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# PERTURBATION-INDEPENDENT METHODS FOR CALCULATING RESEARCH REACTOR KINETIC PARAMETERS

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## TABLE OF CONTENTS

## LIST OF TABLES

## LIST OF FIGURES

**Page**

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## ABSTRACT

The analysis of research reactor transients depends on the effective delayed neutron fraction ($\beta_{eff}$), its family-dependent components ($\beta_{eff,i}$), the prompt neutron lifetime ($l_p$), and the decay constants ($\lambda_i$) for each delayed neutron family. Based on ENDF/B-V data, methods are presented for accurately calculating these kinetic parameters within the framework of diffusion theory but without the need for a perturbation code. For heavy water systems these methods can be extended to include the delayed photoneutron component of $\beta_{eff}$ which results from fission product gamma rays energetic enough to dissociate the deuteron. However, a separate calculation is needed to estimate the fractional loss of fission product gamma rays from leakage, energy degradation, and absorption in fuel and structural materials. These methods are illustrated for a light-water Oak Ridge Research Reactor (ORR) LEU core and for a heavy-water Georgia Tech Research Reactor (GTRR) HEU core where measured and calculated values of the prompt neutron decay constant ($\beta_{eff}/l_p$) are found to be in good agreement. In addition, the calculated $\beta_{eff,i}$ values compare favorably with results obtained from a 3D perturbation code.

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## 1. INTRODUCTION

At ANL, the VARI3D perturbation code[1] is commonly used to calculate delayed neutron fractions ($\beta_{eff,i}$) and the prompt neutron lifetime ($l_p$) from DIF3D[2] flux and adjoint distributions. However, VARI3D is not fully documented nor is it available to the RERTR community. Partly for these reasons, simple, but accurate, methods for calculating research reactor kinetic parameters without perturbation codes are discussed in this paper. The 1/v insertion method, described in the next section, calculates the prompt neutron lifetime more accurately than VARI3D. Two methods are presented for calculating effective delayed fission neutron fractions. The first method (Method I) determines $\beta_{eff,i}$ values from a set of diffusion theory calculations based on cross section sets in which fission yields and the fission spectrum are adjusted in a prescribed manner using ENDF/B-V data. The second method (Method II) is based on the age-diffusion model and is therefore somewhat less accurate than the first. Once the slowing-down ages of prompt and delayed neutrons in $H_2O$ and $D_2O$ have been determined, however, Method II provides a quick way of calculating $\beta_{eff,i}$ values without the need for a computer. Unlike VARI3D, both methods provide a framework for estimating the effective delayed photoneutron fraction in heavy water research reactors.

These techniques are used to calculate $l_p$ and $\beta_{eff,i}$ values for a light-water-moderated and -reflected Oak Ridge Research Reactor core[3] with fresh LEU fuel and for the heavy-water-moderated and -reflected Georgia Tech Research Reactor (GTRR) with 14 fresh HEU fuel elements[4]. For both reactors results for the prompt neutron decay constant $\alpha = \beta_{eff}/l_p$ compare favorably with the measured values. For the delayed fission neutrons, the $\beta_{eff,i}$'s are consistent with VARI3D calculations. These methods and calculations are discussed below. Some of these results were presented at the recent international RERTR meeting[5].

## 2. COMPUTATIONAL METHODS

### 2.1 Prompt Neutron Lifetime

The 1/v insertion method[6] is a simple but accurate way to calculate the prompt neutron lifetime. If the entire reactor (including the reflector) is perturbed by a dilute and uniform distribution of a purely 1/v neutron absorber, the fractional change in the eigenvalue is

$$\delta k / k_p = k_o \int_V \left[ \sum_j \phi_j \phi_j^* \delta\Sigma_a \right] dV / PD = N\sigma_{ao} v_o k_o \int_V \left[ \sum_j \phi_j \phi_j^* / v_j \right] dV / PD = N\sigma_{ao} v_o l_p'$$

where N is the concentration (atoms/b-cm) of the 1/v absorber whose absorption cross section is $\sigma_{ao}$ for neutrons of speed $v_o$. The eigenvalues for the perturbed and unperturbed configurations are $k_p$ and $k_o$, respectively. The prompt neutron lifetime is obtained from this equation in the limit as N approaches zero.

$$l_p = \lim_{N\to 0} l_p' = \lim_{N\to 0} \frac{\delta k}{k_p} / N\sigma_{ao} v_o \tag{1}$$

A good approximation for a purely 1/v absorber is $^{10}B$ for which $\sigma_{a0}$ = 3837 barns at $v_o$ = 2200 m/sec. To determine $l_p$ from this equation a multigroup diffusion theory code, such as DIF3D[2], is used to calculate $k_o$ and $k_p$ eigenvalues for very dilute concentrations of $^{10}B$ (in the range of $10^{-9}$ to $10^{-8}$ atoms/b cm) and tight eigenvalue convergence requirements. For this purpose infinitely dilute $^{10}B$ multigroup cross sections need to be generated for each reactor region (homogenized fuel, side plates, control rods and control rod channels, reflectors, etc.). The prompt neutron lifetime is obtained by linearly extrapolating the concentration-dependent $l_p'$ values to a zero $^{10}B$ concentration.

VARI3D calculations for the prompt neutron lifetime are not as accurate as those obtained from this 1/v insertion method because this perturbation code uses a single set of group-dependent neutron velocities and not region-dependent values. Numerical values for the prompt neutron lifetime are given in Section 5 for a light-water ORR LEU core configuration and for a heavy-water GTRR HEU core, and VARI3D results are compared with those obtained by the 1/v insertion method.

### 2.2 Effective Delayed Fission Neutron Fractions, ${}^{DFN}\beta_{eff,i}$

Since the neutron multiplication factor for a reactor is proportional to the total fission yield ($\nu$), the multiplication factor for prompt neutrons only is proportional to ($\nu$ - ${}^{DFN}E_{ave}\ \nu_d$) = $\nu$(1 - ${}^{DFN}\beta_{eff}$). Here $\nu_d$ is the total delayed fission neutron yield and ${}^{DFN}E_{ave}$ is the average effectiveness of these delayed neutrons. It follows from this that the total effective delayed neutron fraction is

$${}^{DFN}\beta_{eff} = (k - k_p)/k = \Sigma_i\ {}^{DNF}\beta_{eff,\,i} = \Sigma_i\ {}^{DFN}E_i\ \beta_i$$

where k is the eigenvalue for all neutrons, $k_p$ the eigenvalue for prompt neutrons only, and ${}^{DFN}E_i$ the effectiveness of the $i_{th}$ family of delayed fission neutrons (DFN). The summation extends over the N (normally 6) families of delayed fission neutrons. Since delayed fission neutrons have a softer energy spectrum than prompt fission neutrons, they have a lower probability of leakage from the core. Therefore, the effectiveness of the delayed fission neutrons is greater than unity (${}^{DFN}E_i > 1$). For the $i_{th}$ family of delayed fission neutrons

$${}^{DFN}\beta_{eff,i} = (k - k_{p,i})/k \qquad (2)$$

where $k_{p,i}$ is the multiplication factor without contributions from the$i_{th}$ delayed neutron family.

Two approximate methods for calculating $k_{p,i}$, and so ${}^{DFN}\beta_{eff,i}$, are discussed below. Method I is the most accurate because it does not depend on age-diffusion theory approximations used by Method II.

#### 2.2.1 Method I: Fission Yield and Fission Spectrum Adjustments

To calculate $k_{p,i}$ the effects of the $i_{th}$ family of delayed fission neutrons are removed from the multigroup cross section set. This is accomplished by adjusting fission yield and fission spectrum values[7]. For fissionable isotope m, delayed neutron family i, and energy group j, the adjusted fission yield is

$${}^{m}\nu_{p,i,j} = {}^{m}\nu_j - {}^{m}\nu_{d,i} \qquad (3)$$

and the adjusted fission spectrum becomes

$${}^{m}\chi_{p,j} = (\ {}^{m}\chi_j - {}^{m}f_{d,i,j}\ {}^{m}\nu_{d,i} / {}^{m}\nu_j\ )\ {}^{m}N \qquad (4)$$

where ${}^{m}N$ is a normalization factor chosesn so that $\Sigma_j\ {}^{m}\chi_{p,j} = 1.0$ and where ${}^{m}f_{d,i,j}$ is the fraction of the $i_{th}$ family of delayed fission neutrons emitted into energy group j. ENDF/B-V delayed fission neutron data for $^{235}U$ and $^{238}U$ needed for the fission yield and fission spectrum adjustments in Eqs. 3 and 4 are given in Section 3.1 of this report.

For this study multigroup cross sections were generated using the WIMS-D4M code[8]. Fission yield and fission spectrum adjustments (Eqs. 3 and 4) were made by converting the initial multigroup cross section set from binary to ascii, making the required ${}^{m}\nu_{p,i,j}$ and ${}^{m}\chi_{p,j}$ changes, and converting the altered cross section set back to binary. Note that for WIMS-D4M the unaltered fission spectrum is that for ${}^{235}U$ only. Eigenvalues needed to determine $\beta_{eff,i}$ were calculated using these modified cross section sets and the DIF3D[2] code.

### 2.2.2 Method II: Age Approximation

An alternate method for estimating $\beta_{eff,i}$ uses age-diffusion theory expressions for the fast and thermal non-leakage probabilities for a bare homogeneous reactor. Since the infinite multiplication factor is proportional to the number of neutrons emitted per fission event, it follows that

$$^{DFN}\beta_{eff,i} = k_{d,i}/k = [\nu_{d,i}\, P_{NL\text{-}th} \exp(-B^2\, \tau_i)] \,/\, [\nu\, P_{NL\text{-}th} \exp(-B^2\, \tau)] = \beta_i \exp[B^2(\tau - \tau_i)] \qquad (5)$$

Here $\tau$ and $\tau_i$ are the slowing down ages for all neutrons and for the $i_{th}$ family of delayed neutrons, respectively, and where $\beta_i = \nu_{d,i}/\nu$. $B^2$ is the geometric buckling for an equivalent bare homogeneous reactor. For a cylindrical reactor, for example,

$$B^2 = B_r^2 + B_z^2 = (2.405/R)^2 + (\pi/H)^2 \qquad (6)$$

where R and H are the extrapolated boundaries of the cylinder where the flux vanishes. If $R_0$ and $H_0$ are the physical dimensions of the cylindrical core,

$$R = R_0 + \delta_r \text{ and } H = H_0 + 2\delta_z$$

where $\delta_r$ and $\delta_z$ are the radial and axial extrapolation lengths. Since the axial distribution of the total neutron flux varies as $\cos(B_z z)$, a diffusion theory calculation of $\Phi_T(z)$ may be fit to a cosine distribution to determine $B_z$. Similarly, a fit of the $\Phi_T(r)$ distribution to a $J_0(B_r r)$ Bessel function determines $B_r$. These values allow a determination of the geometric buckling to be used in Eq. 5.

