A Non-Hydrostatic Model to Simulate Atmospheric Flows in the Presence of Orography

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Outline of Presentation

- Introduction
- Mathematical Formulation
- Numerical Algorithm
- Results
- Conclusion

Motivation of the Research

Review of Previous Models

	Advection only	Shallow water equations	Compressible or hydrostatic models
No AMR		Qualitative model: • wave equation	• time step restricted by sound/gravity waves • terrain following coordinates • hydrostatic BVP is not well-posed
AMR	•mass conservation equation only • no orography		Current research: • time step restricted by advection only • EB formulation • well-posed BVP for AMR

AMR = Adaptive Mesh Refinement

Time steps

Typical cell: Δx = 1.5km - Δz = 200m

Time step limitation	Wave speed, c	$\Delta t = \frac{\{\Delta x, \Delta z\}}{c}$
Vertical acoustic waves	343 m/s	~0.6 s
Horizontal gravity waves	200 m/s	~7.5s
Horizontal advection	20 m/s	~75 s

Research Objective

- Develop a well-posed boundary value problem for gravitationally stratified flows to use in an AMR framework

- Applications
  - Atmospheric modeling
  - Astrophysics

Algorithmic Requirements

- <u>Use advective time step</u>

=> Implicit treatment of acoustic and gravity waves

- <u>Adaptive Mesh Refinement (AMR)</u>

=> Well-posed boundary-value formulation
of the equations

- <u>Orography</u>

=> Cut-cell methods for irregular boundaries

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Euler Equations for a Compressible Fluid

Mass conservation

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{u}) = 0$$

Momentum

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}.grad(\vec{u}) + \frac{1}{\rho} grad(p) + g\vec{k} = \vec{0}$$

Pressure

$$\frac{\partial p}{\partial t} + \vec{u}.grad(p) + \rho c^2 div(\vec{u}) = 0$$

Separating Out the Acoustic Waves: Hodge Decomposition

$\boldsymbol{u}$	$=$	$\boldsymbol{u}_d$	$+$	$\boldsymbol{u}_p$
total velocity		incompressible vortical motions		compressible motions
		$\mathrm{div}(\boldsymbol{u}_d) = 0$		$\boldsymbol{u}_p = \mathrm{grad}(\varphi)$

Separating Out the Acoustic Waves: Projection Method

Incom-pressible flow and advective transport

$$\frac{\partial \tilde{\rho}}{\partial t} + div(\rho \vec{u}) = 0$$

$$\frac{\partial \vec{u}_d}{\partial t} + A_d \vec{u} + \frac{1}{\rho} grad(\pi) + P_o \left(\frac{1}{\rho} grad\delta + \frac{\tilde{\rho}}{\rho} g\vec{k} \right) = \vec{0}$$

Acoustic equations

$$\frac{\partial \vec{u}_p}{\partial t} + grad\left(\frac{| u_p + u_h |^2}{2} \right) + Q_o \left(\frac{1}{\rho} grad\delta + \frac{\tilde{\rho}}{\rho} g\vec{k} \right) = \vec{0}$$

$$\frac{\partial \delta}{\partial t} + \vec{u}.grad(\pi + \delta) + \rho c^2 div(\vec{u}_p) + w \frac{\partial p_o}{\partial t} \vec{k} + \frac{\partial \pi}{\partial t} = 0$$

Incompressible flow => semi-implicit formulation

Acoustic waves => implicit formulation

Separating Out the Acoustic Waves: Projection Method

- Incompressible equations
  - Poisson-like equation for the pressure
  - Explicit treatment of advection
- Acoustic equations
  - Backward-Euler
  - Implicit treatment => Helmholtz equation for the acoutic pressure

=> Well-posed boundary problems for AMR

Separating Out Fast Gravity Waves

- Isolate incompressible flow and gravity terms

$$\frac{\partial u_d}{\partial x} + \frac{\partial w_d}{\partial z} = 0$$
$$L_z \frac{\partial \pi_H}{\partial t} + \frac{\partial u_d}{\partial x} = f_\rho$$
$$\frac{\partial u_d}{\partial t} + \frac{1}{\rho_o}\frac{\partial \pi_H}{\partial x} = f_u$$
$$\frac{\partial \pi_H}{\partial z} = -\tilde{\rho} g$$

