Chapter 6

String Theory, String Model-Building, and String Phenomenology — A Practical Introduction

Keith R. Dienes

Department of Physics, University of Arizona, Tucson, AZ 85721
E-mail: dienes@physics.arizona.edu

This is the written version of an introductory self-contained course on string model-building and string phenomenology given at the 2006 TASI summer school. No prior knowledge of string theory is assumed. The goal is to provide a practical, "how-to" manual on string theory, string model-building, and string phenomenology with a minimum of mathematics. These notes cover the construction of bosonic strings, superstrings, and heterotic strings prior to compactification. These notes also develop the ten-dimensional free-fermionic construction. A final lecture discusses general features of heterotic string models, Type I (open) string models, and recent trends of string phenomenology. and general features of low-energy string phenomenology.

6.0. Introduction

These lectures were delivered at the 2006 Theoretical Advanced Study Institute (TASI), to an audience of graduate students whose interests were primarily oriented towards high-energy phenomenology. Indeed, this school had a stated focus on neutrino physics, and consequently my goal was to present string theory in a way that ultimately might explain how a specific particle such as a neutrino might ultimately emerge from string theory. Of course, string theory contains a lot more than neutrino physics (and also, in some ways, a lot less!), and in the course of these lectures I will not really focus so much on neutrinos as on string theory as a whole. Nevertheless, I will continue to keep neutrinos as a running theme throughout these lectures as a way of reminding ourselves that our discussion of string theory is ultimately aimed at understanding something real and observable, such as an actual neutrino.

The title of these lectures indicates that these lectures are meant to serve as a practical introduction to string theory, string model-building, and string phenomenology. Let me explain, in a rough sense, what each of these words is meant to convey. We are all familiar with quantum field theory, which is a *language* through which we might construct particular *models* of physics (such as the Standard Model or the Minimal Supersymmetric Standard Model). Such models then have certain physical characteristics, certain *phenomenologies*. String theory, at least as I shall try to present it, can likewise be considered as a *language* for discussing physics: in this sense it replaces quantum field theory (a language based on point-particle physics) with a new language suitable for theories whose fundamental objects are the one-dimensional extended objects known as *strings*. However, from this perspective, string theory is still only a language: it is still necessary to take the next step and *use* this language to construct *models* that describe the everyday world. Therefore, although I will attempt to give a self-contained introduction to the language of string theory, these lectures will primarily focus on the model-building aspects of string theory and on the resulting phenomenologies that these models have. While there already exist many excellent reviews of string theory, there are relatively few that focus on its model-building and phenomenological aspects. These lecture notes will therefore hopefully help to fill the gap, especially for those readers who might care less for the formal aspects of string theory and more for their phenomenological implications.

Finally, I should explain the word "practical" which also appears in the title. The word "practical" refers to actual *practice* — the things that practitioners actually need to know in order to build *bona-fide* string models and/or comprehend their low-energy properties. Of course, string theory is a rich and beautiful subject, with many mathematical aspects that are compelling and ultimately essential for a deep understanding of the subject. However, the goal of these lectures is simply to present the basic features of string theory with a minimum of mathematics — as stated in the abstract, I am seeking to provide a "how-to" manual which cuts the subject to the bone and conveys only that information which will be important for phenomenology. Therefore, in many places the omissions will be substantial. Certainly they do not do justice to the subject. However, these lectures were designed for phenomenologically-oriented graduate students whose desire (I hope) was to learn something of string theory without being deluged by mathematical formalism. It is with them in mind that I designed these lectures to be as elementary as feasible, and to "get to the physics" as rapidly as possible. Therefore, I now issue the following

> **Warning:** These lectures are meant to cover a considerable amount of introductory material very rapidly and without mathematical sophistication. The purpose is to advance quickly to the model-building and phenomenological aspects of string theory, while still conveying an intuitive flavor of the essential issues. The target audience consists of people who have had no prior exposure to string theory, and who wish to understand the basic concepts from a purely phenomenological perspective.

Hopefully, the students came away with a sense that string theory is a real part of physics, one with direct relevance for the real world. Perhaps the reader will too. If so, then these lectures will have served their purpose.

6.1. Lecture #1: Why strings? — an overview

Why should we be interested in string theory? In this lecture, we shall review our present state of knowledge about the underlying constituents of matter, and discuss how string theory has the potential to extend that knowledge in a profoundly new direction. Since this lecture is meant only as an overview, we shall keep the discussion at an extremely superficial level and seek to present the intuitive *flavor* of string theory rather than its substance. We shall deal with the substance in subsequent lectures.

6.1.1. *From atoms to the Standard Model: A quick review*

Certainly we do not need to understand string theory in order to appreciate modern high-energy particle physics, or to understand or interpret the results of collider experiments. Why then should one study string theory, a subject whose connections to observable phenomena are usually considered rather tenuous at best?

The primary reason, of course, is that the goal of high-energy physics has always been to uncover the fundamental "elements" or building-blocks of the natural world. These consist of both the fundamental *particles* that make up the *matter*, and the fundamental *forces* that describe their *interactions*. In this way, we hope to expose the underlying laws of physics in their simplest forms.

But what is "fundamental"? Clearly, the answer depends on the energy scale, or equivalently the inverse length scale, at which these constituents are being probed. In order to establish our frame of reference, recall that 1 eV $\approx 1.6 \times 10^{-19}$ Joules $\approx (10^{-7}$ meters$)^{-1}$. At the eV scale, the fundamental objects are atoms, or nuclei plus electrons. But it turns out that there are many different types of atoms or nuclei — indeed, they fill out an entire periodic table, the complexity but regularity of which suggests a deeper substructure. And indeed such a deeper substructure exists: at the keV to MeV scale, the nuclei are no longer fundamental, but decompose into new fundamental objects — protons and neutrons. Thus, at this energy scale, the fundamental objects are protons, neutrons, and electrons. But once again, it is found that there are many different "types" of protons and neutrons — collectively they are called *hadrons*, and include not only the proton (p) and neutron (n), but also the pions (π), kaons (K), rho (ρ), omega (Ω), and so forth. Indeed, the "periodic table of the elements" at this energy scale is nothing but the Particle Properties Data Book! But once again, the complexity and regularity of these "elementary" particles suggests a deeper substructure, and indeed such a substructure is found, this time at the GeV scale: the proton and neutron are just made of two kinds of *quarks*, the so-called up and down quarks. Thus, at the GeV scale, the fundamental objects are up quarks, down quarks, and electrons. But once again complexity emerges: it turns out that there are many different "types" (flavors) of quarks: up, down, strange, charm, top, and bottom. Likewise, there are many different "types" of electrons (collectively called *leptons*): the electron, the muon, the tau, and their associated neutrinos. And indeed, once again there is a mysterious pattern, usually referred to as a family or generational structure. This once again suggests a deeper substructure.

Unfortunately, this is as far as we've come. Indeed, all of our present-day knowledge down to this energy scale is gathered together into the so-called *Standard Model* of particle physics. The primary features of the Standard Model are as follows. The fundamental *particles* are the quarks and leptons. They are all fermions, and are arranged into three generations of doublets:

$$\begin{array}{ll} \text{quarks}: & \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \\ \text{leptons}: & \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}. \end{array} \tag{6.1.1}$$

The fundamental *forces* also come in three varieties. First, there is the strong (or "color") force, associated with the non-abelian Lie group $SU(3)$.

Its fine-structure constant is $\alpha_3 \approx 1/8$ (as measured at energy scales of approximately 100 GeV), and it is responsible for binding quarks together to form hadrons and nuclei. As such, it is felt only by quarks. Its mediators or carriers are called *gluons*. Second, there is the electroweak force, associated with the non-abelian Lie group $SU(2)$. Its fine-structure constant is $\alpha_2 \approx 1/30$ (indeed, weaker than the strong force!), and it is responsible for β-decay. Unlike the strong force, it is felt by *all* of the fundamental particles. Finally, there is the "hypercharge" force, associated with the abelian Lie group $U(1)$, with fine-structure constant $\alpha_1 \approx 1/59$. Once again, this force is felt by essentially *all* particles, both quarks and leptons. The carriers of the latter two forces are the photon as well as the $W^{\pm}$ and Z particles. Indeed, ordinary electromagnetism is a combination of the electroweak and hypercharge forces, and is the survivor of electroweak symmetry breaking. This breaking is induced by the one remaining particle of the Standard Model, a boson called the *Higgs particle*. An excellent introduction to the physics of the Standard Model can be found in the TASI lectures of G. Altarelli (this volume).