The effectiveness of the $i_{th}$ family of delayed fission neutrons is

$$^{DFN}E_i = {}^{DFN}\beta_{eff,i} \,/\, \beta_i = \exp[B^2(\tau - \tau_i)]. \qquad (7)$$

Since the average energy of fission spectrum neutrons is larger than that of delayed neutrons, $\tau > \tau_i$ and ${}^{DFN}E_i > 1$. However, ${}^{DFN}E_i$ reduces to unity if there is no neutron leakage from the core.

The evaluation of neutron ages in both $H_2O$ and $D_2O$ is discussed the Section 4. Heterogeneous fuel/moderator mixtures have ages larger than that of the pure moderator. Because of neutron moderation from inelastic scattering in uranium, however, the age difference $(\tau - \tau_i)$ is rather insensitive to these heterogeneous effects. This will be illustrated in Section 5.2 for the case of the heavy-water Georgia Tech Research Reactor.

If several fissionable isotopes are present in the fuel,

$$^{DFN}\beta_{eff,i} = \{\Sigma_m\, {}^{m}f\, {}^{m}\nu_{d,i} \exp[B^2({}^{m}\tau - {}^{m}\tau_i)]\} / [\Sigma_m\, {}^{m}f\, {}^{m}\nu] \tag{8}$$

where $^{m}f$ is the fraction of all fission neutrons emitter per sec which come from isotope m. If the fuel consists of only $^{235}U$ and $^{238}U$ and if $^{235}(\tau - \tau_i) \approx {}^{238}(\tau - \tau_i)$, this equation reduces to

$$^{DFN}\beta_{eff,i} = {}^{235}\beta_{eff,i}\, [{}^{235}f + {}^{238}f({}^{238}\nu_{d,i} / {}^{235}\nu_{d,i})] / [{}^{235}f + {}^{238}f({}^{238}\nu / {}^{235}\nu)]. \tag{9}$$

For both HEU and LEU thermal research reactors the denominator in this expression for $^{DFN}\beta_{eff,i}$ is effectively unity.

If the ages and geometric buckling are known, Method II does not require the use of a computer to make reasonable estimates for $^{DFN}\beta_{eff}$ and its family-dependent components $^{DFN}\beta_{eff,i}$ . However, this method is not as accurate as Method I because the continuous slowing down model is not a good approximation for $H_2O$- and $D_2O$-moderated reactors and because of the bare homogeneous reactor model assumed in the derivation ofEq. 5..

### 2.3 Effective Delayed Photoneutron Fraction, $^{DPN}\beta_{eff}$, in $D_2O$

For heavy water reactors there are both delayed fission neutrons (DFN) and delayed photoneutrons (DPN) which contribute to the total effective delayed neutron fraction. In principle, both $\beta_{eff}$ - methods discussed in the previous section for finding $^{DFN}\beta_{eff}$ can be modified to determine $^{DPN}\beta_{eff}$. Since delayed photoneutrons are not present in the initial cross section set, their effects must be added in the adjusted cross section set. For the Method I case,

$$^{DPN}\beta_{eff} = (k_a - k)/k_a \tag{10}$$

where $k_a$ is the eigenvalue based on the cross section set adjusted for delayed photoneutron effects. The adjusted fission yield becomes

$$^{m}\nu_{a,j} = {}^{m}\nu_j + F_\gamma\, \nu_{dpn} \tag{11}$$

and the $^{235}U$ adjusted fission spectrum is

$$\chi_{a,j} = (\chi_j + f_{dpn,\,j}\, F_\gamma\, \nu_{dpn} / \nu_j)\, N. \tag{12}$$

In these equations $F_\gamma$ is the fraction of fission product gamma rays emitted in the fuel with energies above the $D(\gamma,n)$ threshold (2.226 MeV) which are available for photoneutron production in the $D_2O$ channels. Thus, $(1 - F_\gamma)$ is the fraction lost by absorption, energy degradation, and leakage effects. The total delayed photoneutron yield is $\nu_{dpn}$ and the fraction of these emitted into energy group j is $f_{dpn,\,j}$. As before, N is a normalization constant chosen so that the adjusted fission spectrum $\chi_{a,j}$ sums to unity.

For the Method II case,

$$^{DPN}\beta_{eff} = {}^{DPN}\beta\, F_{\gamma}\, \exp[B^2(\tau - \tau_{dpn})] \tag{13}$$

where $^{DPN}\beta$ is the total delayed photoneutron fraction and $\tau_{dpn}$ is the slowing-down age for these delayed photoneutrons. Delayed photoneutron data are given in Section 3.2 and the age in Section 4.

### 2.4 Decay Constants for Combined Sets of Delayed Fission Neutrons

For a given reactor a coalesced set of delayed fission neutron decay constants ($\lambda_i$) is needed if two or more isotopes contribute fission neutrons. The appropriate weighting factor is $^mF\ ^m\nu_{d,i}$ where $^mF$ is the fraction of all fissions which occur in isotope m. Thus, for fresh uranium fuel consisting of only $^{235}U$ and $^{238}U$, $\lambda_i = {}^{235}w_i\ {}^{235}\lambda_i + (1 - {}^{235}w_i)\ {}^{238}\lambda_i$ where $^{235}w_i = [^{235}F\ ^{235}\nu_{d,i}] / [^{235}F\ ^{235}\nu_{d,i} + {}^{238}F\ ^{238}\nu_{d,i}]$.

## 3. DELAYED NEUTRON DATA

The application of Methods I and II for calculating effective delayed neutron fractions requires fundamental delayed neutron data. Such data are provided in the following sections.

### 3.1 ENDF/B-V Delayed Fission Neutron Data for $^{235}U$ and $^{238}U$

Tables 1-4 provide ENDF/B-V delayed fission neutron data for $^{235}U$ and $^{238}U$. Fission yields ($\nu_{d,i}$) and decay constants ($\lambda_i$) for the six families of delayed neutrons are provided in Tables 1 and 2. Energy distributions of delayed fission neutrons are expressed in terms of the probability density P(E) in Table 3 for $^{235}U$ and in Table 4 for $^{238}U$. P(E)dE is the probability that a delayed neutron will be emitted with energies between E and E+dE. The ENDF/B-V interpolation scheme for these probability densities requires that in the energy interval from $E_n$ to $E_{n+1}$ the probability density is constant with the value of $P(E_n)$. However, linear interpolation is used in the first energy interval with E between E=0 and E=$E_1$. With this interpolation scheme the probability densities in Tables 3 and 4 are normalized so that. $\int_0^\infty P(E)dE = 1.0$ while the average energy of the delayed neutrons is $E_{ave} = \int_0^\infty EP(E)dE$.

The Method I technique for finding the effective delayed neutron fractions requires a determination of the fraction of delayed neutrons from family i emitted into energy group j. These fractions are easily evaluated from the probability densities and the interpolation scheme discussed above. Thus,

$$f_{d,i,j} = \int_{Ej,L}^{Ej,U} P_i(E)dE. \tag{14}$$

where $E_{j,L}$ and $E_{j,U}$ are the lower and upper energy boundaries of group j.

Method II requires numerical values for the slowing-down ages of the delayed neutrons. These ages are evaluated in Section 4 of this report.

| Table 1<br><br>Delayed Neutron Data (ENDF/B-V) for the Thermal Fission of $^{235}U$ | | | | |
|---|---|---|---|---|
| Family<br>i | Decay Const.<br>$\lambda_i$ ($sec^{-1}$) | Yield - $\nu_{d,i}$<br>neutrons/fiss | Relative<br>Yield | Fraction<br>$\beta_i = \nu_{d,i} / \nu^*$ |
| 1 | 1.272E-2 | 6.346E-4 | 0.038 | 2.604E-4 |
| 2 | 3.174E-2 | 3.557E-3 | 0.213 | 1.460E-3 |
| 3 | 1.160E-1 | 3.140E-3 | 0.188 | 1.288E-3 |
| 4 | 3.110E-1 | 6.797E-3 | 0.407 | 2.789E-3 |
| 5 | 1.400E+0 | 2.138E-3 | 0.128 | 8.773E-4 |
| 6 | 3.870E+0 | 4.342E-4 | 0.026 | 1.782E-4 |
| Total: | | 1.670E-2 | 1.000 | 6.853E-3 |

* For $^{235}U$ thermal fission, $\nu$ = 2.43670 total neutrons/fission.

Note: For $E_n \geq 7.0$ MeV, $\nu_d$ (total) = 9.000E-3 delayed neutrons/fission but with the above relative yields.

| Table 2<br><br>Delayed Neutron Data (ENDF/B-V) for $^{238}U$ Fission<br>Induced by a Prompt-Neutron Spectrum | | | | |
|---|---|---|---|---|
| Family<br>i | Decay Const.<br>$\lambda_i$ ($sec^{-1}$) | Yield - $\nu_{d,i}$<br>neutrons/fiss | Relative<br>Yield | Fraction<br>$\beta_i = \nu_{d,i} / \nu^*$ |
| 1 | 1.323E-2 | 5.720E-4 | 0.013 | 2.055E-4 |
| 2 | 3.212E-2 | 6.028E-3 | 0.137 | 2.166E-3 |
| 3 | 1.390E-1 | 7.128E-3 | 0.162 | 2.561E-3 |
| 4 | 3.590E-1 | 1.707E-2 | 0.388 | 6.133E-3 |
| 5 | 1.410E+0 | 9.900E-3 | 0.225 | 3.556E-3 |
| 6 | 4.030E+0 | 3.300E-3 | 0.075 | 1.186E-3 |
| Total: | | 4.400E-2 | 1.000 | 1.581E-2 |

* For $\nu$ = 2.7836 total neutrons/fission.

Note: For $E_n \geq 9.0$ MeV, $\nu_d$ (total) = 2.600E-2 delayed neutrons/fission but with the above relative yields.