$\rho_o(z)$: background stratification

$L_z = \frac{\partial}{\partial z}\frac{1}{g\frac{d\rho_o}{dz}}\frac{\partial}{\partial z}$: second order self-adjoint operator

- This set of equations is equivalent to the set of incompressible equations in (u_d,ρ)

Separating Out Fast Gravity Waves

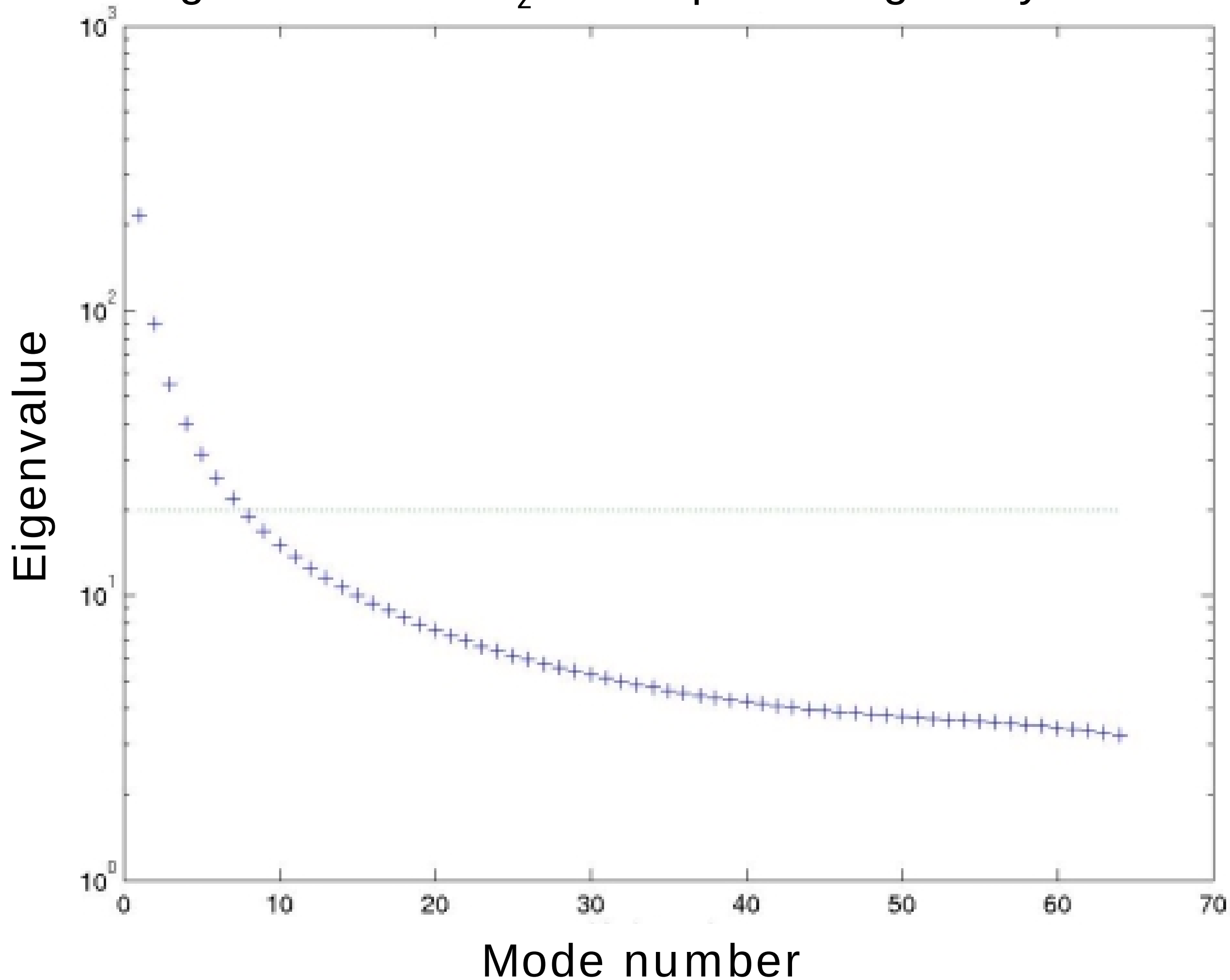
- Decomposition on eigenvectors

$$\begin{pmatrix} u_d \\ \pi_H \end{pmatrix} = \sum_k \begin{pmatrix} u_d^k(x,t) \\ \pi_H^k(x,t) \end{pmatrix} r^k(z)$$