6.1.2. *Beyond the Standard Model: Two popular ideas*

Is that all there is? Clearly, there are lots of reasons to believe in something deeper! First, the Standard Model contains many *arbitrary parameters*, such as the masses and "mixings" of fundamental particles. All of these must ultimately be *fit* to data rather than *explained*. Second, there are many *conceptual* questions. *Why* are there three generations? *Why* are there three kinds of forces? *Why* do these forces have different strengths and ranges? A fundamental theory should explain these features. Finally, there is also another force which we have not yet mentioned: the *gravitational* force. How do we incorporate the gravitational force into this framework? In other words, how do we "quantize" gravity?

There is only one conclusion we can draw from this state of affairs. Just as in each previous case, there must still be a deeper underlying principle. It is important to stress that this is not simply an issue of academic interest. Rather, it is one of practical importance, because the next generation of particle accelerators are being built right now! (Two of the most prominent that will be exploring physics beyond the Standard Model are Fermilab, where upgrades to the TeVatron are being implemented, and CERN, where construction of the Large Hadron Collider (LHC) is already underway.) The pressing question, therefore, is: What do we expect to see at these machines? What will high-energy physics be focusing on over the next ten to twenty years? It turns out that there are two very popular sets of ideas, both of which are thoroughly reviewed in the TASI lectures of N. Polonsky.

6.1.2.1. *Low-energy supersymmetry*

The first idea is *supersymmetry* (SUSY). This refers to a new kind of symmetry in physics, one which relates bosons (particles with integer spin) to fermions (particles with half-integer spin). Thus, for every known particle, there is a predicted new particle, its so-called superpartner:

$$\begin{aligned} \text{quarks} &\Longleftrightarrow \textit{squarks} \\ \text{leptons} &\Longleftrightarrow \textit{sleptons} \\ \text{gauge bosons} &\Longleftrightarrow \textit{gauginos}\ . \end{aligned} \tag{6.1.2}$$

Clearly, this implies the existence of a *lot* of new particles and a *lot* of new interactions! Why then go through all this trouble?

Well, it turns out that supersymmetry can provide a number of striking benefits. First, through supersymmetry, we can explain the relative strengths of the forces ("gauge coupling unification"). Second, we can explain the origin of electroweak symmetry breaking. Third, supersymmetry has a number of favorable cosmological implications (for example, supersymmetry provides a natural set of dark-matter candidates). Finally, it turns out that supersymmetry is the only known answer to certain difficult theoretical puzzles in the Standard Model (chief among them the so-called "gauge hierarchy problem", *i.e.*, the difficulty of explaining the lightness of the Higgs particle, or equivalently to difficulty of explaining the stability of the scale of electroweak symmetry breaking against radiative corrections). In order to serve as an explanation of the gauge hierarchy problem, the energy scale associated with supersymmetry must not be too much higher than the scale of electroweak symmetry breaking. This is therefore called "low-energy supersymmetry", which refers to the common expectation that superparticles should exist at or near the TeV-scale.

Supersymmetry is a beautiful theory, both phenomenologically and mathematically. But it is *not* observed in nature. Therefore, supersymmetry must be broken. The problem, however, is that supersymmetry is very robust! It turns out to be quite hard to find mechanisms that can easily ("spontaneously") break supersymmetry at the expected energy scales. Therefore, we are faced with a major unsolved problem: *How do we break supersymmetry?* Indeed, we often have to resort to introducing

SUSY-breaking by hand, which requires the introduction of *many* additional unknown parameters. This is quite unpleasant, not only from an aesthetic point of view but also a phenomenological (predictive) point of view. However, it is often possible to consider only a minimal supersymmetric extension to the Standard Model (the so-called MSSM) where a minimal number of supersymmetry-breaking parameters are chosen.

6.1.2.2. *Grand unification*

The second popular idea for physics beyond the Standard Model concerns so-called *Grand Unified Theories* (GUTs). This refers to an attempt to realize the different forces and particles in nature as different "faces" or "aspects" of a single GUT force and a single GUT particle. An electromagnetic analogy here might be useful. Recall that the electric force is felt or caused by static charges, and that the magnetic force is felt or caused by moving charges. Are these therefore different forces? As we know, the answer is most definitely "no": we can Lorentz-boost from a rest frame to a moving frame, whereupon the distinction between the electric and magnetic forces melts away and these forces become intertwined. Thus, we conclude that the electric and magnetic forces are merely different aspects of *one* force, the "electromagnetic" force.

Is the same true for the strong, electroweak, and hypercharge forces? Is there a single "strong-weak-hypercharge" GUT force?

At first glance, this doesn't seem possible, because these different forces have different strengths. Recall their fine-structure constants: $\alpha_1 \approx 1/59$, $\alpha_2 \approx 1/30$, and $\alpha_3 \approx 1/8$. However, also recall that in quantum field theory, the strengths of forces ultimately depend on the energy scale through which they are measured. To see why this is so, let us think of placing a positive charge next to a dielectric. The positive charge draws some negative charge from within the dielectric towards it, so that the dielectric medium partially screens the positive charge. Therefore, in a rough sense, the less of the dielectric we see (*i.e.*, the more finely resolved our experimental apparatus to probe the original positive charge), the stronger our original positive charge seems to be. Thus, we see that at shorter distances (corresponding to higher energies), our electric charges (and therefore the corresponding electric forces) appear to be stronger. If this dielectric analogy serves as a good model for the results of a true quantum field-theoretic calculation (and in this case it does), we conclude that the electric force appears to grow stronger with increasing energy.

Of course, this is just a mechanical analogy. However, in the supersymmetric Standard Model, it turns out that the quantum field-theoretic vacuum itself indeed behaves like a dielectric for the hypercharge and weak forces. However, for the strong force, it behaves as an *anti*-dielectric. Thus, while the hypercharge and electroweak forces become *stronger* at higher energies, the strong force becomes *weaker* at higher energies. (This latter feature is the celebrated phenomenon of *asymptotic freedom.*) Together, these observations imply that these three forces have a chance of *unifying* at some energy scale if their strengths become equal, and indeed, carrying out the appropriate calculations, one finds the results shown in Fig. 6.1. We see from this figure that the forces appear to unify at the scale

$$M_{\rm GUT} \approx 2 \times 10^{16}\ {\rm GeV}\ . \tag{6.1.3}$$

This would then be the natural energy scale for grand unification. Note that this unification also requires the existence of weak-scale supersymmetry in the form of weak-scale superpartners. Without such superpartners, the evolution of these fine-structure constants as a function of the energy scale is different, and they fail to unify at any scale. This then serves as another motivation for weak-scale supersymmetry.

GUTs would have numerous important effects on particle physics. First, by their very nature, they would imply new interactions that can mix the three fundamental forces. Second, this in turn implies that GUTs naturally lead to new, rare decays of particles. The most famous example of this is proton decay, the rate for which is experimentally known to be exceedingly small (since the proton lifetime is $\tau_p \gtrsim 10^{32}$ years). Third, GUTs would naturally explain the quantum numbers of all of the fundamental particles. Along the way, GUTs would also explain charge quantization. GUTs might also explain the origins of fermion mass. Finally, because they generally lead to baryon-number violation, GUTs even have the potential to explain the cosmological baryon/anti-baryon asymmetry. By combining GUTs with supersymmetry in the context of SUSY GUTs, it might then be possible to realize the attractive features of GUTs simultaneously with those of supersymmetry in a single theory.

Both the SUSY idea and the GUT idea are very compelling. Certainly, the SUSY idea (and indirectly the GUT idea, through measurements of proton decay and other rare decays) will be the focus of experimental high-energy physics over the next 20 years. But high-energy theorists also have plenty of work to do — we must build theories in order to interpret the data. But *how* do we build realistic SUSY theories? *How* do we build

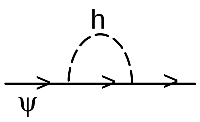

Fig. 6.1. One-loop evolution of the gauge couplings within the Minimal Supersymmetric Standard Model (MSSM), assuming supersymmetric thresholds at the Z scale. Here $\alpha_1 \equiv (5/3)\alpha_Y$, where α_Y is the hypercharge coupling in the conventional normalization. The relative width of each line reflects current experimental uncertainties.

realistic GUT theories? *How* do we incorporate gravity?

Clearly, the possibilities seem endless. And even the SUSY or GUT ideas have not answered our most fundamental questions, such as why there are three gauge forces, or why there are three generations. Therefore, it is natural to hope that there is yet a deeper principle that can provide some theoretical guidance. And that's where string theory comes in.