**Table 3**

**Probability Densities, P(E), for the Emission of Delayed Neutrons from $^{235}$U Fission (ENDF/B-V)**

| E - MeV | Family-Dependent P(E) Values | | | |
|---|---|---|---|---|
| | Family 1 | Family 2 | Family 3 | Families 4-6 |
| 0.00000 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 |
| 0.07973 | 2.70039+ 0 | 1.09860+ 0 | 2.15110+ 0 | 1.32879+ 0 |
| 0.08859 | 2.77519+ 0 | 1.10690+ 0 | 2.61070+ 0 | 1.36589+ 0 |
| 0.09843 | 3.09089+ 0 | 1.14820+ 0 | 2.35640+ 0 | 1.44009+ 0 |
| 0.10937 | 3.25289+ 0 | 1.20870+ 0 | 2.12960+ 0 | 1.48639+ 0 |
| 0.12152 | 3.38999+ 0 | 1.25280+ 0 | 1.99480+ 0 | 1.50799+ 0 |
| 0.13502 | 3.40659+ 0 | 1.24730+ 0 | 1.48310+ 0 | 1.52969+ 0 |
| 0.15002 | 3.37339+ 0 | 1.27210+ 0 | 1.34830+0 | 2.03959+ 0 |
| 0.16669 | 2.69209+ 0 | 1.28860+ 0 | 1.17970+0 | 2.03029+ 0 |
| 0.18522 | 2.94959+ 0 | 1.12610+ 0 | 8.88630- 1 | 1.77069+ 0 |
| 0.20580 | 3.07839+ 0 | 1.09590+ 0 | 8.30410- 1 | 1.72429+ 0 |
| 0.22866 | 2.49269+ 0 | 9.93979- 1 | 1.10620+ 0 | 1.61619+ 0 |
| 0.25407 | 2.04399+ 0 | 1.46480+ 0 | 1.02350+ 0 | 1.52969+ 0 |
| 0.28230 | 1.28370+ 0 | 1.54190+ 0 | 1.02350+ 0 | 1.59769+ 0 |
| 0.31367 | 7.93498- 1 | 1.29690+ 0 | 1.18590+ 0 | 1.47099+ 0 |
| 0.34852 | 7.60258- 1 | 1.32710+ 0 | 1.51370+ 0 | 1.20209+ 0 |
| 0.38724 | 1.04280+ 0 | 1.31890+ 0 | 1.37280+ 0 | 1.18669+ 0 |
| 0.43027 | 5.69159- 1 | 1.49230+ 0 | 1.21650+ 0 | 1.03209+0 |
| 0.47808 | 2.74189- 1 | 1.33540+ 0 | 1.17970+ 0 | 1.01359+ 0 |
| 0.53120 | 2.16029- 1 | 1.30240+ 0 | 9.43790- 1 | 9.85785- 1 |
| 0.59022 | 2.36799- 1 | 8.50800- 1 | 7.99770- 1 | 7.29297- 1 |
| 0.65580 | 2.49269- 1 | 8.86590- 1 | 6.06720- 1 | 4.78988- 1 |
| 0.72866 | 3.28199- 1 | 7.70950- 1 | 4.74960- 1 | 7.01477- 1 |
| 0.80963 | 2.61729- 1 | 5.20390- 1 | 3.49320- 1 | 5.40788- 1 |
| 0.89959 | 1.41250- 1 | 2.69830- 1 | 3.55450- 1 | 3.30659- 1 |
| 0.99954 | 7.89338- 2 | 2.20270- 1 | 2.88040- 1 | 1.45229- 1 |
| 1.11060 | 7.47788- 2 | 8.81089- 2 | 2.29820- 1 | 1.23609- 1 |
| 1.23400 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 |
| $E_{ave}$(MeV): | 0.2680 | 0.4394 | 0.4094 | 0.4017 |

Note: P(E) dE is the probability that delayed neutrons will be emitted with energies between E and E+dE.

**Table 4**

**Probability Densities, P(E), for the Emission of Delayed Neutrons from $^{238}$U Fission (ENDF/B-V)**

| E - MeV | Family-Dependent P(E) Values | | | | |
|---|---|---|---|---|---|
| | Family 1 | Family 2 | Family 3 | Family 4 | Families 5-6 |
| 0.00000 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 |
| 0.07973 | 2.75331+ 0 | 7.68413- 1 | 2.39110+ 0 | 1.40809+ 0 | 1.83650+ 0 |
| 0.08859 | 3.19931+ 0 | 6.65593- 1 | 2.37300+ 0 | 1.43079+ 0 | 2.12731+ 0 |
| 0.09843 | 3.33311+ 0 | 8.38754- 1 | 2.26440+ 0 | 1.62829+ 0 | 2.27691+ 0 |
| 0.10937 | 4.31041+ 0 | 8.52283- 1 | 2.07450+ 0 | 1.46959+ 0 | 2.31011+ 0 |
| 0.12152 | 4.29411+ 0 | 8.71224- 1 | 2.16490+ 0 | 1.78039+ 0 | 2.32670+ 0 |
| 0.13502 | 2.97631+ 0 | 1.19590+ 0 | 1.42920+ 0 | 2.06529+ 0 | 2.12731+ 0 |
| 0.15002 | 2.31541+ 0 | 1.21210+ 0 | 9.40749- 1 | 2.26919+ 0 | 1.90300+ 0 |
| 0.16669 | 1.38270+ 0 | 1.35281+ 0 | 1.31760+ 0 | 1.89369+ 0 | 1.80320+ 0 |
| 0.18522 | 2.04770+ 0 | 1.30140+ 0 | 1.69760+ 0 | 1.81279+ 0 | 1.62040+ 0 |
| 0.20580 | 2.13691+ 0 | 1.34200+ 0 | 1.61920+ 0 | 1.81599+ 0 | 1.62870+ 0 |
| 0.22866 | 2.30731+ 0 | 1.30140+ 0 | 1.54680+ 0 | 1.73829+ 0 | 1.52070+ 0 |
| 0.25407 | 1.73550+ 0 | 1.25810+ 0 | 1.13070+ 0 | 1.66709+ 0 | 1.38770+ 0 |
| 0.28230 | 1.44350+ 0 | 1.35281+ 0 | 7.50789- 1 | 1.57649+ 0 | 1.32130+ 0 |
| 0.31367 | 7.29882- 1 | 1.36090+ 0 | 5.48769- 1 | 1.57969+ 0 | 1.32130+ 0 |
| 0.34852 | 9.36683- 1 | 1.92641+ 0 | 6.03039- 1 | 1.42429+ 0 | 1.34620+ 0 |
| 0.38724 | 1.30970+ 0 | 1.91291+ 0 | 7.92999- 1 | 1.07469+ 0 | 1.30460+ 0 |
| 0.43027 | 8.47483- 1 | 1.84531+ 0 | 8.11089- 1 | 1.18479+ 0 | 1.18830+ 0 |
| 0.47808 | 6.40682- 1 | 1.61800+ 0 | 7.56819- 1 | 1.14919+ 0 | 1.02210+ 0 |
| 0.53120 | 6.24462- 1 | 1.04170+ 0 | 7.92999- 1 | 1.06179+ 0 | 8.55912- 1 |
| 0.59022 | 2.23021- 1 | 6.71003- 1 | 8.68379- 1 | 7.80136- 1 | 7.72822- 1 |
| 0.65580 | 1.94641- 1 | 5.43842- 1 | 9.37729- 1 | 1.74799- 1 | 6.31552- 1 |
| 0.72866 | 3.12231- 1 | 7.33233- 1 | 7.23649- 1 | 4.40238- 1 | 4.73661- 1 |
| 0.80963 | 1.78420- 1 | 4.30202- 1 | 5.66859- 1 | 3.07518- 1 | 2.99151- 1 |
| 0.89959 | 1.33810- 1 | 2.84091- 1 | 4.70369- 1 | 2.68679- 1 | 1.74510- 1 |
| 0.99954 | 9.73172- 2 | 2.59741- 1 | 3.04540- 1 | 2.42779- 1 | 1.32960- 1 |
| 1.11060 | 8.92082- 2 | 1.13640- 1 | 2.80410- 1 | 1.32719- 1 | 6.64791- 2 |
| 1.23400 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 | 0.00000+ 0 |
| $E_{ave}$ (MeV): | 0.2837 | 0.4381 | 0.4341 | 0.4172 | 0.3593 |

Note: P(E) dE is the probability that delayed neutrons will be emitted with energies between E and E+dE.

### 3.2 Delayed Photoneutron Data for $^{235}U$ Fission Product Gamma Rays on $D_2O$.

For heavy water research reactors delayed photoneutrons contribute to the total value of $\beta_{eff}$. Bernstein et al.[9] measured the yield for 8 groups of delayed photoneutrons from $^{235}U$ fission product gamma rays on $D_2O$ relative to the 22-second ($T_{1/2}$) delayed fission neutron family. They corrected their data for gamma ray absorption in structural materials, for energy degradation (from single and multiple Compton scattering) to energies below the D($\gamma$,n) threshold, and for leakage. Using a value of $^{235}\nu_{d,2}$ = 0.00337 n/fiss for the yield of the 22-second delayed fission neutron family, Keepin[10] re-normalized the Bernstein data to absolute values. These data are reproduced in Table 5 which includes Ergen's result[11] for the $T_{1/2}$ = 12.8d family. Because of the Bernstein corrections, the data in Table 5 represent maximum photoneutron yields, for which $F_\gamma = 1$. To account for incomplete saturation, the $\beta_i$ values in this table need to be multiplied by [1 - exp($-\lambda_i\ t_e$)] where $t_e$ is the effective irradiation time The $\beta_i$ delayed neutron fractions in Table 5 are relative to the ENDF/B-V- recommended value for the total neutron yield from the thermal fission of $^{235}U$ ($^{235}\nu_{th}$ = 2.43670).

**Table 5**

**Delayed Photoneutron Data[a] for $^{235}U$ Fission Product Gamma Rays on $D_2O$**

| Family i | Half-Life | Decay Const. $\lambda_i$ (sec$^{-1}$) | Photoneutron Yield - $\nu_{d,i}$ n/fiss ($10^{-5}$) | Relative Yield | Fraction $\beta_i$ ($10^{-5}$) $\beta_i = \nu_{d,i} / \nu^b$ |
|---|---|---|---|---|---|
| 1 | 12.8 d | 6.26E-7 | 0.12 | 0.0005 | 0.05 |
| 2 | 53 h | 3.63E-6 | 0.25 | 0.0010 | 0.103 |
| 3 | 4.4 h | 4.37E-5 | 0.78 | 0.0032 | 0.320 |
| 4 | 1.65 h | 1.17E-4 | 5.65 | 0.0231 | 2.32 |
| 5 | 27 m | 4.28E-4 | 5.01 | 0.0205 | 2.06 |
| 6 | 7.7 m | 1.50E-3 | 8.14 | 0.0333 | 3.34 |
| 7 | 2.4 m | 4.81E-3 | 17.0 | 0.0695 | 6.98 |
| 8 | 41 s | 1.69E-2 | 49.5 | 0.2025 | 20.31 |
| 9 | 2.5 s | 2.77E-1 | 158.0 | 0.6463 | 64.84 |
| Total: | | | 244.45 | 1.0000 | 100.32 |

[a] Data taken from Ref. 9. These data have been corrected for gamma ray absorption, energy degradation, and leakage effects and so represent maximumphotoneutron yields for which $F_\gamma = 1$.

[b] For $^{235}U$ thermal fission, $\nu$ = 2.43670 total neutrons/fission.

.

For the heavy water GTRR a combination of ORIGEN[12] and MCNP[13] Monte Carlo gamma ray source calculations were done to estimate $F_\gamma$, the spectrum of delayed photoneutrons in the $D_2O$ channels, and the $f_{dpn,j}$ fraction of delayed photoneutrons emitted into energy group j. These calculations are discussed in Section 5.2.

## 4. EVALUATION OF SLOWING-DOWN AGES FOR $^{235}U$ PROMPT AND DELAYED NEUTRONS IN $H_2O$ AND $D_2O$

Numerical values for neutron ages in both light and heavy water are needed if Method II is to be used to calculate effective delayed neutron fractions in research reactors. Some measured and calculated ages are given in Ref. 14. However, the information is incomplete and the calculated ages are not based on the ENDF/B-V data. Therefore, Monte Carlo calculations have been performed to determine the ages of both prompt and delayed neutrons in $H_2O$ and $D_2O$.