- $\begin{pmatrix} u_d^k \\ \pi_H^k \end{pmatrix}$ satisfy the wave equation in x with wave speed λ_k= eigenvalues of $L_z^{-1/2}$

Separating Out Fast Gravity Waves

Eigenvalues of $L_z^{-1/2}$ = speed of gravity waves



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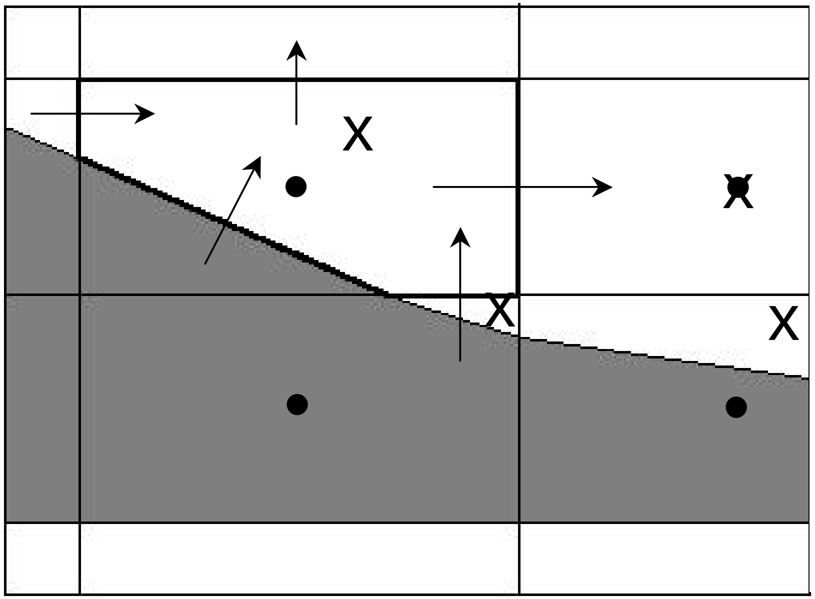
Overview of the Algorithm

$$\frac{\partial \vec{V}}{\partial t} + \vec{A}\left(\vec{V}, \frac{\partial \vec{V}}{\partial x}, \frac{\partial \vec{V}}{\partial z}\right) = \vec{0}$$

- Implicit treatment of acoustic waves using splitting
- Semi-implicit (explicit for advection) for incompressible advection
- Splitting of fast horizontal gravity waves from dynamics
- Use embedded boundaries for orography

Cartesian Grid Embedded Boundary methods



- PDEs written in conservation form

$$\frac{\partial U}{\partial t} + \nabla . F(u) = 0 \qquad \nabla . \vec{F} \approx \frac{1}{V} \int_V \nabla . \vec{F} dV = \frac{1}{V} \oint_S \vec{F} . \vec{n} dS$$

- Away from boundaries: standard finite-difference discretization

Cartesian Grid Embedded Boundary methods

Advantages of underlying rectangular grid

- Grid generation is tractable (T. Deschamps' talk)
- Well-understood
- Straightforward coupling to structured AMR