6.1.3. *So what is string theory?*

The basic premise of string theory is very simple: all elementary particles are really closed vibrating loops of energy called *strings*. The length scale of these loops of energy is on the order of 10^{-35} meters (corresponding to 10^{19} GeV), so it is not possible to probe this stringy structure directly.

This idea has great power, because it provides a way to unify all of the particles and forces in nature. Specifically, each different elementary particle can be viewed as corresponding to a different vibrational mode of the string. A pictorial representation of this idea is given in Fig. 6.2, where we are schematically associating higher vibrational string modes with string loops containing more "wiggles". From the point of view of a low-energy observer who cannot make out this stringy structure, the different excitations each appear to be point particles. However, to such an observer, the states with more underlying "wiggles" appear to have higher spin. Thus, in this way we find that string theory predicts not only spin-1/2 and spin-1 states (which can be associated with the fermions and gauge bosons of the Standard Model respectively), but also a spin-2 state (which can naturally be associated with the graviton). Thus, through string theory, we see that the gauge interactions, particles, *and also gravity* are unified into a common quantized description as corresponding to different excitation modes of a single fundamental entity, the string itself.

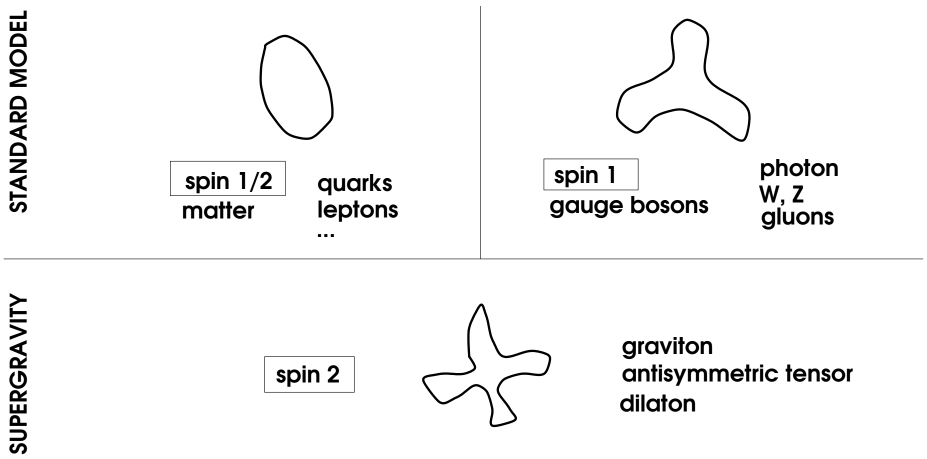

Fig. 6.2. The basic hypothesis of string theory is that the different elementary particles correspond to the different vibrational modes of a single fundamental entity, a closed loop of energy called a string. In this way one obtains not only spin-1/2 and spin-1 states which can be associated with the matter and gauge bosons of the Standard Model, but also a spin-2 state which can be identified with the graviton. Thus, string theory provides a way of unifying the Standard Model with gravity.

Of course, this is not the end of the story. Just as a violin string has an infinite number of harmonics, so too does a string give rise to an infinite tower of states corresponding to higher and higher vibrational modes. Since it takes more and more energy to excite these higher vibrational string modes, such states are increasingly massive. Indeed, because the fundamental string scale is on the order of $M_{\text{string}} \approx 10^{18}$ GeV, these string states are quantized in units of M_{string}. The states which we have illustrated in Fig. 6.2 are all *massless* with respect to M_{string}, and correspond, in some sense, to the ground states of the string. These are the so-called "observable states", and include not only the (supersymmetric) Standard Model and (super)gravity, but also may include various additional states (often called "hidden-sector states" which contain their own matter and gauge particles). However, there also exists an infinite tower of massive states with masses $M_n \approx \sqrt{n} M_{\text{string}}$, $n \in \mathbb{Z}^+$. In most discussions of the phenomenological properties of string theory, these massive states are ignored (since they are so heavy), and one concentrates on the phenomenology of the massless states. One then presumes that they accrue (relatively small) masses through other means, such as through radiative corrections.

Nevertheless, the passage from point particles to strings has tremendous consequences. Not only have we replaced the physics of zero-dimensional objects (elementary point particles) with the physics of one-dimensional objects (strings), but we have also replaced the physics of the one-dimensional worldlines that they sweep out with the physics of two-dimensional so-called *worldsheets*. Likewise, we have replaced the physics of Feynman diagrams with the physics of two-dimensional *manifolds*, so that a tree diagram corresponds to a genus-zero manifold (a sphere) and a one-loop diagram corresponds to a genus-one manifold (a torus). These comparisons are illustrated in Fig. 6.3. Note that the latter descriptions as spheres and tori correspond to shrinking the external strings to points, essentially "pinching off" the external legs. This is a valid description for reasons to be discussed in Lecture #2.

This is clearly a new language for doing physics. However, as we have seen, because string theory also includes gravity (which is exceedingly weak compared with the other forces), its fundamental mass scale is very high. Indeed, since the fundamental energy scale for gravity is the Planck mass

$$M_{\text{Planck}} \equiv \sqrt{\frac{\hbar c}{G_N}} \approx 10^{19} \text{ GeV} \approx (10^{-33} \text{ cm.})^{-1} , \tag{6.1.4}$$

the string scale must also be very high. Indeed, to a first approximation, it turns out that

$$M_{\text{string}} \approx g_{\text{string}} M_{\text{Planck}} \tag{6.1.5}$$

where g_{string} is the string coupling constant, typically assumed to be $\sim \mathcal{O}(1)$. Thus, we see that string theory is ultimately a theory of Planck-scale physics.

There are lots of "formal" reasons for being excited about string theory. First, it turns out that string theory requires the existence of extra spacetime dimensions in order to be consistent, and consequently we now have to consider physics in different numbers of dimensions as well as all sorts of geometric questions pertaining to different possible "compactification" scenarios. Second, string theory gives us a new perspective on the structure of spacetime itself. For example, string theory gives rise to many novel Planck-scale effects. One of these is called T-duality: the physics of a closed string in a spacetime one of whose dimensions is compactified on a circle of radius R turns out to be *equivalent* to the physics of the same string in a spacetime in which the radius is M_{string}^2/R. Thus, T-duality interchanges large radii and small radii, and suggests that our naïve view of spacetime and its linear hierarchy of energy and length scales cannot ultimately be correct. Third, string theory also provides new types of strong/weak coupling dualities. These have proven useful for elucidating the strong-coupling dynamics of not only string theory, but also *field* theory. Finally, there have even been novel applications to black-hole physics. The most famous example of this is the fact that various non-perturbative string structures called D-branes have provided the first *statistical* (*i.e.*, microscopic) derivation of the Bekenstein-Hawking entropy formula $S = A/4$ that relates the entropy S of a black hole to its surface area A. Indeed, the above list only begins to scratch the surface of all of the many exciting recent formal developments in string theory.

But we are phenomenologists, so it is natural to ask about the rest of high-energy physics. How does string theory connect with the rest of particle physics?

Some of the answers to this question have already been given above. We have seen, in particular, that string theory is capable of reproducing the Standard Model as its low-energy limit. Moreover, as we have also seen, the Standard Model naturally emerges coupled with gravity. Furthermore, in many cases this entire structure is also joined with *supersymmetry.* Finally, this entire structure is also often joined with many properties of GUTs (such as gauge coupling unification). All of this comes out of the low-energy limit of string theory, in some sense automatically.

There are also many other benefits to considering the application of string theory to particle physics. First, string theory provides us with new kinds of symmetries (so-called "worldsheet symmetries") which lead to powerful new constraints on the resulting low-energy phenomenology. Second, in principle* string theory has *no free parameters*, which leads to a very predictive theory. Third, string theory has no divergences — in some sense, string theory is a completely *finite* theory in which many of the troublesome divergences associated with field theory are simply absent. Finally, it turns out that string theory can even give rise to a new perspective on the Standard Model itself, and often provides new and simpler ways to perform calculations.

These last three points (absence of free parameters, absence of divergences, and new ways to perform calculations) are truly remarkable. Therefore, let us pause to explain in an intuitive way why these features arise. First, let us explain why string theory has fewer free parameters. To do this, let us consider a Feynman diagram for a typical tree-level decay $A \to B+C$, as shown in Fig. 6.4(a). In field theory, such a process depends on many separate parameters ultimately associated with the separate propagators and vertices. Specifically, even though the propagators are determined once the masses and spins of the particles are specified, there still remains an *independent* choice as to the form of the vertex interaction. Thus, in a given field theory, there still remain many independent parameters to choose. In string theory, by contrast, there is no sharp distinction between propagators and vertices; they melt into each other, and are essentially the same. Thus, once the propagators are determined, the vertices are also intrinsically determined. This is one of the underlying reasons why string theory contains fewer free parameters than field theory.

