Low Dimensional Dynamics in a Pulsating Star *

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Abstract

We report the discovery of a low dimensional dynamical system in a 5.5 hr Hubble Space Telescope High Speed Photometer observation of a rapidly oscillating star. The topological description of the phase space orbits is given, as well as a dynamical model which describes the results. This model should motivate theorists of stellar pulsations to search for a three-dimensional system with the same topological structure to describe the mechanisms for pulsation. The equations are compatible with recently proposed nonlinear mode interaction models.

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The rapidly oscillating peculiar A stars (hereafter *roAp* stars) were identified as a class of pulsating single stars in 1978 by Kurtz (cf, [1]). They exhibit nearly regular fluctuations, on the order of milli-magnitudes, on time scales of 3 to 25 minutes, believed to be low order, high overtone, non-radial p-mode oscillations. The oscillating material is localized near the magnetic poles; the strength of the oscillations is modulated by the rotation of the star since the rotation and magnetic axes of the body are typically not aligned. Many *roAp* stars have a rich spectrum of modes of oscillation, and in these cases there are commonly additional modulations present which are not so well understood, but are believed to be related to the non-linear interaction of the active modes [2], [3],[4].

The bulk of research involving these stars concerns the determination of stellar parameters [5]. It is possible to determine the angle between the rotation and magnetic axes, as well as the angle between the rotation axis and the observer, through measurement of the frequency splitting present in very long, often discontinuous, data sets. This analysis involves generating a single power spectrum from a large time series. The *evolution* of the frequencies present has not gained as much attention, but is at least as interesting, because it can give insight into the physical mechanism(s) responsible for the oscillations themselves.

The High Speed Photometer (HSP) was a first generation instrument aboard the Hubble Space Telescope, uniquely capable of making high time-resolution observations (up to 10,000 Hz) in the ultraviolet-to-visible regions of the spectrum. Due to its location in space, it was also the only photometer free from the effects of atmospheric scintillation.

The HSP observed the *roAp* star HD 60435 as a calibration target for 5.5 hours 1991 August 22, since the rapid fluctuations are only a small fraction of its otherwise constant magnitude. The data were taken with 82.4 millisecond time resolution through a broad-band filter with peak transmission at 2400 Angstroms and full width at half-maximum of 550 Angstroms. Preliminary analysis showed modulation of the main frequency present (11.6 min) on the time scale of the observation, which was identified as consistent with the beating of 5 frequencies [6]. Closer analysis revealed that this frequency underwent an abrupt switch from on to off, inconsistent with beating and consistent with nonlinear behavior [7].

Effects due to the telescope itself were eliminated according to the method described elsewhere [6]. Since we are interested in only the behavior of the relatively long 11.6 minute period, we eliminated high frequency noise by the following method. The data were rebinned to 30 second bins, and a spline interpolation was used to resample back to the original number of points. This faithfully represents the longer period oscillations (a few minutes or longer) while ignoring random point-to-point fluctuations. This data is shown if Fig. 1. Even to the eye, there is a marked difference in the behavior of the first 2/3 of this data set compared to the last 1/3. In Fig. 2 we display a contour plot of the Gabor transform of this data. This windowed Fourier analysis method clearly shows the power in the 11.6 min frequency undergo a sudden decrease approximately 200 min into the observation. Also evident is a weaker structure with slowly increasing frequency (near 80 cycles early in the observation, increasing toward 120 cycles at later times).

The basic equations describing the dynamics of a pulsating star have been discussed thoroughly in the literature. These account for the conservation of energy, momentum, heat gain and losses, as well as for the equations of state that relate pressure to density and temperature. Altogether, these equations describe the competition between radiation and gravity that is ultimately responsible for the pulsating dynamics. Even so, the lack of a qualitative theory of partial differential equations, combined with the difficulty of measuring the appropriate parameters, makes it difficult to treat this problem from first principles. From a theoretical standpoint, simplified models have been proposed which assume either linearizations around equilibrium states or weakly nonlinear interactions between active modes. In the latter case, a reduction of the dynamics (from partial differential equations to a finite set of ordinary differential equations describing the behavior of the mode amplitudes) makes the problem tractable. In support of these simplifying assumptions, it has been reported recently that the dynamics of pulsating star R Scuti is actually a low dimensional one [8].