Large aspect ratio (1/10) introduces new issues

- Line solver for multigrid method

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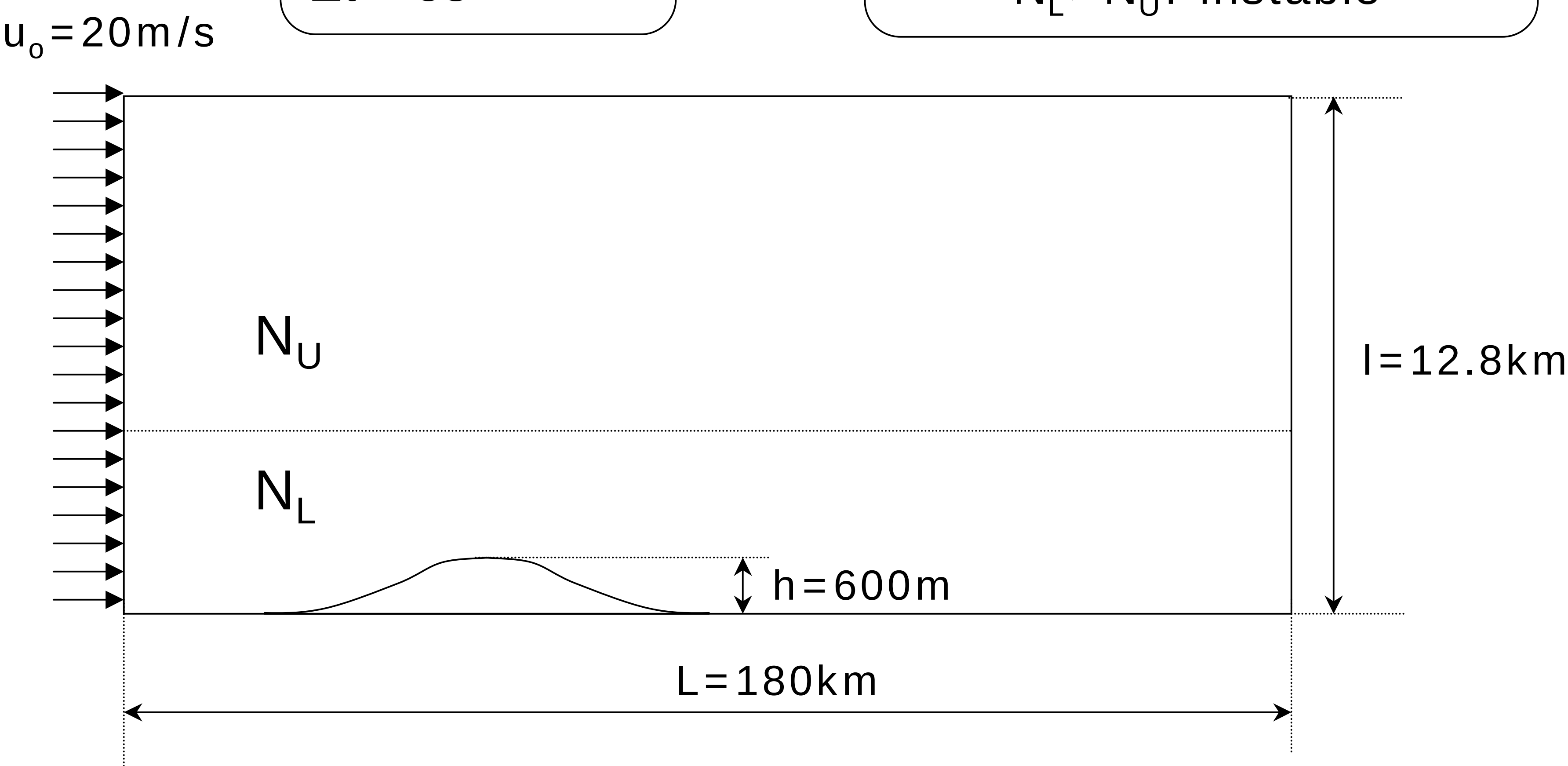
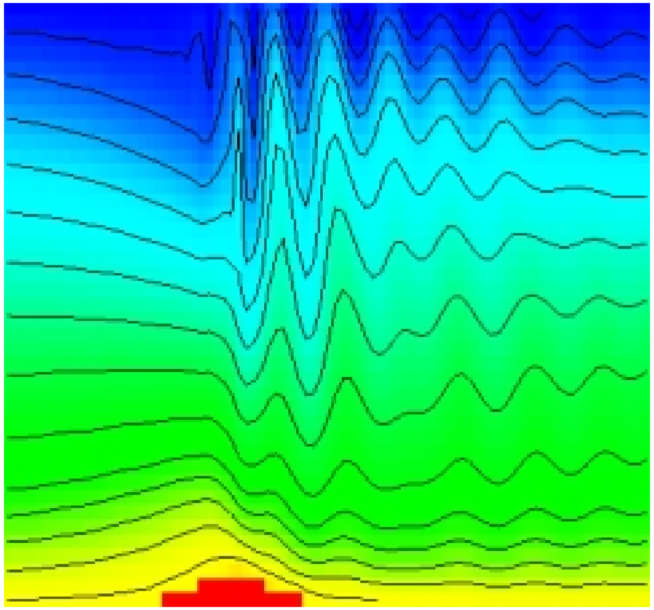
Results for a 2-Layered Atmosphere