Next, let us discuss why string theory is more finite than field theory. To do this, let us consider a typical one-loop Feynman diagram, as shown in Fig. 6.4(b). In field theory, the virtual interactions occur at sharp spacetime locations x and y. This is ultimately the origin of the ultraviolet (*i.e.*, short-

*In this connection, we hasten to emphasize the phrase "in principle". Unfortunately, our relative inability to understand the non-perturbative structure of string theory often means that the pragmatic consequences of having no free parameters cannot be realized, and in practice one is often forced to introduce many parameters to reflect our ignorance of the underlying dynamics. This will be discussed in subsequent lectures. This situation is rather analogous to one that arises in the MSSM: we do not know how supersymmetry is broken, so we typically parametrize our ignorance through the introduction of various supersymmetry-breaking parameters. Likewise, in string theory, there are analogous questions which come under the heading of "vacuum selection": we do not know how the non-perturbative dynamics of string theory selects a particular vacuum state. Thus, in order to proceed to make phenomenological predictions, we are often forced to *assume* a certain vacuum state, or to parametrize the vacuum via the introduction of essentially unfixed parameters. The important point, however, is that string theory is a complete theory in that it should *in principle*, by virtue of its dynamics, uniquely fix the values of all of its fundamental parameters.

distance) divergence as $x \to y$. In string theory, by contrast, we have seen that there are no such sharp interaction points — essentially the interaction is "smoothed out" by the presence of the string. Thus, there is no sense in which the dangerous $x \to y$ limit exists, for there are no precise means by which one can define such interaction locations x and y. It is in this manner that string theory automatically removes ultraviolet divergences: the string itself, through its extended geometry, acts as a (Planck-scale) ultraviolet regulator.

Finally, let us discuss why string theory can often give us simpler ways to perform calculations than in field theory. To do this, let us consider the total tree-level amplitude for a typical process $A + B \to C + D$, as illustrated in Fig. 6.4(c). As we know, in field theory there are two separate topologies of Feynman diagram that must be separately considered: the s-channel diagram and the t-channel diagram. In general, at any given order, there are many separate diagrams to evaluate, and one often finds that great simplifications and cancellations occur only when these individual contributions are added together. In string theory, by contrast, there is only *one* corresponding diagram to evaluate at any given order. Thus, the sorts of simplifications or cancellations that might occur in field theory are automatically "built into" string theory from the very beginning. In some sense, string theory manages to find a way to reorganize the field-theory diagrams in a perturbative expansion in a useful and potentially profitable way. Indeed, this observation has even led to the development of many new techniques for evaluating complicated field-theoretic processes, particularly in QCD where the number of diagrams and the number of terms in each diagram can easily grow to otherwise unmanageable proportions.

We thus see that in a number of ways, string theory is a very useful language in which we might consider thinking about particle physics. Indeed, in various aspects (such as finiteness, fewer parameters, *etc.*) it is superior to field theory. But overall, the fundamental fact remains that if we are thinking about strings, we are abandoning our usual four-dimensional point of view of particle physics. Specifically, since each different particle in spacetime is now interpreted as a different quantum mode excitation of an underlying string, we see that four-dimensional (spacetime) physics is now ultimately the consequence of two-dimensional (worldsheet) physics. Thus, everything we ordinarily focus on in field theory (such as the four-dimensional particle spectrum, the gauge symmetries, the couplings, *etc.*) are now all ultimately determined or constrained by worldsheet symmetries.

And this brings us to string phenomenology.

6.1.4. *So what is string phenomenology?*

In order to understand what string phenomenology is, we can draw a useful analogy. Just as we are replacing the *language* of high-energy physics from field theory to string theory, we likewise replace field-theory phenomenology with string-theory phenomenology. The goals of string phenomenology are of course the same as those of ordinary field-theory phenomenology: both seek to reproduce, explain, and predict observable phenomena, and both seek to suggest or constrain new physics at even higher energy scales. Indeed, only the language in which we will carry out this procedure has changed. Thus, in some sense, string phenomenology is the "art" of using the new insights from string theory in order to understand, explain, and predict what physics at the next energy scale is going to look like. Or, recalling that string theory is ultimately a theory of Planck-scale physics, we can say that string phenomenology is the "interplay" or "meeting-ground" between Planck-scale physics and GeV-scale physics.

It is important to understand that we are not abandoning field theory completely. Nor would we want to. Field theory automatically incorporates many desirable features such as causality, spin-statistics relations, and CPT invariance (which in turn implies the existence of antiparticles). These are all generic predictions of field theory, and are the underlying reasons why field theory is the appropriate language for particle physics. However, since string theory ultimately reduces to field theory in its low-energy limit, all of these features will still be retained in string theory. Moreover, as we have seen, string theory *additionally* predicts or explains gravity, supersymmetry, and the absence of ultraviolet divergences. Furthermore, as we shall see, string theory also automatically predicts the existence of gauge symmetry, and even incorporates features such as gauge coupling unification. These are all generic predictions of string theory. It is for these reasons to believe that a change in language from field theory to string theory might be useful.

String theory will also provide us with new tools for model-building, new mechanisms and new guiding principles. Let us give some examples. In field theory, there are many well-known ideas that are part and parcel of the model-building game: one must enforce ABJ anomaly cancellation (to preserve gauge symmetries); one can employ the Higgs mechanism (to generate spontaneous symmetry breaking and give masses to particles); one has the GIM mechanism (to preserve flavor symmetries); and one has supersymmetry (to cancel quadratic divergences). Likewise, in string theory there are analogous sets of ideas, many of which are extensions of their field-theory counterparts. For example, one has the so-called "Green-Schwarz" mechanism for anomaly cancellation (to preserve gauge symmetries); one has string vacuum shifting via pseudo-anomalous $U(1)$ gauge symmetries (to generate spontaneous symmetry breaking and generate particle masses); one has spacetime compactification (to generate gauge symmetries); one has hidden string sectors (to break supersymmetry and impose selection rules); and one has massive towers of string states (to enforce finiteness). Thus, model-building proceeds, but with a different set of principles.

There is also a much more subtle effect of changing our language from field theory to string theory. Ultimately, since four-dimensional physics is now derived from an underlying two-dimensional (worldsheet) theory, string phenomenology is ultimately much more constrained than field-theory phenomenology. One given worldsheet symmetry, which might serve as an "input", can have various seemingly unrelated effects in the resulting spacetime phenomenological "output". Thus, string theory not only leads to unexpected connections or correlations between seemingly disparate spacetime phenomena, but can also give rise to entirely new phenomenological scenarios that could not have been anticipated within field theory alone. We will see many examples of this in the coming lectures.

Thus, we see that string phenomenology does many things and has many goals:

- to provide a new framework for addressing and answering numerous phenomenological questions;
- to provide a rigorous test of string theory as a theory of physics;
- to explore the interplay between worldsheet physics and spacetime physics (*i.e.*, to ultimately determine which "patterns" of low-energy phenomenology are allowed or consistent with being realized as the low-energy limit of an underlying string theory); and
- to augment field theories of "low-energy" physics into the string framework so as to give them the full benefits of the language of string theory.

Because of these different roles, string phenomenology occupies a rather central position in high-energy physics: it allows the transmission of ideas from high-scale string theory to guide "low"-scale particle physics, and vice versa. This situation is illustrated in Fig. 6.5. At the lowest energies (lower left), string phenomenology has direct relevance for the Standard Model, where it can potentially explain features such as the choice of the gauge group, the number of generations, and numerous other parameters such as the masses and mixings of Standard-Model particles. At slightly higher energies (lower right), we see that string phenomenology can also suggest or constrain various extensions to the Standard Model, such as SUSY and SUSY-breaking, grand unification, and hidden-sector physics. At the highest energies (upper left), string phenomenology is also concerned with the more formal aspects of string theory: such important questions include string vacuum selection, non-perturbative string dynamics, string duality, and new mathematical structures and techniques. And string phenomenology even has relevance outside the strict confines of particle physics. For example, string theory should have a profound impact on cosmology (upper right), where important stringy issues include the role of the dilaton, the effects of many other light degrees of freedom (the so-called *moduli*), the possibility of extra spacetime dimensions, the cosmological constant problem, and even more exotic ideas such as topology change. As illustrated in Fig. 6.5, string phenomenology sits at the center of this web of ideas. Exploring the connections between the different corners of this figure is, therefore, the job of the string phenomenologist. Indeed, through string phenomenology, one "uses" string theory in order to open a window into the possibilities for physics beyond the Standard Model.