Whenever a complicated invariant set coexists with a set of unstable periodic orbits, the trajectory corresponding to an initial condition within invariant set will reflect their influence in the existence of $p-$close return segments [9]. These are segments of the scalar time series $x(i), i = 1, ..., n$ such that $x(i+k) \sim x(i+p+k)$, for some sequence of values $k = 0, 1, 2, ...$ The relationship between the existence of unstable periodic orbits buried within the invariant set and the close returns is the following: whenever a state point is near an unstable periodic orbit, it can evolve along the stable manifold of the orbit until it gets expelled along the unstable manifold. If the period of the unstable periodic orbit is reasonably small, and the orbit is not highly unstable, the evolution in the neighborhood of the orbit can be long enough to remain in an epsilon neighborhood of the starting point for at least one cycle of the periodic orbit. Selecting pieces of the scalar data which are good close returns (almost close themselves and continue behaving similarly for a while), results in a collection of approximations of the unstable orbits coexisting with the complex trajectory under study. Periodic orbits can be embedded in three dimensions with the use of techniques which aim at reconstructing a phase space for the orbits. Among these, the time delay technique is the most widely used. This consists of creating a ariate environment from $x(t) \rightarrow (x(t), x(t-\tau), x(t-2\tau))$. Some close returns found in the HSP observation of HD 60435, after applying the above embedding, are displayed in Figs. 3a, 4a and 5a.

We propose that the following low dimensional dynamical system is capable of reproducing the unstable periodic orbits in the observational time series:

$$
\begin{aligned}
X' &= Y \\
Y' &= \mu(1+\epsilon cos(\phi)) - Y - X^2 + XY \\
\phi' &= \omega,
\end{aligned}
\tag{1}
$$

where X and Y are dynamical variables, ϵ and μ are control parameters and ω is a constant. In Figs. 3b, 4b and 5b we display different (embedded) periodic orbits that are attracting solutions of equations 1, for parameter values listed in the figure captions. Notice the striking similarity between the observational orbits and the theoretical ones.

To understand why equations 1 reproduce the observed orbits, we describe the dynamics of the equations in the absence of forcing ($\epsilon = 0$). This set of two nonlinear coupled equations constitutes the unfolding of a Takens-Bogdanov bifurcation [10]. For negative values of μ, there are no fixed points. For $0 < \mu < \mu_c$, a pair of fixed points arise: a saddle (at $x_1 < 0, y_1 = 0$) and a sink (at $x_2 > 0, y_2 = 0$). As μ is increased, the x coordinate distance between the two fixed point increases; at $\mu_c = (\frac{10-\sqrt{(96)}}{2})^2$, the fixed point (x_2, y_2) becomes an attracting focus. At $\mu = 1$, this fixed point undergoes another qualitative change, and a periodic orbit bifurcates from the fixed point. If the value of μ is further increased, the distance between the two fixed points continues to increase and the periodic orbit gets closer to the fixed point at (x_1, y_1). As the period of this periodic orbit increases, the time spent close to this fixed point is a large fraction of the period. This phenomenon is known as critical slowing down.

The addition of a forcing term changes the dynamics qualitatively. As the forcing is in the parameter that controls the distance between the two fixed points, the dynamics manifests itself as oscillations around a point that is itself positioned at an oscillating coordinate. Moreover, when e trajectory gets close to the saddle fixed point, the evolution is slower than when the trajectory is far away from it, explaining the uneven distribution of "curls" in the orbits. A rich variety of dynamical structures can be found in the equations, from periodic orbits to chaotic trajectories.