For a point source in an infinite medium the age of neutrons slowed down to energy E is defined by the equation

$$\tau(E) = (1/6)\int_0^\infty r^4\Phi(E,r)dr \Big/ \int_0^\infty r^2\Phi(E,r)dr \tag{15}$$

where $\Phi(E,r)$ is the spatial distribution of the neutron flux at energy E. Historically, the age to indium resonance (1.46 eV) was commonly measured from the distribution of cadmium-covered indium foil activations about a point or plane source.

At large r the asymptotic flux for $H_2O$ and $D_2O$ moderators behaves as

$$\Phi_{asy}(r,E) = A\,[\exp(-r/\lambda)]\,/\,r^2\,. \tag{16}$$

In this asymptotic range where $r > r_m$

$$\int_{rm}^\infty r^2\Phi(E,r)dr = A\lambda\exp(-r_m/\lambda) \equiv I_{D2} \tag{17}$$

and

$$\int_{rm}^\infty r^4\Phi(E,r)dr = A\lambda[\exp(-r_m/\lambda)][r_m^2 + 2r_m\lambda + 2\lambda^2] \equiv I_{N2}\,. \tag{18}$$

For a point neutron source with a specified energy distribution and located at the center of a large spherical medium of $H_2O$ or $D_2O$, MCNP[13] Monte Carlo calculations were performed to determine the $\Phi(E,r)$ flux distribution. For these calculations the energy bin E was chosen to be in the range from 1 eV to 2 eV in order to span the indium resonance energy. For a given energy distribution of source neutrons, $\tau(E)$ was determined from Eq. (15) using numerical integration techniques. A non-linear least squares analysis of the Monte Carlo fluxes at large r determines the fitting coefficients A and $\lambda$. Figure 1 illustrates this procedure as well as the asymptotic behavior of the flux at large r (Eq. 16). Since the

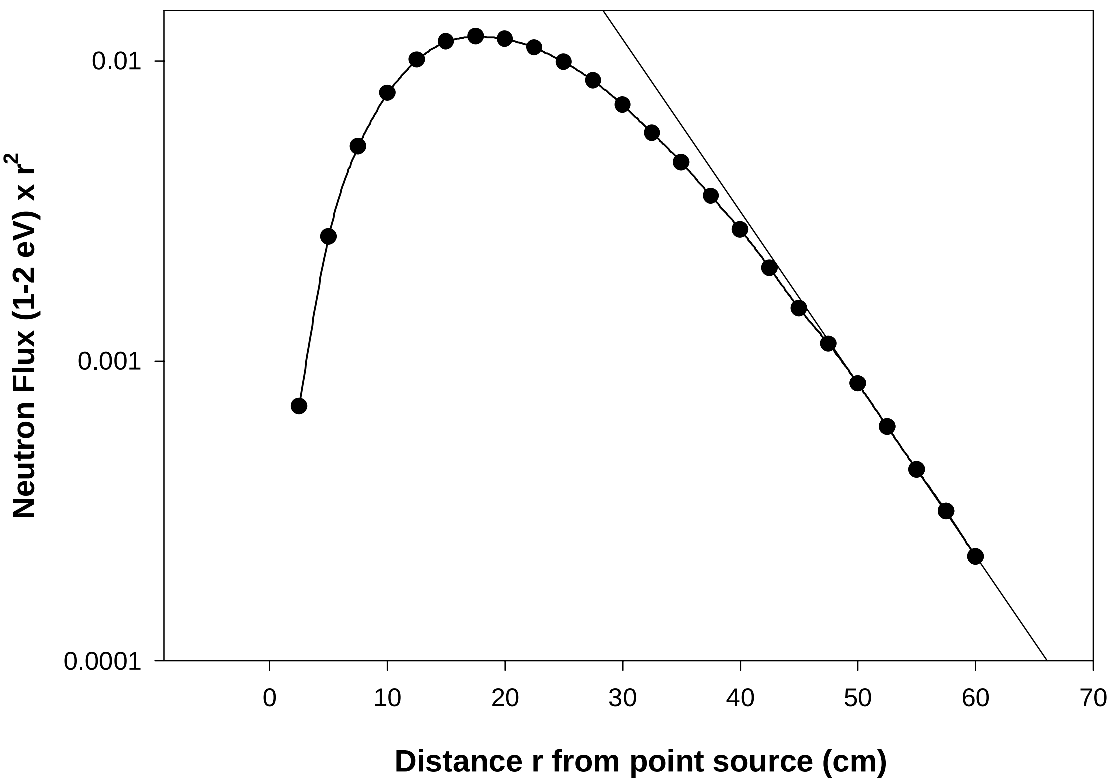
Monte Carlo fluxes were evaluated at equally-spaced radial intervals, Simpson's Rule was used to evaluate the integrals from zero to $r_m$.

$$\int_0^{rm} r^4 \Phi(E,r)dr \equiv I_{N1} \qquad \text{and} \qquad \int_0^{rm} r^2 \Phi(E,r)dr \equiv I_{D1} \tag{19}$$

Thus,

$$\tau(E) = (1/6)[I_{N1} + I_{N2}] / [I_{D1} + I_{D2}]. \tag{20}$$

**Flux Distribution About Point Source of**
**$^{235}U$ Fission Spectrum Neutrons in $D_2O$**




**The flux of near indium resonance neutrons in $D_2O$ is multiplied by $r^2$ to show the exponential decrease at large r as a straight line.**

**Figure 1**

The point source energy distribution for fission neutrons is accurately represented by the Watt fission energy spectrum, namely

$$P(E) = C \exp(-E/a) \sinh(bE)^{1/2}. \qquad (21)$$

For both thermal and fast fission of $^{235}U$, a = 0.988 MeV and b = 2.249 $MeV^{-1}$. C is a normalization constant chosen so that the integral of P(E) over the entire energy range is unity. For the $^{235}U$ fission spectrum C = 0.43962 $MeV^{-1}$. The average energy of this source distribution is

$$^{235}E_{ave} = \int_0^\infty EP(E)dE = 2.2008 \text{ MeV}.$$

Constants for the fast fission of $^{238}U$ are a = 0.89506 MeV and b = 3.2953 $MeV^{-1}$.

The energy distribution of delayed fission neutrons is expressed in terms of the tabulated values for the probability density P(E). (See Tables 3 and 4 for family-dependent P(E) values for the fission of $^{235}U$ and $^{238}U$, respectively.) These probability densities were used for point source energy distributions in the Monte Carlo calculations of the slowing down ages of each family of delayed neutrons from $^{235}U$ fission.

An evaporation spectrum was used to describe the energy distribution of delayed photoneutrons from fission product gamma rays on heavy water. For this case

$$P(E) = C\,E \exp(-E/a). \qquad (21)$$

The average energy of delayed photoneutrons in the GTRR was found to be 0.51 MeV (see Section 5.2). Corresponding to this average energy the constants for the evaporation spectrum have the values a = 0.255 $MeV^{-1}$ and C = 15.379 $MeV^{-2}$.

With these energy distributions and the methods described earlier, the slowing down ages for $^{235}U$ prompt and delayed fissions neutrons were calculated for $H_2O$ and $D_2O$ moderators. In addition, the age of delayed photoneutrons in heavy water was determined. For these calculations it was assumed that the heavy water moderator contained 0.25 atom % $H_2O$. Results from these calculations are summarized in Table 6 and, where possible, are compared with values given in Ref. 14.

Table 6 also includes age calculations for an homogenized mixture of fuel and heavy water for the GTRR for $^{235}U$ prompt fission and delayed (Family 2) neutrons. For these point source calculations nu-bar in the fuel-moderator mixture was set equal to zero so as to avoid neutron production in the slowing down medium. Mostly because of slowing down from inelastic scattering in uranium, the age difference ($\tau - \tau_2$) from these calculations is nearly the same as that for the pure heavy water moderator. Therefore, no additional age calculations were performed for fuel/moderator mixtures.

**Table 6**

**$^{235}$U Neutron Ages to Near Indium Resonance (1-2 eV) in $H_2O$ and $D_2O$**

| Spect. | $E_{ave}$ MeV | Medium | $r_m$ cm | $I_{N1}$ | $I_{N2}$ | $I_{D1}$ | $I_{D2}$ | Age τ cm² MCNP[a] | Age τ cm² Ref.14 |
|---|---|---|---|---|---|---|---|---|---|
| Watt | 2.0308 | $H_2O$ | 40.0 | 5.89576 | 0.52133 | 4.0412-2 | 2.2723-4 | 26.32 | 26.5±.9 |
| Delayed Fiss n's | | | | | | | | | |
| Fam. 1 | 0.2680 | $H_2O$ | 25.0 | 1.90513 | 0.00753 | 4.1009-2 | 9.7868-6 | 7.77 | 7.1 |
| Fam. 2 | 0.4394 | $H_2O$ | 30.0 | 2.32499 | 0.00508 | 4.1159-2 | 4.6670-6 | 9.43 | 9.1 |
| Fam. 3 | 0.4094 | $H_2O$ | 30.0 | 2.21574 | 0.00489 | 4.0787-2 | 4.4816-6 | 9.07 | 8.6 |
| F's 4-6 | 0.4017 | $H_2O$ | 30.0 | 2.23201 | 0.00444 | 4.1060-2 | 4.0864-6 | 9.08 | 9.6 |
| Watt | 2.0308 | $D_2O$[b] | 60.0 | 188.209 | 7.81250 | 0.303804 | 1.6909-3 | 106.94 | 109±3 |
| Delayed Fiss n's: | | | | | | | | | |
| Fam. 1 | 0.2680 | $D_2O$[b] | 60.0 | 155.559 | 1.93373 | 0.306510 | 4.4196-4 | 85.51 | |
| Fam. 2 | 0.4394 | $D_2O$[b] | 60.0 | 161.997 | 1.99718 | 0.308380 | 4.6033-4 | 88.50 | |
| Fam. 3 | 0.4094 | $D_2O$[b] | 60.0 | 159.623 | 2.16778 | 0.307127 | 4.9506-4 | 87.66 | |
| F's 4-6 | 0.4017 | $D_2O$[b] | 60.0 | 160.009 | 2.11018 | 0.307620 | 4.8255-4 | 87.70 | |
| Delayed Photo n's | 0.51 | $D_2O$[b] | 60.0 | 161.811 | 2.35205 | 0.306828 | 5.3560-4 | 89.02 | |
| Watt | 2.0308 | Hmg'd Core[c] | 60.0 | 214.436 | 12.5533 | 0.307764 | 2.6901-3 | 121.86 | |
| Delayed Fiss n's: | | | | | | | | | |
| Fam. 2 | 0.4394 | Hmg'd Core[c] | 60.0 | 186.884 | 4.42844 | 0.310954 | 9.9957-3 | 102.21 | |

[a] $\tau = (1/6)(I_{N1} + I_{N2}) / (I_{D1} + I_{D2})$. See text for definition of integrals.
[b] Contains 0.25 atom % $H_2O$.

