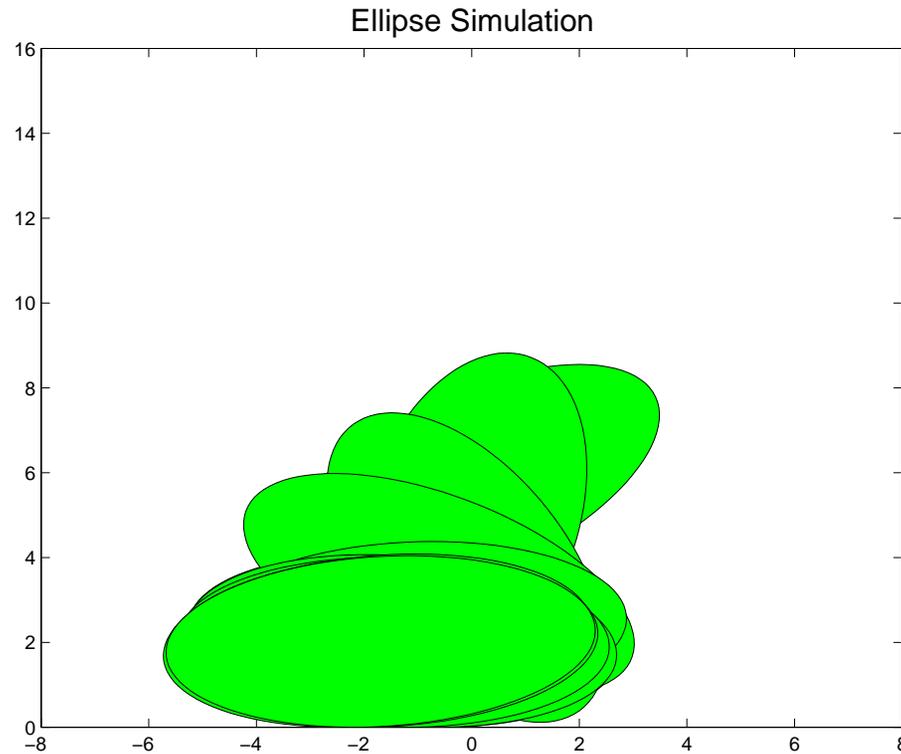
We present ten frames of the simulation of an elliptic body that is dropped on the table. There is an initial angular velocity of 3, the body has axes 4 and 8 and is dropped from a height of 8.

Infeasibility behavior unstabilized versus stabilized metod

Constraint infeasibility unstabilized

Constraint infeasibility stabilized
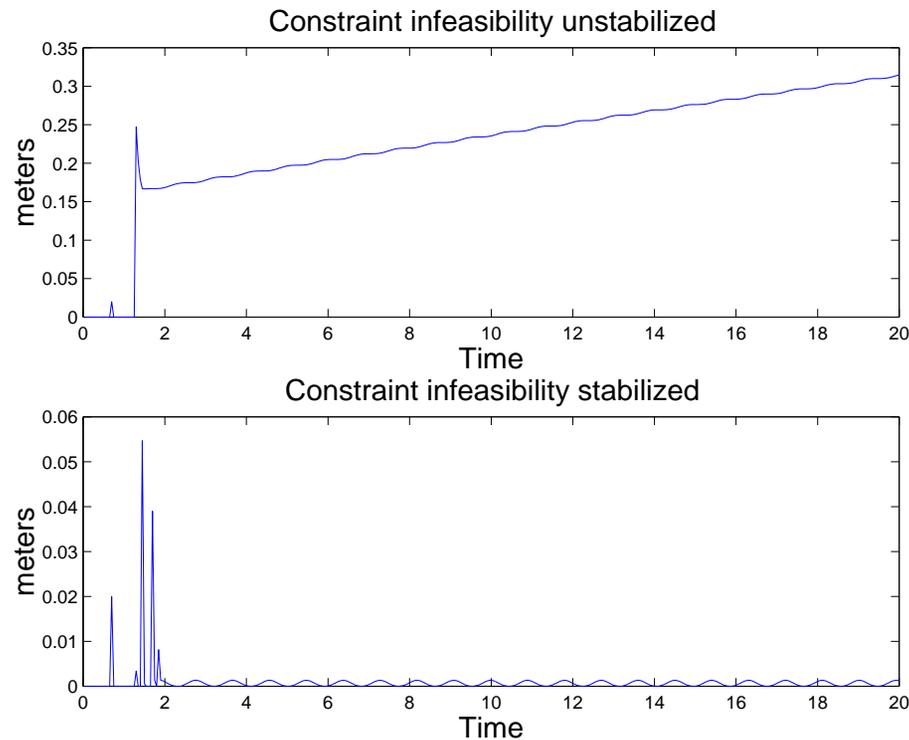
We see that drift becomes catastrophic for the unstabilized method, whereas remains in a narrow range for the stabilized method.

