Proof of Proposition 3. Assume that $T_j^{NB}(q_j)$ is differentiable almost everywhere. For convenience, we repeat conditions (21) and (22) in the main body of the article:

$$v_1(q_1) = \max_{\mathbf{q_{-1,2}}} R(q_1, 0, \mathbf{q_{-1,2}}) - F_1 - \sum_{j \neq 1,2} T^{NB}(q_j), \tag{1}$$

$$v_2(q_2) = \max_{\mathbf{q_{-1,2}}} R(0, q_2, \mathbf{q_{-1,2}}) - F_2 - \sum_{j \neq 1,2} T^{NB}(q_j). \tag{2}$$

Assume that the solutions to the maximization problems in (1) and (2) are unique.

The strategy of the proof is to show that even if the non-merging firms' contracts are not differentiable, the functions $v_i(\cdot)$, $i \in \{1,2\}$, are differentiable at the equilibrium quantities and, in particular, condition (25) in the text holds. The analysis presented in the text then establishes Proposition 3.

Let $\mathbf{q^1_{-1,2}}(q_1)$ solve the maximization problem in (1) and $\mathbf{q^2_{-1,2}}(q_2)$ solve the maximization problem in (2). Define

$$r_j^1(q_1, \mathbf{q_{-1,j}}) \equiv \arg\max_{q_j} R(q_1, 0, q_j, \mathbf{q_{-1,2,j}}) - F_1 - \sum_{j \neq 1,2} T^{NB}(q_j),\ j \in \{3, 4, ..., N\}. \tag{3}$$

The solution to the maximization problem in (1) can be characterized as the simultaneous solution to the $N-2$ "sub-maximization" problems in (3). Define $r_j^2(q_2, \mathbf{q_{-2,j}})$, $j \neq 1, 2$, symmetrically as the quantities that solve the analogous sub-maximization problems that correspond to (2).

Step 1. If $T_j^{NB}(q_j^{NB})$ is discontinuous in either direction from q_j^{NB}, it must jump upward; otherwise the retailer could increase its profits by choosing a different quantity.

Step 2. Suppose $T_j^{NB}(q_j^{NB})$ jumps up to both the right and the left of q_j^{NB}. Consider an arbitrarily small change in q_1^{NB} to $q_1^{NB} + x$. This will have an arbitrarily small effect on the marginal revenue of product j, so the solution to firm j's maximization problem in (3) will not change. That is, $r_j^1(q_1^{NB}, \mathbf{q_{-1,j}}) = r_j^1(q_1^{NB} + x, \mathbf{q_{-1,j}})$ provided that x is small. Similarly, $r_j^2(q_1^{NB}, \mathbf{q_{-2,j}}) = r_j^2(q_1^{NB} + x, \mathbf{q_{-2,j}})$ for small x.

Step 3. Suppose $T_j^{NB}(q_j)$ jumps up to the right of q_j^{NB}, but is continuous to the left. At the solution to (3), it must be true that

$$\left[\frac{\partial R(q_1^{NB}, 0, r_j^1, \mathbf{q^{NB}_{-1,2,j}})}{\partial q_j} - \frac{\partial T_j^{NB}(r_j^1)}{\partial q_j}\right]_- \geq 0 \tag{4}$$

where the notation $[\ ]_-$ indicates the left-hand derivative. Suppose the inequality in (4) is strict. Consider an arbitrarily small change in q_1 to $q_1^{NB} + x$. Since the marginal revenue function is continuous, the inequality in (4) will still hold at $r_j^1(q_1^{NB} + x, \mathbf{q^{NB}_{-1,j}})$. Therefore, $r_j^1(q_1^{NB}, \mathbf{q^{NB}_{-1,j}}) = r_j^1(q_1^{NB} + x, \mathbf{q^{NB}_{-1,j}})$. Suppose that (4) holds with equality. This means that the first-order condition holds for movements of q_j in the leftward direction. Movements in the rightward direction will not occur given small changes in marginal revenue because T_j^{NB} jumps upward in that direction. Analogous conditions hold for $r_j^2(q_1^{NB} + x, \mathbf{q^{NB}_{-2,j}})$.

Step 4. Suppose $T_j^{NB}(q_j)$ jumps up to the left of q_j^{NB}, but is continuous to the right. At the solution to (1), it must be true that

$$\left[\frac{\partial R(q_1^{NB}, 0, r_j^1, \mathbf{q^{NB}_{-1,2,j}})}{\partial q_j} - \frac{\partial T_j^{NB}(r_j^1)}{\partial q_j^1}\right]_+ \leq 0 \tag{5}$$

where $[\ ]_+$ denotes the right hand derivative. Suppose the inequality in (5) is strict. By the same argument as in the preceding paragraph, a small change in q_1 to $q_1^{NB} + x$ will leave r_j^1 unchanged, i.e., $r_j^1(q_1^{NB}, \mathbf{q^{NB}_{-1,j}}) = r_j^1(q_1^{NB} + x, \mathbf{q^{NB}_{-1,j}})$. Suppose that (5) holds with equality. This means that the first-order condition holds for movements of q_j in the rightward direction. Movements in the leftward direction will not occur given small changes in marginal revenue because T_j^{NB} jumps upward in that direction. Analogous conditions hold for $r_j^2(q_1^{NB} + x, \mathbf{q^{NB}_{-2,j}})$.

Step 5. Steps 1-4 establish how the solution to each product's sub-maximization problem changes in response to small changes in q_1 starting at the equilibrium quantity q_1^{NB}. In particular, product j's quantity either does not change or it changes to satisfy its first order condition. We now establish that this is true for the solutions to the maximization problems in (1) and (2).

The solution to the problem in (1) is given by the simultaneous solution to the $N-2$ sub-maximization problems in (3). For a given change in q_1 to $q_1^{NB}+x$, let $\mathcal{S}$ be the subset of products for which the solution to the product's sub-maximization problem changes according to its first-order condition. By the implicit-function theorem, the simultaneous solution to the sub-maximization problems for products in $\mathcal{S}$ (holding constant the quantities of products not in $\mathcal{S}$) are continuous functions of q_1 on the interval $(q_1^{NB}, q_1^{NB}+x)$. This means that a small change x results in a small change in these quantities, and hence a small change in the marginal revenues of the other products whose sub-maximization solutions do not change in response to changes in q_1. Since the change in marginal revenue from all the adjustments for products in $\mathcal{S}$ is small, the quantities of the products not in $\mathcal{S}$ will not change in response to a small change in q_1 and the associated adjustments in quantities for products in $\mathcal{S}$. Therefore, in the solution to (1), the quantity q_j either does not respond to a small change in q_1, or it responds according to its first-order condition.

Now differentiate (23) in the text, and recognize that $\frac{\partial v_1(q_1^{NB})}{\partial q_1} = \frac{\partial R(q_1^{NB},0,\tilde{\mathbf{q}}_{\mathbf{-1,2}}(q_1^{NB}))}{\partial q_1}$ regardless of whether $T_j^{NB}(q_j)$ is smooth or continuous at q_j^{NB}. Proposition 3 follows as explained in the text. *Q.E.D.*