$\Delta x = 1406.25m$
$\Delta z = 200m$
$\Delta t = 5s$

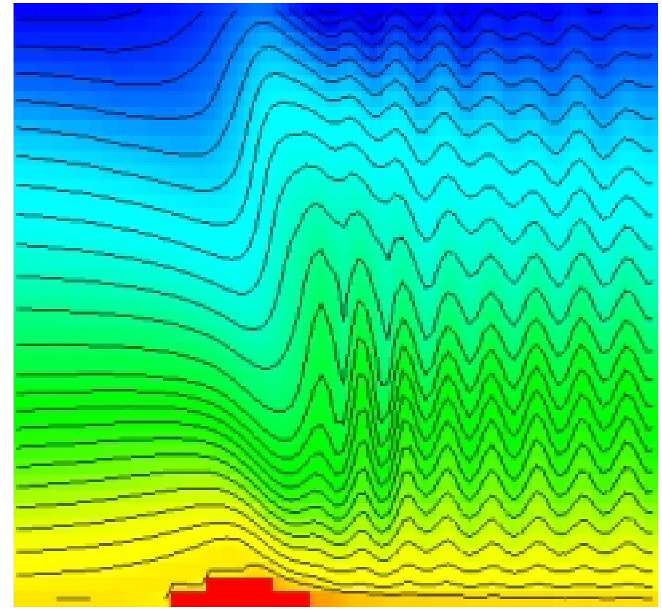
N: Brunt-Väisälä frequency

- $N_U > N_L$: stable
- $N_L > N_U$: instable



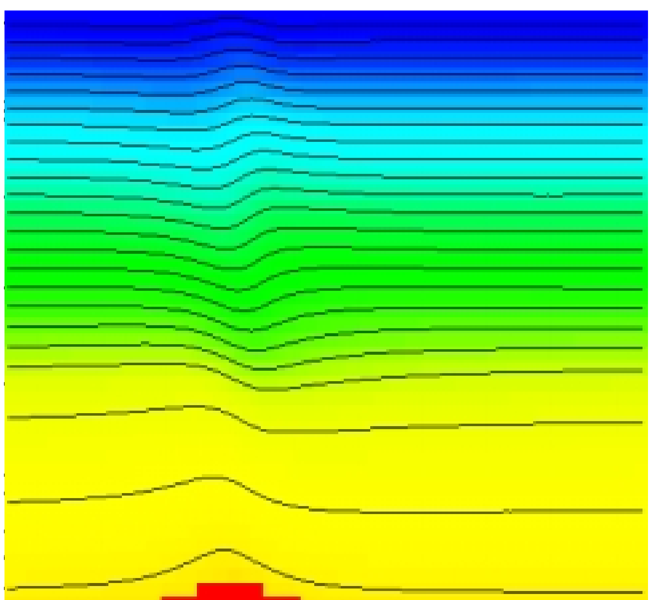


$N_L > N_U$

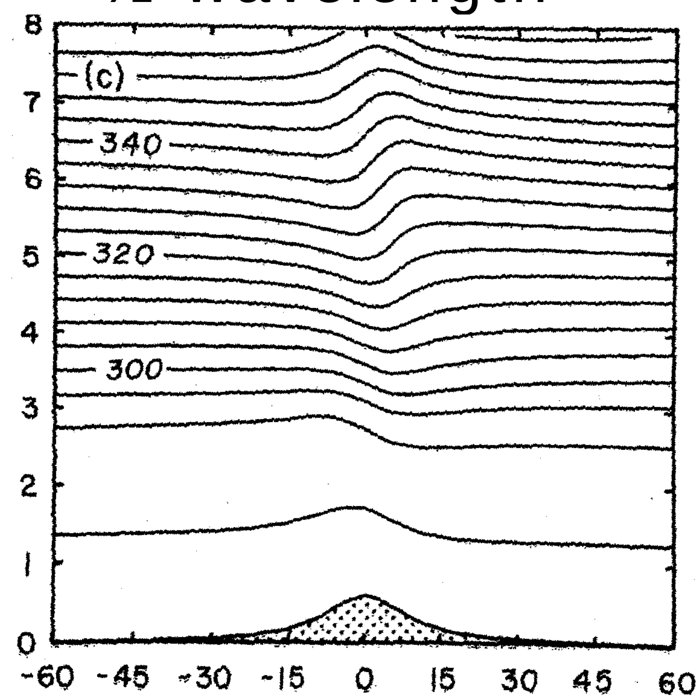


¼ wavelength

½ wavelength



$N_U > N_L$



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Status and Summary

- Formulation of well-posed initial-boundary problem
  - ✓ Stiff acoustic waves
  - ❑ Still working on implicit treatment of gravity waves
  - ❑ How does 2D implicit gravity waves couple with 3D AMR slow dynamics?

- EB treatment of orography
  - ✓Good results on mountain lee-waves

- AMR
  - ❑ Not yet, but substantial progress on well-posed boundary-value problem

Future Work

- Split the gravity waves
- Go to 3D
- Use an AMR framework
- Parallelize the code

First-Aid Kit

Separating Out the Acoustic Waves: Hodge Decomposition

0	0	0	0
u_o $\boxed{u}$ 0 =	0 $\boxed{u_d}$ 0 +	0 $\boxed{u_p}$ 0 +	u_o $\boxed{u_h}$ 0
0	0	0	0
	$div(u_d)=0$	$curl(u_p)=0$	$div(u_d)=0$ $curl(u_p)=0$
total velocity	incompressible	compressible	