[c]Homogenized core composition for the GTRR (see Section 5.2). Nu-bar set equal to zero to avoid neutron sources in the slowing down medium.

## 5. NUMERICAL ILLUSTRATIONS

These methods for calculating kinetic parameters ($l_p$, $\beta_{eff}$, and its family-dependent components) for research reactors are illustrated in this section for the light-water Oak Ridge Research Reactor (ORR) core 179-AX5 and for the heavy-water Georgia Tech Research Reactor (GTRR) with 14 fresh HEU fuel elements. These examples were chosen because for each case an experimental measurement of the prompt neutron decay constant ($\beta_{eff}$ / $l_p$) was made with which the analytical results can be compared.

### 5.1 Light-Water ORR LEU Core 179-AX5

This ORR core was operated in 1986 as part of the Whole-Core LEU $U_3Si_2$-Al Fuel Demonstration[3]. The $H_2O$-cooled and -reflected core (179-AX5) is fully described in Ref. 3 (pp.3-7, 79-81). It consisted of a $U_3Si_2$-Al fresh LEU core with 14 standard 19-plate fuel elements (340 g $^{235}$U/element) and four 15-plate fuel follower control elements (200 g $^{235}$U/element) arranged in a 5 x 4 lattice with two vacant corner positions. Figure 2 shows a core map of the 179-AX5 assembly. The lattice dimensions for each fuel element are 7.70890 cm x 8.10006 cm. For the diffusion calculations the four cadmium box-type shim/safety rods were withdrawn to the experimentally-determined critical elevation.

Results for the prompt neutron lifetime obtained from the 1/v-insertion method (Eq. 1) for this core are shown below (see Ref. 3, p. 56).

**Table 7**

**Prompt Neutron Lifetime for the ORR LEU Core 179-AX5**

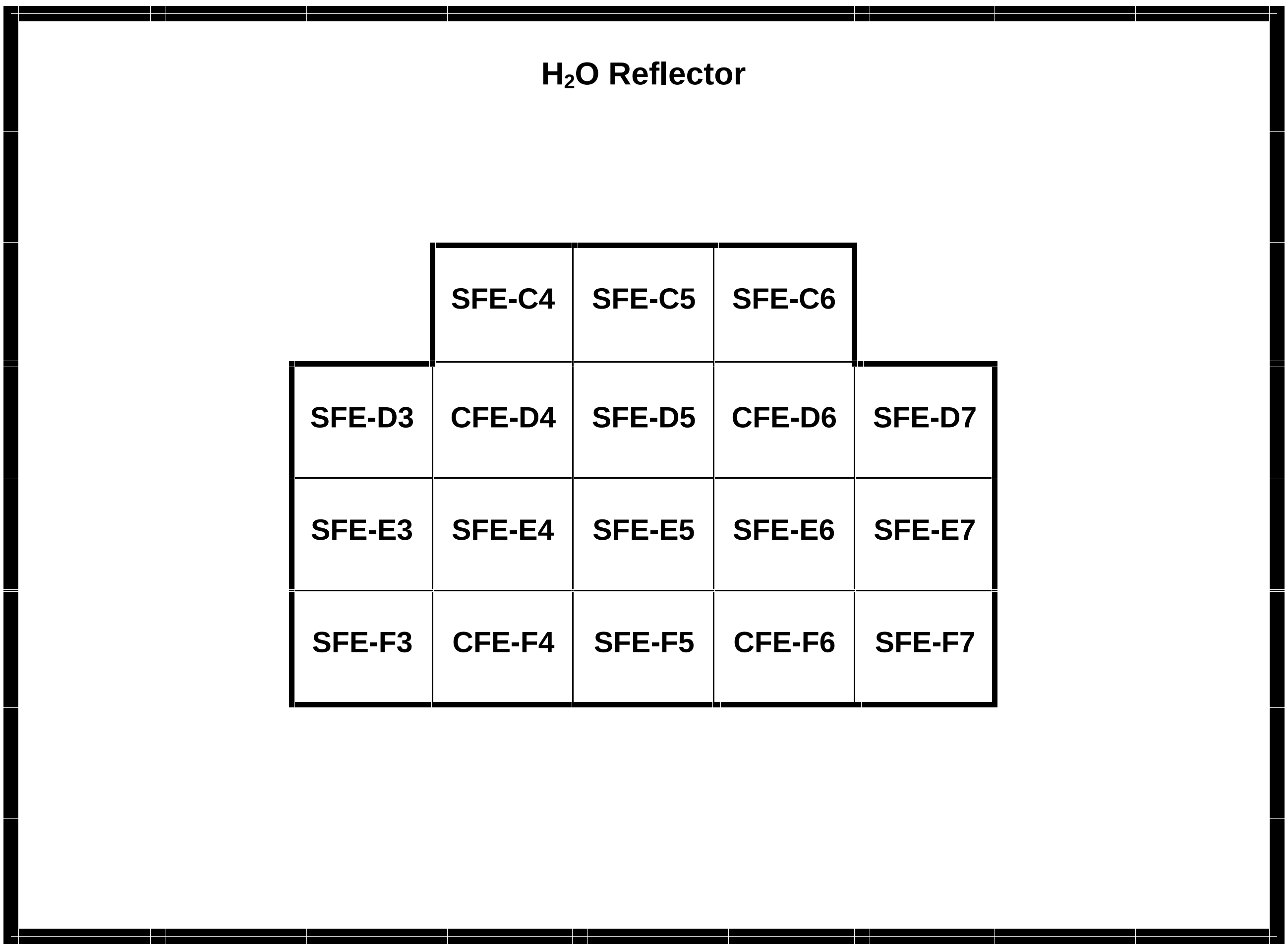
| N($^{10}$B) atoms/barn-cm | $k_{eff}$ | $l_p'$ μ-sec |
|---|---|---|
| 5.0E-08 | 0.992428 | 41.441 |
| 2.5E-08 | 0.993294 | 41.497 |
| 0.0 | 0.994164 | 41.553 |

These $l_p'$ values were obtained using infinitely dilute $^{10}$B multigroup cross sections generated for each reactor region (fuel, side plate, reflector, etc.). The prompt neutron lifetime was calculated by linearly extrapolating the concentration-dependent $l_p'$ values to zero $^{10}$B concentration. If the prompt neutron generation time is needed, it is $\Lambda = l_p / k_{eff}$ = 41.797 μ–sec.

Methods I and II were used to calculate the effective delayed neutron fractions. Method I (Eqs. 3 and 4) requires group-dependent values for the neutron yields and for the neutron spectra. Total neutron yields are included in the WIMS-D4M[8] cross section set and the delayed neutron yields are given in Tables 1 and 2. The delayed neutron spectra ($f_{d,i,j}$ values) were calculated from Eq. (14) using the probability densities P(E) shown in Table 3 and the interpolation methods discussed in Section 3.1. Table 8 shows these results. Since the WIMS-D4M $\chi$-spectrum is that for $^{235}U$, only $^{235}f_{d,i,j}$ values are needed.

**Core Map of the $H_2O$-Cooled and $H_2O$-Reflected ORR**

**Assembly 179-AX5 with Fresh LEU Fuel**




**SFE = MTR-type Standard Fuel Element, 19 plates, 340 g $^{235}U$**

**CFE = MTR-type Control Fuel Element, 15 plates, 200 g $^{235}U$**

**Figure 2**

| Table 8 | | | | | | |
|---|---|---|---|---|---|---|
| **$^{235}$U Fission Spectrum ($\chi_j$) and Delayed Neutron Spectra ($f_{d,i,j}$) ORR-179AX5 Core** | | | | | | |
| Group, j | $E_U$ - MeV | Fission Spectrum | Delayed Neutron Spectra by Family,i | | | |
| | | | Family 1 | Family 2 | Family 3 | Family 4-6 |
| 1 | 1.00E+1 | 7.5979E-1 | 5.26813- 2 | 1.03203- 1 | 1.23330- 1 | 1.06933- 1 |
| 2 | 8.21E-1 | 2.4004E-1 | 9.46801- 1 | 8.96587- 1 | 8.76258- 1 | 8.92812-1 |
| 3 | 5.53E-3 | 1.8038E-4 | 5.17875- 4 | 2.10687- 4 | 4.12533- 4 | 2.54833-4 |
| 4 | 6.25E-7 | 0.0 | 6.6151-12 | 2.6912-12 | 5.2695-12 | 3.2551-12 |
| | 1.00E-11 | | | | | |

Equations (8) and (9) form the basis for calculating the effective delayed neutron fractions by Method II. The required ages are given in Table 6. The geometric buckling $B^2$ (Eq. 6) was obtained from DIF3D[2] axial and radial total neutron flux distributions from an "equivalent" RZ homogenized core model of the ORR179-AX5 assembly. The height of this cylindrical core is $H_0$ = 60.01 cm which corresponds to the height of the ORR fuel column and the radius $R_0$ = 18.91 cm was determined by equating the circular area to the lattice area occupied by the 18 fuel elements. Volume-weighting of each distinct core composition (fuel meat, clad, coolant, side plates and moderator) was used to determine the homogenized core composition. By means of the least squares process, axial and radial neutron flux distributions at several radial and axial positions were fit to $\cos(B_z z)$ and $J_0(B_r r)$ functions to determine $B_r$ and $B_z$ values. From these values the geometric buckling was found to be $B^2 = B_{r,ave}^2 + B_{z,ave}^2$ = 9.43555E-3 $cm^{-2}$. For these fits the control rods were withdrawn and fluxes within a few cm from core/reflector interfaces were excluded from the fits. Figure 3 shows a pair of fits for axes through the origin. The radial fit was obtained by expanding the $J_0(B_r r)$ Bessel function into an infinite series and keeping the first 6 terms (see the Appendix for this expansion). The $^m f$ and $^m\nu$ values needed in Eq. 9 were determined from WIMS-D4M reaction rates. For this ORR core they have the values $^{235}f$ = 0.998577, $^{238}f$ = 0.001423, $^{235}\nu$ = 2.43803, and $^{238}\nu$ = 2.81010.

Table 9 summarizes the kinetic parameters for the light-water ORR LEU core 179-AX5. The VARI3D[1] result for the prompt neutron lifetime is 38.18 μ-sec which is smaller and not as accurate as that obtained by the 1/v insertion method because the perturbation code uses a single set of group-dependent neutron velocities and not region-dependent values. The effective delayed neutron fractions obtained from Methods I and II are in favorable agreement with those calculated by the VARI3D perturbation code. However, the Method I results are somewhat more accurate than those obtained from Method II for the reasons discussed earlier (see Section 2.2.2).

The prompt neutron decay constant $\alpha$ is the ratio of the effective delayed neutron fraction $\beta_{eff}$ to the prompt neutron lifetime $l_p$. Using reactor noise techniques, Ragan and Michalczo[15] measured $\alpha$ for the ORR 179-AX5 core. Based on the non-perturbation methods discussed above, the calculated values of

$\alpha$ are about 1% larger than the measured one. Differences between VARI3D and the calculated coalesced set of delayed neutron decay constants ($\lambda_i$) are negligible.