6.1.5. *Plan of these lectures*

For much of the past decade, string phenomenology has been practiced assuming a particular type of underlying string theory, the so-called *perturbative heterotic string*. Therefore, this string will be the focal point of most of these lectures. However, it turns out that the heterotic string is built directly on the foundations of two other kinds of strings, the *bosonic string* and *Type II superstring*. Indeed, in a sense to be made more precise in Lecture #5, one can view the heterotic string as the "sum" of the bosonic string and the superstring string. Therefore, in these lectures, we will have to start at the beginning by studying first the bosonic string, then the Type II string, and finally the heterotic string. Indeed, this situation is analogous to the way in which one often studies quantum field theory: first one learns how to quantize the Klein-Gordon field, then the Dirac field, and finally the gauge field. In a certain sense, the bosonic string is the analogue of the Klein-Gordon field, while the Type II superstring is the analogue of the Dirac field and the heterotic string is the analogue of the gauge field. Of course, this analogy is only a pedagogical organizational one, since the heterotic string itself will ultimately contain *all* of the phenomenological properties (*e.g.*, scalars, fermions, and gauge symmetries) that we desire.

In Lecture #2, we will therefore give a brief introduction to the bosonic string, stopping only long enough to develop the ideas and techniques we will need for later applications. In Lectures #3 and #4, we will then proceed to develop the Type II superstring, once again focusing on only those aspects that will be useful for later applications. Finally, in Lecture #5, we will arrive at our destination: the heterotic string. In Lecture #6 we will construct some ten-dimensional heterotic string models, and in Lecture #7 we will develop a useful set of rules for heterotic string model-building.

It is important to note, however, that all of string phenomenology is not based on the heterotic string. Particularly over the past decade, there has been a profound shift in our understanding of both string theory and its phenomenological implications. One of the consequences of this so-called "second superstring revolution" has been a new emphasis on yet another class of strings, the *Type I (open) strings*. Within this class, so-called *intersecting D-brane models* have shown great promise in yielding chiral, Standard-Model-like spectra. Indeed, there has even emerged a new superstructure which promises to relate all of these strings to each other: this structure is called *M-theory*, and is deeply tied to many non-perturbative aspects of string theory which are still being understood. Needless to say, these recent developments have the potential to completely change the way we think about string theory and string phenomenology. We will therefore discuss some of these modern developments in the final Lecture #8. Nevertheless, the bulk of these lectures will primarily be focused on the more traditional aspects of string phenomenology that concern the weakly coupled heterotic string. Indeed, this affords the best introduction to string theory and string phenomenology, regardless of the future directions that string theory and string phenomenology might ultimately take.

We also remind the reader that our goal here is to provide an introduction to string theory that avoids mathematical complications wherever possible, and which "gets to the physics" as rapidly as possible. Therefore, in many places, we will simply assert a mathematical result to be true, leaving its derivation to be found in various textbooks on the subject. For this purpose, we recommend Volume I of the textbook *Superstring Theory*, by M.B. Green, J.H. Schwarz, and E. Witten (henceforth to be referred to as GSW[†]). In fact, our initial approach will be very similar to that of GSW,

[†]Not to be confused with another great GSW trio, namely Glashow, Salam, and Weinberg. One can only hope that someday string theory will be as well-established, both theoretically and experimentally, as the GSW electroweak theory. This may sound a

and we will continually refer back to this textbook as we proceed. Another recommended textbook with a more modern mathematical perspective is *Introduction to String Theory*, by J. Polchinski. Likewise, *A First Course in String Theory* by B. Zwiebach is particularly useful for students who may lack a full background in relativistic quantum field theory.

6.2. Lecture #2: Strings and their spectra: The bosonic string

6.2.1. *The action*

We begin by studying the simplest string of all: the bosonic string. As we discussed in Lecture #1, the physics of a string is ultimately described by the shape it takes (*e.g.*, its vibrational mode of oscillation) as it propagates through an external spacetime and thereby sweeps out a two-dimensional worldsheet. Therefore, we must first have a way of describing the shape of this worldsheet. To this end, we parametrize the worldsheet by two worldsheet coordinates (σ_1, σ_2) as illustrated in Fig. 6.6, and describe the embedding of this worldsheet into the external spacetime by giving the spacetime coordinates X^μ of any location (σ_1, σ_2) on the worldsheet. Thus, the physics of the string is ultimately encapsulated in the embedding functions $X^\mu(\sigma_1, \sigma_2)$, where $\mu = 0, 1, ..., D-1$. Here D is the total spacetime dimension, which we shall keep arbitrary for now.

Given these embedding functions, we can attempt to write down an appropriate action for the string. To do this, we first note that as we might expect, strings have *tension* — *i.e.*, strings generically have a non-zero energy per unit length. In other words, it takes energy to stretch a string and to give the worldsheet a larger area. Thus, as the string propagates along in spacetime, we expect on physical grounds that this string should choose a configuration that minimizes the area of the worldsheet. This leads us to identify the string action with the area of the corresponding worldsheet. Indeed, this results in the so-called *Nambu-Goto action*, which involves a non-trivial square root of the X^μ coordinates. For certain calculational purposes, however, this square root is often problematic. Fortunately, however, there exists an alternative action, the so-called *Polyakov action*, which is classically equivalent to the Nambu-Goto action but which does not involve

bit optimistic, but a possible new *experimental* direction for string theory and string phenomenology will be discussed in Lecture #8 in the context of the brane world.

fractional powers of the X coordinates. This action is given by

$$S \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{h}\,h^{\alpha\beta}\,g_{\mu\nu}\,\partial_\alpha X^\mu\partial_\beta X^\nu\ . \qquad (6.2.1)$$

Here $g_{\mu\nu}$ is the metric of the external spacetime, $h_{\alpha\beta}$ is the metric of the worldsheet, the worldsheet derivative is given by $\partial_\alpha \equiv \partial/\partial\sigma^\alpha$, and $h \equiv \det h_{\alpha\beta}$. In the prefactor, α' is a dimensionful constant (called the *Regge slope*) with units of (length)^2. Since these units are equivalent to length/energy, we see that α' is an inverse tension, and indeed the string tension T turns out to be related to α' via $T = (2\pi\alpha')^{-1}$. We shall discuss the numerical value of α' below. Note that the action (6.2.1) is manifestly spacetime Lorentz-invariant.

Before proceeding further, it may be useful to draw an analogy between this action and the analogous action for a *point* particle propagating through spacetime and sweeping out a *worldline* rather than a worldsheet. The worldline can be parametrized by a single coordinate σ, which functions as a proper time along the worldline. The point-particle action can then be written in the form

$$S_{\text{point particle}} \;=\; \tfrac{1}{2}\int d\sigma\;\left(e^{-1}\,g_{\mu\nu}\partial_\sigma X^\mu\partial_\sigma X^\nu - e\hat{m}^2\right) \qquad (6.2.2)$$

where $\hat{m}$ is the mass of the point particle and where $e(\sigma)$ is an auxiliary field (a so-called *einbein*). Solving for $e(\sigma)$ through its equation of motion and substituting back into (6.2.2) yields an action proportional to the length of the worldline and involving a square root. Thus, we see that the string action (6.2.1) is nothing but the generalization of the point-particle action (6.2.2), where we have associated

$$e^{-1}(\sigma) \;\Longleftrightarrow\; h^{\alpha\beta}(\sigma_1,\sigma_2)\ , \qquad \hat{m} \;=\; 0\ . \qquad (6.2.3)$$

In other words, the string action (6.2.1) is the two-dimensional generalization of the action of a *massless* point particle, where the worldsheet metric functions as an auxiliary field (a "zweibein"). This masslessness property will be crucial shortly.

It is now possible to make some simplifications. Perhaps the most obvious is to restrict our attention to a flat spacetime and take $g_{\mu\nu} = \eta_{\mu\nu}$. We shall do this throughout these lectures. A much more subtle simplification, however, is to simplify the worldsheet metric. Let us therefore pause to discuss how this can be done.