Definitions of Projections

$$
\begin{aligned}
\vec{u}_d &= P_o(\vec{u}) \\
\vec{u}_p &= Q_o(\vec{u}) \\
Q_\rho &= \frac{1}{\rho}\, grad\left(L_\rho\right)^{-1} div \\
P_\rho &= I - Q_\rho \\
Q_o &= grad\left(L_o\right)^{-1} div \\
P_o &= I - Q_o \\
L_\rho &= div\left(\frac{1}{\rho}\, grad\right) \\
L_o &= div\left(grad\right)
\end{aligned}
$$

Definitions of Other Terms

$$A_d u = \vec{u}.grad(\vec{u}) - grad\left(\frac{|\, u_p + u_h \,|^2}{2}\right)$$

$$\frac{1}{\rho} grad(\pi) = -Q_\rho (A_d u)$$

$$\delta = p - p_o(z) - \pi$$

Separating Out Fast Gravity Waves

$$\frac{\partial \pi_H}{\partial z} = -\tilde{\rho} g$$

$$\boxed{\begin{aligned} L_z \frac{\partial \pi_H}{\partial t} + \frac{\partial u_d}{\partial x} &= f_\rho \\ \frac{\partial u_d}{\partial t} + \frac{1}{\rho_o} \frac{\partial \pi_H}{\partial x} &= f_u \end{aligned}}$$

Wave equation

Project on fast eigenmodes $\longrightarrow$

$$\lambda_k \frac{\partial \hat{\pi}_H^k}{\partial t} + \frac{\partial \hat{u}_d^k}{\partial x} = \hat{f}_\rho$$

$$\frac{\partial \hat{u}_d^k}{\partial t} + \frac{\partial \hat{\pi}_H^k}{\partial x} = \hat{f}_u$$