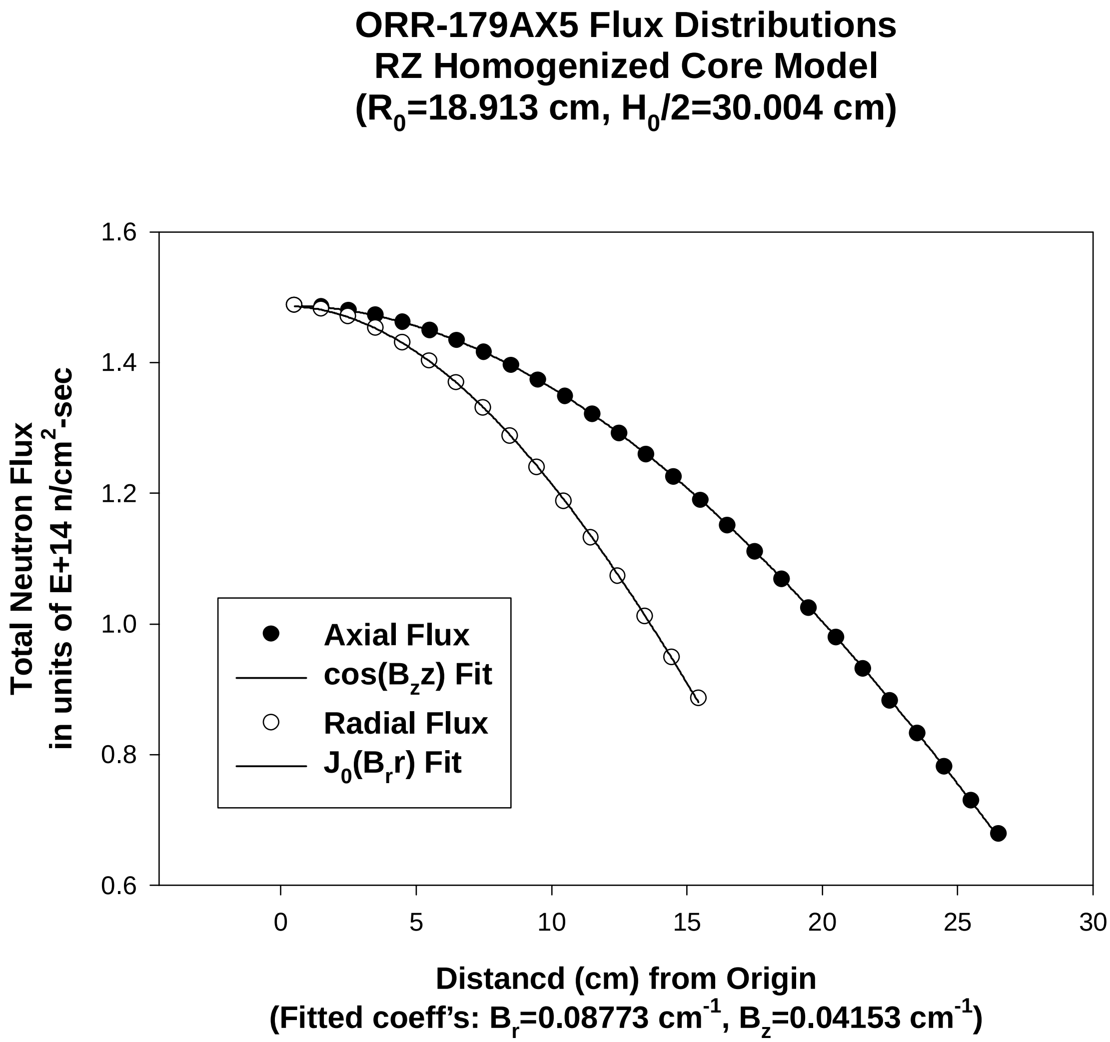



Figure 3

| Table 9<br>Kinetic Parameters for the Light-Water ORR LEU Core 179-AX5 | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Prompt Neutron Lifetime** | | | | | | | |
| | VARI3D | l/v Ins. | Ratio | | | | |
| $l_p$ - $\mu$sec | 38.18 | 41.55 | 1.088 | | | | |
| **Effective Delayed Neutron Fractions** | | | | | | | |
| | VARI3D | Method I | Method II[a] | Ratios Relative to VARI3D | | | |
| | | | | Meth. I | Meth. II | | |
| $\beta_{eff,\,1}$ | 3.044E-4 | 3.033E-4 | 3.101E-4 | 0.996 | 1.019 | | |
| $\beta_{eff,\,2}$ | 1.694E-3 | 1.707E-3 | 1.713E-3 | 1.008 | 1.011 | | |
| $\beta_{eff,3}$ | 1.493E-3 | 1.512E-3 | 1.518E-3 | 1.013 | 1.017 | | |
| $\beta_{eff,\,4}$ | 3.246E-3 | 3.279E-3 | 3.288E-3 | 1.010 | 1.013 | | |
| $\beta_{eff,\,5}$ | 1.030E-3 | 1.047E-3 | 1.037E-3 | 1.016 | 1.007 | | |
| $\beta_{eff,\,6}$ | 2.120E-4 | 2.166E-4 | 2.116E-4 | 1.022 | 0.998 | | |
| Total $\beta_{eff}$ | 7.980E-3 | 8.064E-3 | 8.078E-3 | 1.011 | 1.012 | | |
| **Prompt Neutron Decay Constant, $\alpha = \beta_{eff} / l_p$** | | | | | | | |
| | Exp. | VARI3D | Meth. I | Meth. II | C/E Ratios | | |
| | | | | | VARI3D | Meth. I | Meth. II |
| $\alpha$ - sec$^{-1}$ | 192.3±1.2 | 209.0 | 194.1 | 194.4 | 1.087 | 1.009 | 1.011 |
| **Coalesced Set of Delayed Neutron Decay Constants, $\lambda_i$ (sec$^{-1}$)** | | | | | | | |
| | Family 1 | Family 2 | Family 3 | Family 4 | Family 5 | Family 6 | |
| VARI3D | 1.272E-2 | 3.174E-2 | 1.162E-1 | 3.115E-1 | 1.400 | 3.875 | |
| $^{m}F\ ^{m}\nu_{d,i}$ wt'd | 1.272E-2 | 3.174E-2 | 1.163E-1 | 3.117E-1 | 1.400 | 3.877 | |

[a]$\beta_i$'s relative to $^{235}\nu$ = 2.43804 total neutrons/fission.

## 5.2 Heavy-Water GTRR HEU Core with 14 Fresh Fuel Elements

The 5-MW heavy-water Georgia Tech Research Reactor (GTRR) is described in Ref.'s 4 and 16. Each of the HEU MTR-type fuel assemblies contains 16 fueled and 2 unfueled plates and about 188 g $^{235}U$. Fuel assemblies are spaced 6 inches apart in a triangular array and the height of the fuel column is $H_0$ = 59.69 cm. During the startup phase of the GTRR, the prompt neutron decay constant ($\beta_{eff}/l_p$) was measured by the pile oscillator technique while $\beta_{eff}$ was determined using a method of reactor noise analysis and an absolute calibration of reactor power. The measurements were done for a fresh 14-element core (see Fig. 4) and are described in Ref. 15. Since this report does not include the elevation of the control rods during the measurements, they were not modeled in the calculations described below.

Using region-dependent 7-group WIMS-D4M ENDF/B-V cross sections, the prompt neutron lifetime was calculated for the 14-element core using the 1/v insertion method (Eq. 1). As for the case of the ORR, Method I (Eqs. 3 and 4) and Method II (Eqs. 8 and 9) were used to calculate the delayed fission neutron fractions $^{DFN}\beta_{eff,i}$ for each delayed fission neutron family. Table 10 gives the prompt and delayed fission neutron spectra obtained from Eq. (14) and P(E) values from Table 3 and using the same methods as were used for the ORR. An evaporation model (Eq. 21) for $E_{ave}$ = 0.51 MeV was used to calculate the spectrum of delayed photoneutrons for $^{235}U$ fission product gamma rays on $D_2O$. This spectrum is included in Table 10 while the method used to estimate $E_{ave}$ is described below.