One of the first things we realize is that the ultimate physics of the string should not depend on the particular choice of coordinate system

(σ_1, σ_2) on the string worldsheet. After all, on purely physical grounds, we know that the particular choice of worldsheet coordinate system cannot have a physical effect, for the same worldsheet geometry can ultimately be described using an infinite variety of coordinate systems which differ from each other through relative reparametrizations or rescalings. (Indeed, in the point-particle case, we are likewise free to reparametrize our proper-time variable along the particle worldline.) Therefore, the string action should have a symmetry that makes it invariant under reparametrizations and rescalings of the worldsheet coordinates. Note, in particular, that the invariance under *rescalings* follows from the fact that we chose our string action (6.2.1) to generalize that of a *massless* point particle. In other words, we have taken $\hat{m} = 0$ in (6.2.3). While it is possible to add terms to the action of the bosonic string which mimic the effects of possible mass terms and which explicitly break the scale invariance of the bosonic string, we shall not need to consider such theories in these lectures.

The symmetry that comprises both reparametrizations and rescalings of the worldsheet coordinates is called *conformal symmetry*, and the bosonic string action (6.2.1) is thus said to be "conformally invariant". Clearly, this symmetry must hold not only at the classical level, but also at the quantum level, for we would not have a consistent theory if this symmetry were broken by quantum anomalies. Conformal invariance of the action is a very powerful physical tool which will play an important role throughout these lectures, and indeed the mathematical structure underlying conformal symmetry and its implications is a deep and beautiful subject which we will not have time or space to discuss here. A recommended starting point is *Applied Conformal Field Theory* (Proceedings of Les Houches, Session XLIX, 1988), by P. Ginsparg. Therefore, in order to proceed, we will have to make the first of many "great leaps", and take certain results on faith. Our first great leap will therefore be the following:

> **Great Leap #1:** Conformal invariance of the string action allows us to replace the string metric $h_{\alpha\beta}$ with the two-dimensional Minkowski metric $\eta_{\alpha\beta}$ without loss of generality.

This then results in the simplified bosonic string action

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \; \partial_\alpha X^\mu \partial^\alpha X_\mu \; . \tag{6.2.4}$$

Looking at the action (6.2.4), we see that it has two possible interpretations. The first interpretation is the one that we have already been following: minimizing this action is classically equivalent to minimizing the worldsheet area. This follows directly from the interpretation of $X^\mu(\sigma_1, \sigma_2)$ as the *spacetime coordinates* of a given worldsheet position (σ_1, σ_2). Note that this action is invariant under $SO(D-1,1)$ Lorentz transformations of the spacetime coordinates, with the index μ interpreted as a spacetime vector index relative to the Lorentz group. We shall refer to this as the *spacetime interpretation*.

There is, however, a completely different interpretation of (6.2.4): this is the action of a *two-dimensional quantum field theory* where the two dimensions refer to the worldsheet coordinates and where the "fields" are nothing but the functions $X^\mu(\sigma_1, \sigma_2)$, $\mu = 0, 1, ..., D-1$. Indeed, we see that these spacetime coordinate functions are simply a collection of D different massless bosonic Klein-Gordon fields which happen to exhibit an internal $SO(D-1,1)$ rotation symmetry (analogous to a gauge symmetry) between them. In such a case, the index μ is simply an internal symmetry index which tells us that the X^μ fields transform as vectors with respect to the internal $SO(D-1,1)$ symmetry. We shall refer to this as the *worldsheet interpretation*. Indeed, it is because this string action contains only bosonic worldsheet fields that we call this the *bosonic string*. In such a description, spacetime is not a fundamental concept but rather a "derived" concept: it results from the interpretation of various worldsheet fields as spacetime coordinates, and from the interpretation of an internal symmetry as a spacetime Lorentz symmetry. It is indeed remarkable that such different interpretations can be made of the same physics, and we shall often go back and forth between these different worldsheet and spacetime points of view.

Given these two descriptions of the action, we can also understand the origin of the Regge slope parameter α' on dimensional grounds. Let us first take the worldsheet point of view, so that our length dimensions are determined with respect to the coordinates (σ_1, σ_2). In such a case, we know that the ordinary Klein-Gordon action does not require any dimensionful prefactor, for $\int d^2\sigma (\partial_\alpha X^\mu)^2$ is indeed dimensionless when the Klein-Gordon field X^μ is itself dimensionless. However, from the spacetime point of view, we see that X^μ cannot be dimensionless, for we ultimately need to interpret this field as a spacetime coordinate with units of length. Thus, we are forced to compensate by inserting a dimensionful prefactor α' in front of the action. In other words, *the need for the dimensionful prefactor α' arises from the need to interpret our dimensionless (scale-free) worldsheet* *theory as a dimensionful (spacetime) theory.* Or, to put it slightly differently, the parameter α' is the dimensionful conversion factor that describes the overall scale of the embedding of the dimensionless worldsheet physics into the dimensionful spacetime. We shall see this phenomenon very often throughout these lectures: the worldsheet physics is by itself scale-invariant (since it generalizes the physics of a massless point particle with $\hat{m} = 0$), and it is only in the conversion to dimensionful *spacetime* quantities that the overall scale α' plays a role. Thus, α' sets the overall spacetime mass scale of string theory, often called the *string scale*:

$$M_{\text{string}} \equiv \frac{1}{\sqrt{\alpha'}} . \tag{6.2.5}$$

A priori, this mass scale is unfixed, but we shall see shortly how this scale is ultimately determined.

Now that we have established the worldsheet picture and the spacetime picture, it is easy to see how they are related to each other: each quantum excitation of the Klein-Gordon worldsheet fields X^μ corresponds to a different particle in spacetime. Thus, the study of string theory can be reduced to the study of a two-dimensional quantum field theory! For example, particle scattering amplitudes in spacetime can be re-interpreted as the correlation functions of our two-dimensional worldsheet fields, evaluated on various two-dimensional manifolds. Of course, as we have stated above, this is not just *any* two-dimensional quantum field theory, for physical consistency also requires the presence of conformal symmetry. Thus, from this point of view, string theory is the study of two-dimensional *conformal* field theories. In two dimensions, it turns out conformal symmetry is extremely powerful, for it gives rise to an infinite number of conserved currents. Indeed, two-dimensional conformal symmetry is often sufficiently powerful to permit the *exact* evaluation for many scattering amplitudes.

In the case in question, the particular conformal field theory that concerns us is that of D free massless bosonic fields X^μ, $\mu = 0, 1, ..., D-1$. However, just as with any symmetry, there is always the danger of quantum anomalies. Nevertheless, it is straightforward to show that

> **Great Leap #2:** Conformal invariance of the string action is preserved at the quantum level (*i.e.*, all quantum anomalies are cancelled) if and only if $D = 26$.

This is clearly a big result, and we will not have space to provide a proper mathematical derivation of this fact. At the very least, however, we can give a guide as to the most useful way of thinking about this result. Note that our D bosonic fields are identical to each other and essentially decoupled from each other. Therefore, each contributes the same amount to any potential anomaly. This amount is called the *central charge*, and the central charge c of each bosonic field X will be denoted c_X. It turns out that $c_X = 1$, and therefore the total central charge from the D bosonic fields is $c_{\text{fields}} = D$. However, it can be shown that there also exists a "background" central charge (*i.e.*, a background quantum anomaly) of magnitude $c_{\text{background}} = -26$. Thus, the total anomaly is cancelled only if $D = 26$. Clearly, the most mysterious part of this discussion is the origin of this "background" central charge. In technical terms, it reflects the contributions of the conformal ghosts that arose when we used the conformal symmetry to set (or "gauge-fix") the worldsheet metric $h_{\alpha\beta} \to \eta_{\alpha\beta}$. However, all we will need to know for the future is that the value of the "background" anomaly $c_{\text{background}}$ depends on only the particular symmetry of the worldsheet action that we are dealing with. In the present case, this worldsheet symmetry is simply conformal invariance, and the corresponding background central charge corresponding to conformal invariance is $c_{\text{background}} = -26$. Therefore, we see that the total conformal anomaly is cancelled only if $D = 26$. This is typically called the *critical dimension* of the bosonic string.

We see, then, that string theory is able to determine the spacetime dimension as the result of an *anomaly cancellation argument*! It is worth reflecting on how this happened by considering an analogous situation in field theory, namely the cancellation of the triangle axial anomaly. We know that this anomaly is cancelled only for very particular combinations of particle representations (*e.g.*, we require complete generations of Standard-Model fields, with three colors of quark for every lepton). So we are used to the idea that anomalies are extremely sensitive to the field content of the theory. In string theory, however, we have seen that the analogous worldsheet field content is parametrized by the spacetime dimension. More worldsheeet fields correspond to more spacetime dimensions. Therefore, just as triangle anomaly cancellation requires three colors, conformal anomaly cancellation requires 26 dimensions.