recompose $\downarrow$

$$\hat{\pi}_H^{Fast} = \sum_1^K \hat{\pi}_H^k r_k$$

$$\hat{u}_d^{Fast} = \sum_1^K \hat{u}_d^k r_k$$

Back to original variables $\longleftarrow$

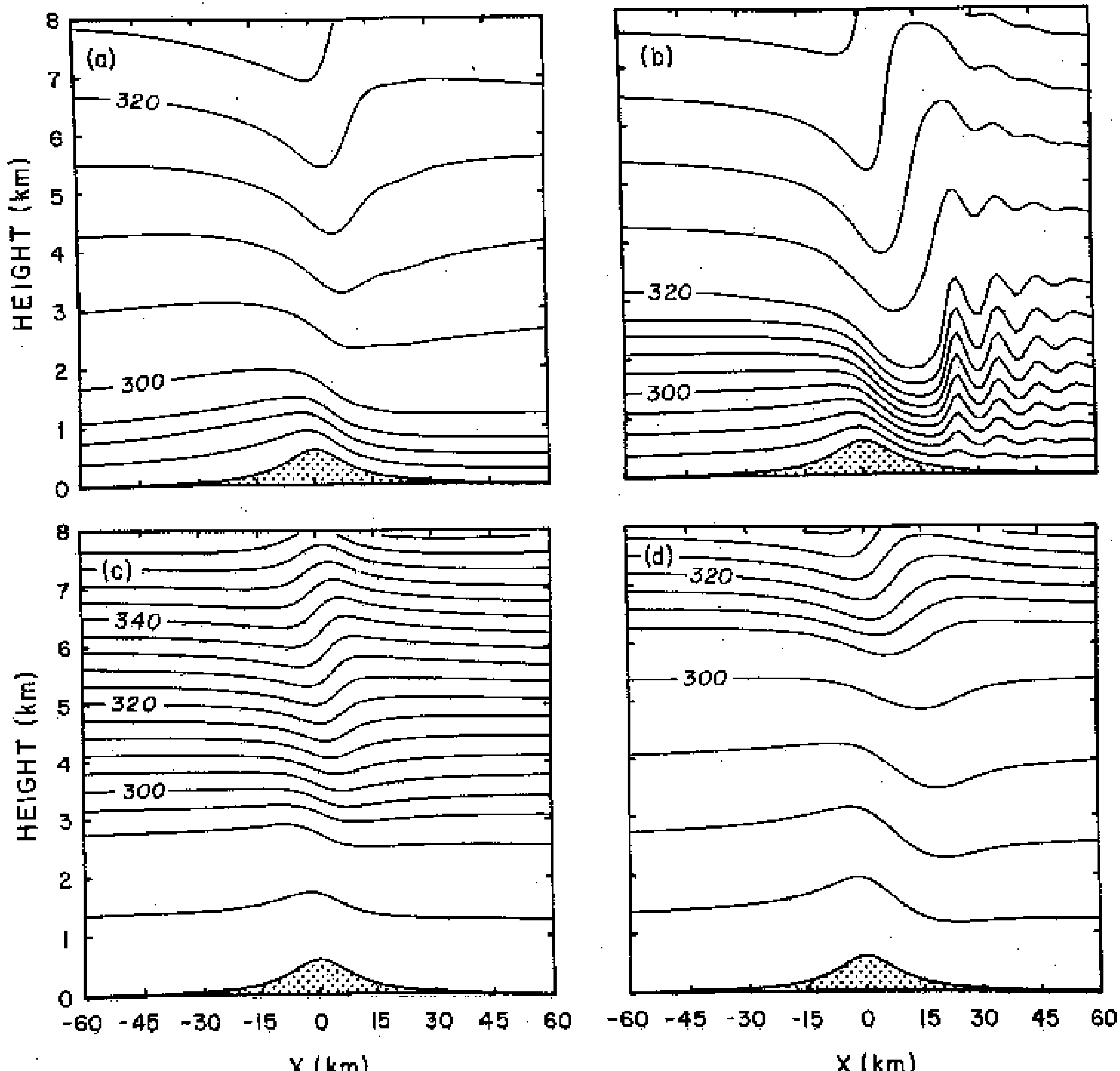
$$u_d^{Fast}, \pi_H^{Fast}$$

Outline of Algorithm

Recall: unknows are u_d, u_p, ρ, π, δ

Find the eigenvalues and eigenvectors of L_z

1. Advance $\boldsymbol{u}_\mathrm{d}$ to half time step and to face centers
2. Advance ρ to half time step and to face centers
3. Partially advance $\boldsymbol{u}_d$ using only the advective terms
4. Solve for π_H^{Fast}, u_d^{Fast}
5. Project out instable (fast) modes
6. Solve for auxiliary pressure π
7. Advance acoustic pressure δ implicitly
8. Advance curl-free velocity $\boldsymbol{u}_p$
9. Update ρ
10. Add missing terms in update for $\boldsymbol{u}_d$



(a)
(b)
(c)
(d)
320
300
340
HEIGHT (km)
X (km)