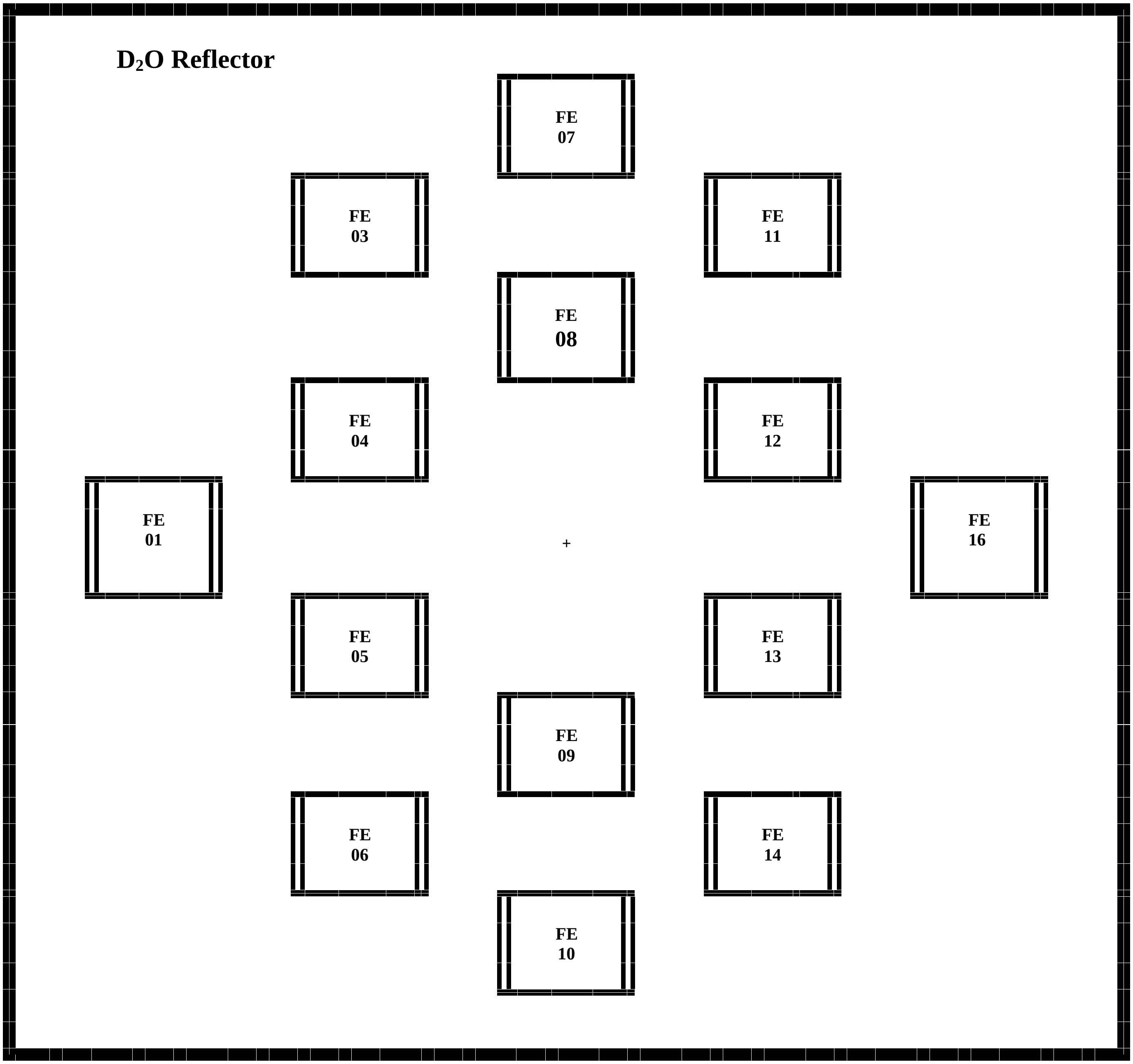
**Table 10**

**$^{235}U$ Fission Spectrum ($\chi_j$) and Delayed Neutron Spectra ($f_{d,i,j}$)**
**GTRR 14-Fuel-Element Core**

| Group, j | $E_U$ MeV | Fission Spectrum | Delayed Neutron Spectra by Family,i | | | | Delayed Photo n's |
|---|---|---|---|---|---|---|---|
| | | | Family 1 | Family 2 | Family 3 | Fam. 4-6 | |
| 1 | 1.00E+1 | 7.5979-1 | 5.26813-2 | 1.03203-1 | 1.23330-1 | 1.06933-1 | 1.68661-1 |
| 2 | 8.21E-1 | 2.4004-1 | 9.46801-1 | 8.96587-1 | 8.76258-1 | 8.92812-1 | 8.31107-1 |
| 3 | 5.53E-3 | 1.8038-4 | 5.17875-4 | 2.10687-4 | 4.12533-4 | 2.54833-4 | 2.31775-4 |
| 4 | 2.10E-6 | 0.0 | 6.8066-11 | 2.7692-11 | 5.4221-11 | 3.3494-11 | 6.1813-11 |
| 5 | 6.25E-7 | 0.0 | 5.5567-12 | 2.2606-12 | 4.4264-12 | 2.7343-12 | 5.0461-12 |
| 6 | 2.50E-7 | 0.0 | 1.0014-12 | 4.0742-13 | 7.9774-13 | 4.9278-13 | 9.0944-13 |
| 7 | 5.80E-8 | 0.0 | 5.6968-14 | 2.3176-14 | 4.5380-14 | 2.8032-14 | 5.1734-14 |
| | 1.00E-11 | | | | | | |

**Core Map of the $D_2O$-Cooled and $D_2O$-Reflected GTRR with**

**14 MTR-Type Fresh HEU Fuel Assemblies**




**Fuel Element: 16 fueled, 2 unfueled plates, 188 g $^{235}U$**

**Figure 4**

As for the ORR example, the geometric buckling $B^2$ was obtained from axial and radial fits of DIF3D[2] total neutron flux distributions for an RZ homogenized core model of the GTRR. For the 6-inch triangular array of fuel element locations, the radius of the cylindrical core was determined to be $R_0$ = 30.99 cm. The height of the homogenized core is that of the fuel column length which is $H_0$ = 59.69 cm. Again, volume weighting of each distinct core composition (fuel meat, clad, coolant, side plates and the $D_2O$ moderator between fuel elements) was used to calculate the homogenized core composition. For this $D_2O$-reflected RZ model of the GTRR, radial and axial total neutron flux distributions in the homogenized core region were fit by the least squares method to $J_o(B_r r)$ and $\cos(B_z z)$ functions. Fluxes at points within a few cm of the core/reflector interfaces were excluded from the fits. Figure 5 shows axial and radial fits for axes through the origin of the RZ model. Fits were made at several axial and radial positions to determine core-averaged values for $B_r$ and $B_z$. On the basis of these average values the geometric buckling was determined to be $B^2 = B_{r,ave}^2 + B_{z,ave}^2$ = 1.86614E-3 $cm^{-2}$. With the ages given in Table 6, Eq. 9 determines the Method II values for ${}^{DFN}\beta_{eff,i}$. For this HEU fuel ${}^{235}f$ = 0.999978, ${}^{238}f$ = 0.000022, ${}^{235}\nu$ = 2.43694, and ${}^{238}\nu$ = 2.81488 and so contributions to ${}^{DFN}\beta_{eff,i}$ from $^{238}U$ fission are negligible.

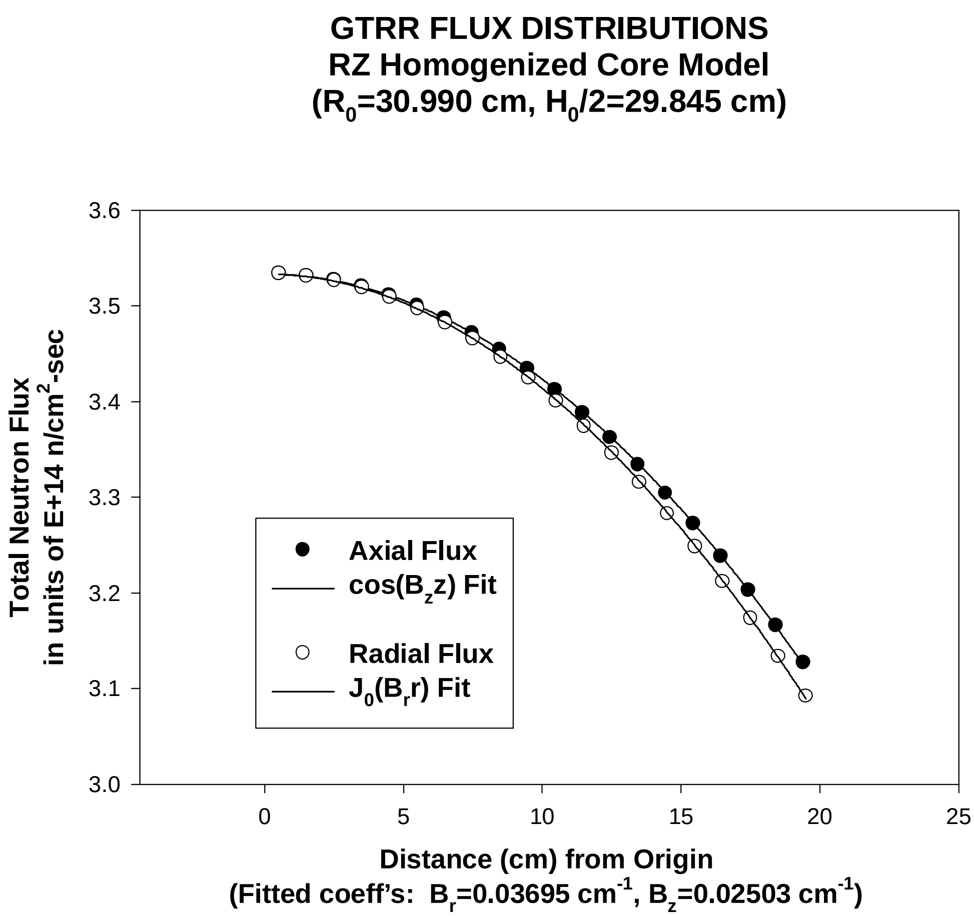



Figure 5

An estimate of the effective delayed photoneutron fraction was obtained in the following manner. A combination of ORIGEN[12] and MCNP[13] calculations was used to estimate the spectrum of the delayed photoneutrons in the $D_2O$ coolant channels and to determine $F_\gamma$ which, as mentioned earlier, is needed to correct the maximum photoneutron yields (Table 5) for γ-ray attenuation from leakage, absorption, and energy degradation effects. The ORIGEN calculation determined the energy distribution of fission product gamma rays at shutdown with energies above the deuterium (γ,n) threshold (2.226 Mev). For this calculation the irradiation time was long enough so that all the delayed photoneutron precursor activities were saturated prior to shutdown. This gamma ray energy distribution was used as a source term in the meat of each fuel plate in subsequent MCNP Monte Carlo photon calculations. For these γ-calculations the entire 14 fuel element assembly with thick axial and radial $D_2O$ reflectors was modeled. Energy-dependent gamma ray fluxes resulting from the ORIGEN source terms were calculated for the $D_2O$ coolant channels and for the heavy water surrounding each fuel element. For convenience, these fluxes were combined to determine the average gamma ray flux in the heavy water regions in and around the core. Two Monte Carlo calculations were needed to determine $F_\gamma$. The first was done using normal atom densities for all the materials and with leakage boundary conditions applied at the external surfaces. To determine the amount of gamma ray attenuation from leakage, energy degradation, and absorption in the core, a second Monte Carlo calculation was needed where fuel meat, clad, and side plate atom densities were reduced to effectively zero and where reflective boundary conditions were applied at the extrapolated surfaces of the approximately cylindrical core to eliminate γ–leakage effects. $F_\gamma$ is the ratio of the fission product gamma ray fluxes in the $D_2O$ regions from these two MCNP calculations for energies above 2.226 MeV. In this way it was found that $F_\gamma$ = 0.538. The energy of photoneutrons from the (γ, n) reaction on deuterium is approximately (see Ref. 6, p. 713)

$$E_n(\text{MeV}) = 0.5035[E_\gamma - 2.226 - 5.296\text{E-4}\ E_\gamma^{\,2}] + 0.4965\ E_\gamma[1.0816\text{E-3}(E_\gamma - 2.226)]^{1/2} \cos\theta$$

where $E_\gamma$ is the gamma ray energy in MeV and θ is the angle between the directions of the incident γ-ray and the emitted neutron. The threshold energy is just the binding energy of the deuteron (2.226 MeV). Assuming an isotropic gamma ray source distribution, the energy spectrum of the delayed photoneutrons was calculated from this equation using the energy-dependent γ-fluxes in the $D_2O$ core regions and D(γ,n) cross sections for the photodisintegration of the deuteron (see Ref. 10, p.143). Table 11 shows this spectrum from which the average energy of the delayed photoneutrons was found to be 0.51 Mev. An evaporation spectrum (Eq. 21) corresponding to this average neutron energy was used in Eq. 14 to calculate the delayed photoneutron fractions $f_{dpn,j}$ given in Table 10. Note that this average energy is larger but not very different from the average energy of the delayed fission neutrons (see Tables 3 and 4). With these values for $F_\gamma$ and $f_{dpn,j}$ and data in Tables 5 and 6, Method I (Eqs. 10-12) and Method II (Eq. 13) were used to estimate ${}^{DPN}\beta_{eff}$.