There is a quantitative way to compare these orbits, by means of topology [11]. These one dimensional curves can be embedded in three dimensions, and can therefore be characterized by the way in which they are knotted. It is possible to associate with each class of knots (a class of knots defined as the closed curves that can be deformed into each other) a *knot invariant*. This invariant need not characterize the class uniquely, but if the invariants associated with two knots are not equal, then the knots cannot be deformed one into the other (in knot theoretical language, they are not isotopic). A regular isotopy invariant is the *HOMFLY* polynomial (the term regular relates to the kind of moves allowed in the process of deforming one knot into the other).

To build this invariant for a given knot, it is necessary first to obtain a *diagram* for the knot, *i.e.* a two dimensional projection of the knot in which small segments of the projected curve are deleted in order to keep information on the overcrossings taking place in the real 3-dimensional space (see Figs. 3c, 4c and 5c). Then, one iteratively changes crossings and unwinds curls until one gets a trivial knot, keeping track of all moves by the construction of a polynomial on two variables (α and z), one for crossings and the other for curls, following prescribed rules [11].

If K_1, K_2 and K_3 denote the knots displayed in Figs. 3, 4 and 5 respectively, then the polynomials associated with the orbits are

$$H_1(\alpha, z) = 1 \tag{2}$$
$$H_2(\alpha, z) = \alpha^{-1} \tag{3}$$
$$H_3(\alpha, z) = \alpha^{-3}. \tag{4}$$

More important than this particular expression is the fact that *the computation of the invariants gives the same results for the orbits reconstructed from the observational time series and for the theoretical orbits.*

In order to reproduce, in the order of their appearance, the close returns uncovered in the data equations 1 must be integrated with slightly increasing frequencies, as indicated in the captions of Figs. 3-5. This is compatible with the structure that shows steadily increasing frequency displayed in the Gabor transform plot (Fig. 2). The last $\frac{1}{3}$ of the observational data contains almost no dynamics near the $\sim$11 minute period, but does present close returns near the $\sim$ three minute period. In equations 1, this dynamical behavior is reproduced when the forcing is eliminated.

We have shown that a low dimensional dynamical system can account for the pulsations of a rapidly oscillating star. Moreover, we proposed a model in terms of the normal form of a co-dimension two bifurcation plus a forcing. Some nonlinear interactions between bifurcating modes of the basic equations of stellar dynamics have been discussed theoretically [12]. The equations describing those interactions are compatible with our results. This agreement indicates that, despite the large number of complex effects taking place, pulsating stars can display the nonlinear interaction of a very small number of spatially coherent modes.

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Figure Captions

Figure 1. The HST High Speed Photometer observation of roAp star HD 60435. Orbital effects have been eliminated and the data has been rebinned to 30 second sample time. High frequency components have been eliminated. For clarity, every tenth point is plotted.

Figure 2. Contour plot of the Gabor transform power of the entire observation. The 11.6 minute period is the fairly flat feature across the bottom of the plot. It's strength suddenly diminishes at about 200 minutes into the observation. A second feature, with steadily increasing frequency, can be seen in the upper part of the plot.

Figure 3. Embedded orbit with HOMFLY polynomial of 1. In a) the orbit extracted from the observational data at t=1070 sec is shown. The numerical integration of equations 1 yield the orbit shown in b), with $\mu = 0.028$. The schematic of the knot, with undercrossings indicated, is displayed in c).

Figure 4. Embedded orbit with HOMFLY polynomial of α^{-1}. In a) the observational result at t=2430 sec. Equations 1 yeild the orbit shown in b), with $\mu = 0.030$. The schematic is displayed in c).

Figure 5. Embedded orbit with HOMFLY polynomial of α^{-3}. In a) the observational result at t=13960 sec. Equations 1 yeild the orbit shown in b), with $\mu = 0.033$. The schematic is displayed in c).

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