Constraint stabilization is accomplished!

Infeasibility comparison for 21 body simulation

21 disks simulation, unstabilized constraint infeasibility

21 disks simulation, stabilized constraint infeasibility
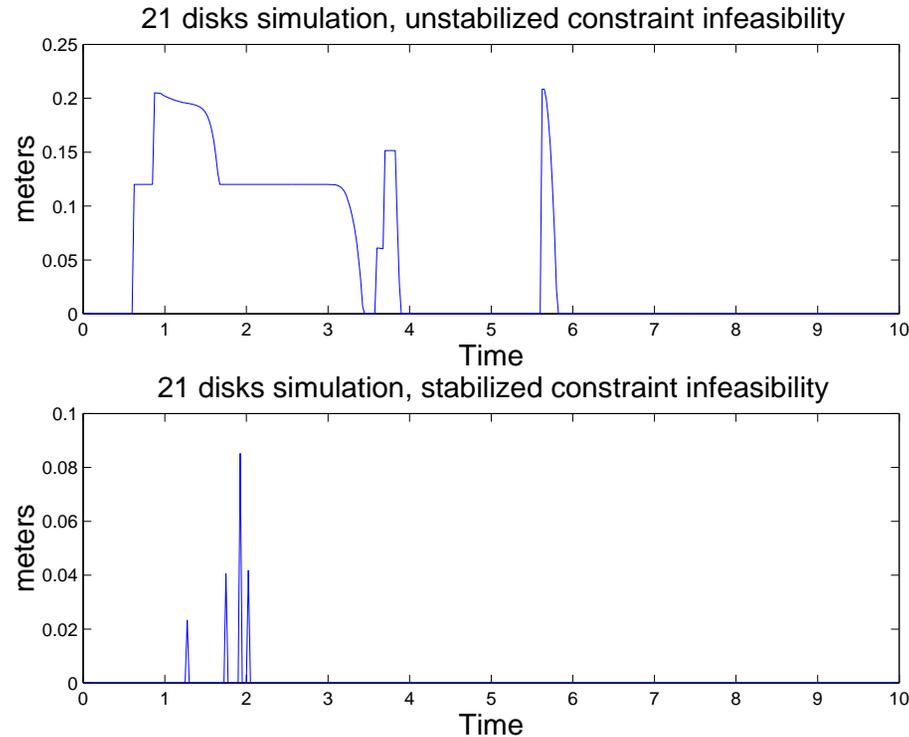
21 disks of radius 3 on a plank starting from the cannoball arrangement at rest. Stabilized method still has a much lower infeasibility. The time-step is 50ms, and the LCP is solved in at most 30 ms at every step.

Conclusions and future work

- We define a method that achieves constraint stabilization while solving only linear complementarity problem per step.

- Our method does not need to stop and detect collisions explicitly and can advance with a constant time step and predictable amount of effort per step. Good property for real-time applications.

- Our method is parameter-free in so far as constraint stabilization is concerned. Very good property for real time computations.

- In future work we plan to extend these results to include partially elastic contact, nondifferentiably-shaped bodies, and parameter dependent approaches, such as the one in (Miller and Christiansen) that uses: $\widehat{\Delta} = \gamma\Delta$, and $\widehat{\Upsilon} = \gamma\Upsilon$ for some $0 < \gamma < 1$, but for which no good guidelines for choosing γ exist as of yet.