Of course, our world does not consist of 26 flat spacetime dimensions, and we shall ultimately need to find a way of reducing this to a four-dimensional theory. For now, however, we can just think of the present bosonic string as a 26-dimensional toy model.

6.2.2. *Quantizing the bosonic string*

Let us now quantize this theory. Having already noted that the action (6.2.4) is nothing but the action of a set of 26 Klein-Gordon fields X^μ, we already know how to proceed: in the usual fashion, we introduce a Fourier-expansion of the fields X^μ, and interpret the coefficients of this expansion as creation and annihilation operators obeying canonical quantization relations.

Because we ultimately wish to interpret the fields X^μ as spacetime coordinates, we must first impose the constraint

$$X^\mu(\sigma_1+\pi,\sigma_2) \;=\; X^\mu(\sigma_1,\sigma_2) \tag{6.2.6}$$

where we have chosen to normalize the length of the closed string as π. In other words, the spacetime coordinates must be single-valued as we make one complete circuit around the closed string. This is the first place where we have essentially incorporated the requirement that we are dealing with closed strings whose topology is that of a circle. Moreover, because of this topology (and because of the linear nature of the wave equation resulting from the action (6.2.4)), we know that we can also decompose any possible quantum excitation of the wiggling string into a superposition of modes that travel clockwise around the string (in the direction of, say, decreasing σ_1) and those that travel counter-clockwise (in the direction of increasing σ_1). These are respectively called *left-movers* and *right-movers*. We can therefore decompose each of our Klein-Gordon fields into the form

$$X^\mu(\sigma_1,\sigma_2) \;=\; X^\mu_L(\sigma_1+\sigma_2) \;+\; X^\mu_R(\sigma_1-\sigma_2)\;. \tag{6.2.7}$$

The most general mode-expansion consistent with the boundary condition (6.2.6) is then

$$X^\mu(\sigma_1,\sigma_2) \;=\; x^\mu + \ell^2 p^\mu \sigma_2 + \frac{i}{2}\ell \sum_{n\neq 0}\left[\frac{\alpha_n^\mu}{n} e^{-2in(\sigma_1+\sigma_2)} \;+\; \frac{\tilde\alpha_n^\mu}{n} e^{+2in(\sigma_1-\sigma_2)}\right], \tag{6.2.8}$$

which decomposes into

$$\begin{aligned}
X^\mu_L(\sigma_1+\sigma_2) &= \tfrac{1}{2}x^\mu + \frac{\ell^2}{2}p^\mu(\sigma_1+\sigma_2) + \frac{i}{2}\ell\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-2in(\sigma_1+\sigma_2)}\\
X^\mu_R(\sigma_1-\sigma_2) &= \tfrac{1}{2}x^\mu - \frac{\ell^2}{2}p^\mu(\sigma_1-\sigma_2) + \frac{i}{2}\ell\sum_{n\neq 0}\frac{\tilde\alpha_n^\mu}{n}e^{+2in(\sigma_1-\sigma_2)}
\end{aligned} \tag{6.2.9}$$

Here $\ell \equiv \sqrt{2\alpha'}$ is a fundamental length that has been inserted on dimensional grounds.

It is easy to interpret the different terms in (6.2.8) and (6.2.9). Clearly the final terms in each line represent the internal quantum vibrational oscillations of the string, where α_n^μ and $\tilde{\alpha}_n^\mu$ are the left-moving and right-moving creation/annihilation operators corresponding to vibrational modes of a given frequency n. We shall discuss these operators shortly. Note that the contribution from the "zero-mode" has been separated out and written explicitly in the form $x^\mu \pm \frac{1}{2}\ell^2(\sigma_1 \pm \sigma_2)$ for the left- and right-movers respectively. In the case when there are no quantum excitations (so that we can ignore the final exponential terms), these "zero-modes" are all that remain of the mode-expansion, whereupon we see from (6.2.8) that the total X^μ field takes the form $X^\mu = x^\mu + \ell^2 p^\mu \sigma_2$. Interpreting σ_2 as the timelike coordinate on the string worldsheet, we thus see that x^μ is nothing but the center-of-mass position of the string, and p^μ its center-of-mass momentum.

Let us now consider the quantization rules that we must impose. The first one (for the zero-modes) is easy: we simply impose the usual commutation relation $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}$. We shall henceforth set $\hbar = 1$. The excited modes also have a similar commutation relation. First, note that because the X fields are interpreted as spacetime coordinates, they are necessarily *real*. This implies that we must identify $\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger$, with a similar result for the right-moving oscillator modes. In other words, the negative modes *create* excitations, while the positive modes *annihilate* the same excitations. Given this, we then can immediately write down the commutation relation for the creation/annihilation operators:

$$[\alpha_m^\mu, \alpha_n^\nu] \;=\; m\,\delta_{m+n}\,\eta^{\mu\nu}\;, \qquad [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] \;=\; m\,\delta_{m+n}\,\eta^{\mu\nu}\;. \tag{6.2.10}$$

Here we have introduced the notation $\delta_x = \delta_{x,0} \equiv 1$ if $x = 0$, and $\equiv 0$ if $x \neq 0$. Note that these are exactly the harmonic oscillator commutation relations, except that we have rescaled each mode α_n by its corresponding frequency n in (6.2.9). Thus, $a_n \equiv \alpha_n/\sqrt{n}$ obey the usual harmonic oscillator commutation relations. This rescaling has become conventional in string theory, and we shall retain it here. Likewise, it is often conventional to define the zero-mode $\alpha_0^\mu \equiv \frac{1}{2}\sqrt{\alpha'}p^\mu$.

Given this mode-expansion, we can now construct the corresponding number operators

$$n > 0: \qquad N_n \;=\; \frac{1}{n}\,\alpha_{-n}^\mu \alpha_{n\mu}\;, \qquad \tilde{N}_n \;=\; \frac{1}{n}\,\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu} \tag{6.2.11}$$

which count the number of excitations of the n^{th} frequency modes of the string. Once again, this is completely analogous to the harmonic-oscillator creation/annihilation modes, after we take into account the rescaling $\alpha_n \equiv \sqrt{n} a_n$ and the hermiticity condition $\alpha_{-n} = \alpha_n^\dagger$.

Likewise, we can also write down the total *energy* of the system. To do this, let us consider the different contributions to the total energy. First, there is the energy associated with the internal quantum vibrational oscillations of the string. As we might expect, this is given by

$$
\begin{aligned}
L_0^{(\mathrm{osc})} &\equiv \sum_{n=1}^{\infty} n N_n \;=\; \sum_{n=1}^{\infty} \alpha^{\mu}_{-n} \alpha_{n\mu} \\
\bar{L}_0^{(\mathrm{osc})} &\equiv \sum_{n=1}^{\infty} n \tilde{N}_n \;=\; \sum_{n=1}^{\infty} \tilde{\alpha}^{\mu}_{-n} \tilde{\alpha}_{n\mu} \ .
\end{aligned}
\tag{6.2.12}
$$

For convenience, we are defining these energy operators in such a way that they are dimensionless numbers (*i.e.*, they are *worldsheet* energies). These L_0 operators are often called *Virasoro generators*, which are more generally defined $L_m \equiv \sum_n \alpha^{\mu}_{m-n} \alpha_{n\mu}$. These generators are nothing but the different frequency modes of the total worldsheet stress-energy tensor, and together they satisfy the so-called *Virasoro algebra*. We shall only consider L_0 in these lectures.

Next, there is the energy of the zero-modes, which correspond to the net center-of-mass motion of the string. This is given by

$$
\begin{aligned}
L_0^{(\mathrm{com})} &\equiv \alpha_0^{\mu} \alpha_{0\mu} \;=\; \frac{\alpha'}{4} p^{\mu} p_{\mu} \\
\bar{L}_0^{(\mathrm{com})} &\equiv \tilde{\alpha}_0^{\mu} \tilde{\alpha}_{0\mu} \;=\; \frac{\alpha'}{4} p^{\mu} p_{\mu} \ .
\end{aligned}
\tag{6.2.13}
$$

Note that factors of α' must appear in order to counter-balance the fact that the center-of-mass momentum p^{μ} is a spacetime quantity, and hence dimensionful.