**Table 11**

**Spectrum of Delayed Photoneutrons in $D_2O$ Channels for the 14-Element GTRR with HEU Fuel ($11.75g$ $^{235}U$ per Plate)**

| $E_\gamma$ - MeV | $\chi_\gamma$ | $E_n$ (ave) -MeV | $\sigma_{\gamma,n}$(ave) - mb[a] | $\chi_{pn}$ |
|---|---|---|---|---|
| < 2.226 | 0.57335 | | 0.0 | 0.0 |
| 2.226 - 2.50 | 0.19384 | 0.066 | 0.65 | 0.2327 |
| 2.50 - 3.00 | 0.09250 | 0.262 | 1.35 | 0.2306 |
| 3.00 - 3.50 | 0.07108 | 0.513 | 1.87 | 0.2455 |
| 3.50 - 4.00 | 0.01323 | 0.764 | 2.17 | 0.0530 |
| 4.00 - 4.50 | 0.03553 | 1.014 | 2.30 | 0.1509 |
| 4.50 - 5.00 | 0.01635 | 1.265 | 2.32 | 0.0701 |
| 5.00 - 5.50 | 0.00412 | 1.515 | 2.24 | 0.0172 |
| >5.50 | 0.0 | | | 0.0 |

[a]Taken from Ref. 10, p. 143.
Note: The average photoneutron energy is $\Sigma_i E_{n,i} \chi_{pn,i} = 0.51$ MeV.

Table 12 summarizes the calculations for the prompt neutron lifetime and the effective delayed neutron fractions (fission and photo). For reasons stated earlier, the VARI3D result for the prompt neutron lifetime is too small. Effective delayed fission neutron fractions from both Method I and Method II are in very good agreement with the VARI3D perturbation results. Table 12 shows that the Method II value for $^{DFN}\beta_{eff,2}$ is increased by about 0.2% if ages for the homogenized core are used (see Table 6). VARI3D does not calculate effective delayedphotoneutron fractions.

No errors are quoted in Ref. 17 for the $\beta_{eff}$ and $\beta_{eff}/l_p$ measurements. However, the measured value of $\beta_{eff}$ depends on a combination of reactor noise analyses and an absolute calibration of the reactor power (or fission rate). Ref. 17 quotes an error of 5.6% for this calibration. Since $\beta_{eff}$ depends inversely on the square root of the fission rate, the minimum error in the measurement of the effective delayed neutron fraction is 3%. Typical errors in pile oscillator measurements of $\beta_{eff}/l_p$ are in the 1-2% range. The experimental value for the prompt neutron lifetime is inferred from the $\beta_{eff}$ and $\beta_{eff}/l_p$ measurements.

The calculated values for the total effective delayed neutron fraction obtained by both Methods I and II are within the $1\sigma$ uncertainties of the measurement. This is also true for the prompt neutron decay constant $\beta_{eff}/l_p$ measurement. There may be more attenuation of the photoneutron fission product gamma ray precursors than are accounted for in the ORIGEN/MCNP calculations. Some of these gamma rays are lost in the GTRR cadmium shim/safety rods which were not included in the above

| **Table 12**<br>**Kinetic Parameters for the Heavy-Water GTRR HEU Core**<br>**with 14 Fuel Elements** | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Prompt Neutron Lifetime** | | | | | | | |
| | Exp. | VARI3D<br>. | 1/v Ins. | C/E Ratios | | | |
| | | | | VARI3D | 1/v Ins. | | |
| $l_p$ - $\mu$sec | 770.4 | 728.2 | 770.3 | 0.945 | 1.000 | | |
| **Effective Delayed Neutron Fractions** | | | | | | | |
| Delayed Fiss. n's | VARI3D | Method I | Method II[a] | Ratios Relative to VARI3D | | | |
| | | | | Meth. I | Meth. II | | |
| $\beta_{eff, 1}$ | 2.713E-4 | 2.721E-4 | 2.710E-4 | 1.003 | 0.999 | | |
| $\beta_{eff, 2}$ | 1.516E-3 | 1.521E-3 | 1.511E-3<br>1.514E-3[b] | 1.003 | 0.997<br>0.999[b] | | |
| $\beta_{eff,3}$ | 1.337E-3 | 1.339E-3 | 1.335E-3 | 1.001 | 0.998 | | |
| $\beta_{eff, 4}$ | 2.896E-3 | 2.901E-3 | 2.891E-3 | 1.002 | 0.998 | | |
| $\beta_{eff, 5}$ | 9.110E-4 | 9.140E-4 | 9.094E-4 | 1.003 | 0.998 | | |
| $\beta_{eff, 6}$ | 1.851E-4 | 1.859E-4 | 1.847E-4 | 1.004 | 0.998 | | |
| $\Sigma_i \beta_{eff, i}$ | 7.116E-3 | 7.133E-3 | 7.102E-3 | 1.002 | 0.998 | | |
| Delayed Photo n's<br>$^{DPN}\beta_{eff}$ | | 5.734E-4 | 5.581E-4 | | | | |
| | Exp. | Method I | Method II | C/E Ratios | | | |
| | | | | Meth. I | Meth. II | | |
| Total $\beta_{eff}$ | 7.55E-3<br>$1\sigma > 3\%$ | 7.706E-3 | 7.660E-3 | 1.021 | 1.015 | | |
| **Prompt Neutron Decay Constant, $\alpha = \beta_{eff} / l_p$** | | | | | | | |
| | Exp. | VARI3D | Method I | Method II | C/E Ratios | | |
| | | | | | VARI3D | Meth. I | Meth. II |
| $\alpha$ - $sec^{-1}$ | 9.8<br>$1\sigma \approx 1–2\%$ | | 10.0 | 9.94 | | 1.021 | 1.015 |
| **Coalesced Set of Delayed Neutron Decay Constants, $\lambda_i$ ($sec^{-1}$)** | | | | | | | |
| | Family 1 | Family 2 | Family 3 | Family 4 | Family 5 | Family 6 | |
| VARI3D | 1.272E-2 | 3.174E-2 | 1.160E-1 | 3.110E-1 | 1.400 | 3.870 | |
| $^mF\ ^m\nu_{d,i}$ wt'd | 1.272E-2 | 3.174E-2 | 1.160E-1 | 3.110E-1 | 1.400 | 3.870 | |

[a]$\beta_i$'s relative to $^{235}\nu$ = 2.43694 total neutrons/fission. [b]Based on hmg'd core ages (see Table 6).
calculations. This cadmium absorption would somewhat lower $F_\gamma$ but probably not enough to reduce the $\beta_{eff}/l_p$ C/E ratio to unity. However, these calculated kinetic parameters are thought to be accurate enough to analyze the transient behavior of the GTRR. Because this GTRR core contains fresh HEU fuel, the coalesced set of delayed fission neutron decay constants are just those for $^{235}U$ (Table 1). Decay constants for the delayed photoneutrons are given in Table 5.

## 6. SUMMARY AND CONCLUSIONS

Methods have been presented for calculating reactor kinetic parameters (the prompt neutron lifetime and family-dependent effective delayed neutron fractions) from a series of multigroup diffusion calculations which are independent of perturbation techniques. Accurate results for the prompt neutron lifetime are obtained from the 1/v insertion method. Two methods are given for calculating effective delayed fission neutron fractions. Method I depends on a diffusion theory eigenvalue calculation in which the spectrum and the yield of fission neutrons are adjusted to remove delayed fission neutron contributions. Method II is based on the continuous slowing down model and and on an homogenized core model. It requires slowing-down ages for both prompt and delayed fission neutrons. Although Method II is less rigorous than Method I, once the ages and geometric buckling have been determined only a pocket calculator is needed to compute the effective delayed neutron fractions. Monte Carlo methods were used to calculate the ages of prompt and delayed neutrons in both $H_2O$ and $D_2O$. The geometric buckling was found from least squares fits to the axial and radial neutron flux distributions in the homogenized core. Modified forms of Methods I and II are presented which may be used to determine the effectiveness of delayed photoneutrons. However, an independent calculation is needed to account for the loss from absorption, leakage, and energy degradation of fission product gamma rays with energies above the D($\gamma$,n) threshold.

These methods are illustrated by calculating $l_p$ and $\beta_{eff,i}$ values for the $H_2O$-moderated and $H_2O$-reflected ORR 179-AX5 core with fresh LEU fuel and for the $D_2O$-moderated and $D_2O$-reflected GTRR core with fresh HEU fuel. Results are compared with VARI3D perturbation calculations and, where possible, with measured values. The prompt neutron lifetime obtained from the 1/v insertion method is more accurate than the VARI3D result because this code does not use region-dependent multigroup neutron speeds. Family-dependent effective delayed fission neutron fractions obtained from both Methods I and II agree very well with results of VARI3D perturbation calculations for the ORR and GTRR cores. This close agreement was unexpected in view of the approximations upon which Method II depends. For the ORR core the calculated prompt neutron decay constant ($\beta_{eff}/l_p$) was found to be in good agreement with the directly-measured value. For the GTRR, however, the calculated $\beta_{eff}/l_p$ ratio, which includes delayed photoneutrons, is about 2% larger than the measured value, but within the estimated error limits. This calculation of $\beta_{eff}/l_p$ was done without modeling the control rods whereas the pile oscillator measurement was made with the GTRR at critical. Calculations done for the ORR core showed that the $\beta_{eff}/l_p$ ratio decreased by about 1.5% with the control rods at their critical elevation as compared with the no control rod model. This suggests that modeling the control rods in the GTRR would improve the C/E ratio. However, the control rod elevations during the pile oscillator measurements are unavailable.

The effectiveness of delayed photoneutrons in heavy water reactors depends on the attenuation of fission product gamma rays with energies greater than the binding energy of the deuteron and on the energy spectrum of photoneutrons in the $D_2O$. Using the ORIGEN code to determine a fission product γ-ray source distribution in the fuel meat, MCNP Monte Carlo photon calculations were performed to calculate the fission product gamma ray attenuation factor and to estimate, from D(γ,n) cross sections, the photoneutron energy spectrum in the $D_2O$ coolant channels. This procedure was used to calculate the effective delayed photoneutron fraction in the heavy-water GTRR with fresh HEU fuel.

These methods, including Method II, should be sufficiently accurate to determine the parameters needed to calculate the transient response of research reactors. Connolly, et al.[18] analyzed the transient behavior of the SPERT II BD22/24 heavy water moderated core with kinetic parameters taken from Ref. 10. They assumed the effectiveness of both delayed fission neutrons and delayed photoneutrons to be unity and even with this over-simplified approximation found good agreement between measured and calculated transients in the BD22/24 core.

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## APPENDIX: Expansion of the $J_0(x)$ Bessel Function

$$J_0(x)=\Sigma_{k=0}\,[(-1)^k\,(x/2)^{2k}\,/\,(k!)^2] = 1 - x^2/(2^2) + x^4/(2^4(2!)^2) - x^6/(2^6(3!)^2) + x^8/(2^8(4!)^2) - x^{10}/(2^{10}(5!)^2) + \ldots$$

For fits to a radial flux distribution in RZ geometry, $x = B_r r$.