Finally, there is the possibility of an overall non-zero vacuum energy for both the left-movers and the right-movers. In other words, there is no reason to assume that the vacuum state (the state without any excitations) is exactly at zero energy. This is important, of course, since string theory is ultimately a theory which will contain gravity, and it is precisely in theories containing gravity that the overall zero of energy becomes important. Indeed, mathematically, one can imagine that due to the commutation relations (6.2.10), there can be an overall normal-ordering ambiguity in the definitions in (6.2.12), and this overall normal-ordering constant would be our "vacuum energy".

Thus, denoting the left- and right-moving vacuum energies as $a_{L,R}$, we have the total left- and right-moving energies

$$H \equiv L_0^{(\text{com})} + L_0^{(\text{osc})} + a_L , \qquad \bar{H} \equiv \bar{L}_0^{(\text{com})} + \bar{L}_0^{(\text{osc})} + a_R . \quad (6.2.14)$$

These are the total worldsheet Hamiltonians.

Clearly, the important thing to do at this stage is to determine the vacuum energies $a_{L,R}$. Of course, the symmetry between left-movers and right-movers requires $a_L = a_R$. Calculating this vacuum energy can be done in numerous ways, each of which would take too much space for our purposes. Once again, we refer the reader to Chapter 2 of GSW, where a full calculation is given. Therefore, it is time for another

> **Great Leap #3:** Conformal invariance of the string action implies that $a_L = a_R = -1$.

Finally, in order to determine the total *spacetime mass* of a given string state, we must have a *mass-shell condition* for the string. Rather than provide a rigorous derivation (for which we again refer the curious reader to GSW), we can instead give an intuitive argument which suggests the proper answer. In a quantum field theory of point particles, the mass $\hat{m}$ is a parameter that appears in the Lagrangian through an explicit mass term that might be generated in some separate manner, *e.g.*, through the Higgs mechanism. Since a point particle has no internal degrees of freedom beyond those associated with its center-of-mass motion, such a mass parameter $\hat{m}$ would then be directly identified with M, the resulting physical mass of the particle. Such a physical mass M is the quantity satisfying the condition $p^\mu p_\mu = -M^2$, or equivalently the condition $L_0^{(\text{com})} = \bar{L}_0^{(\text{com})} = -\alpha' M^2/4$. In the special case of a massless particle (for which $\hat{m} = M = 0$), this mass-shell condition then takes the simple form $L_0^{(\text{com})} = \bar{L}_0^{(\text{com})} = 0$.

A similar condition emerges in string theory. We have already seen that our string action (6.2.1) generalizes that of a massless particle, which again suggests that our effective Lagrangian mass parameter $\hat{m}$ vanishes. Indeed, as we have discussed, this is the root of the scale invariance of the string action (6.2.1). However, unlike the point-particle case, a string *does* have additional, purely internal degrees of freedom — these are the oscillations of the string itself, whose additional energy contributions are represented by $L_0^{(\text{osc})}$, $\bar{L}_0^{(\text{osc})}$, and $a_{L,R}$. Thus, even though $\hat{m} = 0$, the resulting string state can still have a non-zero physical mass M in spacetime. Indeed, just as the mass-shell condition for massless point particles is given by $L_0^{(\text{com})} = \bar{L}_0^{(\text{com})} = 0$, the mass-shell condition for our scale-invariant string is generalized to $H = \bar{H} = 0$. This then becomes our scale-invariant mass-shell condition in string theory. Of course, spacetime Lorentz invariance still allows us to identify the physical spacetime mass M of a given string state via the relations $L_0^{(\text{com})} = \bar{L}_0^{(\text{com})} = -\alpha' M^2/4$. Thus, the string mass-shell conditions $H = \bar{H} = 0$ lead to the identifications

$$\frac{1}{4}\alpha' M^2 \; = \; L_0^{(\text{osc})} - 1 \; , \qquad\qquad \frac{1}{4}\alpha' M^2 \; = \; \bar{L}_0^{(\text{osc})} - 1 \; . \tag{6.2.15}$$

Note that these two conditions can also be written in the form

$$\alpha' M^2 \; = \; 2\left(L_0^{(\text{osc})} + \bar{L}_0^{(\text{osc})} - 2 \right) \tag{6.2.16}$$

where we must obey the constraint

$$L_0^{(\text{osc})} \; = \; \bar{L}_0^{(\text{osc})} \; . \tag{6.2.17}$$

Interpreting the conditions (6.2.16) and (6.2.17) is easy. The condition (6.2.16) simply tells us that the physical spacetime mass M of a given string state (and thus the square of its center-of-mass momentum) is generated *solely* from its internal left- and right-moving vibrational excitations. The condition (6.2.17), by contrast, tells us that the mass of the string must come *equally* from left-moving and right-moving excitations. The latter condition (6.2.17) is often referred to as the *level-matching condition*, since it implies that a given string oscillator state is considered to be "on shell" (or "physical") only if the total excitation level of the left-movers matches the total excitation level of the right-movers. This condition implies that the string does not have an unbalanced "wobbling", for if such a wobbling existed, it could ultimately be used to determine a preferred coordinate system on the worldsheet (thereby breaking conformal invariance). Indeed, demanding invariance under shifts in the σ_1 variable leads directly to the condition (6.2.17). We remark, however, that states not satisfying (6.2.17) are nevertheless important for understanding the "off-shell" or "virtual" structure of string theory. Such "virtual" states contribute, for example, within loop amplitudes. In these lectures, however, we shall focus on only the so-called "tree-level" string spectrum for which the level-matching constraint (6.2.17) is imposed and the corresponding physical masses are given by (6.2.16).

6.2.3. *The spectrum of the bosonic string*

Having discussed the quantization of the bosonic string, we can now examine its spectrum. The procedure is simple: we simply consider all possible combinations of left- and right-moving mode excitations of the string worldsheet, subject to the level-matching constraint (6.2.17), and then we tensor these left- and right-moving states together to form the total resulting string state. The spacetime mass of this string state is then given by (6.2.16), and the properties of the state are deduced directly from the underlying vibrational configuration of the string.

The simplest state, of course, is the string vacuum state

$$|0\rangle_R \otimes |0\rangle_L \tag{6.2.18}$$

in which the right- and left-moving vacuum states are tensored together. This state trivially satisfies (6.2.17), which indicates that this state is indeed part of the physical string spectrum. Unfortunately, we see from (6.2.16) that this state has a negative squared mass — *i.e.*, the spacetime mass of this state is imaginary! This state is thus a *tachyon*. Making sense of this string state is problematic, and is one of the reasons that we shall not ultimately be interested in the bosonic string.

Let us continue, however. The first excited string state is

$$\tilde{\alpha}^{\mu}_{-1}|0\rangle_R \otimes \alpha^{\nu}_{-1}|0\rangle_L \ . \tag{6.2.19}$$

This state has $L_0^{(\mathrm{osc})} = \bar{L}_0^{(\mathrm{osc})} = 1$, and according to (6.2.16) is therefore massless. As evident from its Lorentz index structure, this state transforms under the spacetime Lorentz group as the tensor product of two spin-one Lorentz vectors. We can therefore decompose this tensor product into a spin-two state (the symmetric traceless component), a spin-one state (the antisymmetric component), and a spin-zero state (the trace). Mathematically, this is equivalent to the tensor-product rule for Lorentz transverse $SO(24)$ vector representations:

$$\mathbf{V}_{24} \otimes \mathbf{V}_{24} = \mathbf{1} \oplus \mathbf{276} \oplus \mathbf{299} \tag{6.2.20}$$

where $\mathbf{V}_8$ is the eight-dimensional vector representation, and where the $\mathbf{1}$ representation is the spin-zero state, the $\mathbf{276}$ representation is the spin-one state, and the $\mathbf{299}$ representation is the spin-two state.

How can we interpret these states? A massless spin-two state must, by Lorentz invariance, have equations of motion which are equivalent to the Einstein field equations of general relativity. Thus, we are forced to identify the spin-two (traceless symmetric) component of the state (6.2.19) as the *graviton* $g_{\mu\nu}$, which is the spin-two mediator of the gravitational interactions. The spin-one (antisymmetric) state within (6.2.19) is an antisymmetric tensor field, often denoted $B_{\mu\nu}$, and the spin-zero (trace) component is the so-called *dilaton*, denoted ϕ. Together, $(g_{\mu\nu}, B_{\mu\nu}, \phi)$ are called the *gravity multiplet*.

By identifying (6.2.19) with the gravity multiplet, we see that string theory becomes a theory that contains gravity! This in turn allows us to determine the value of our previously unfixed mass scale α'. We shall now sketch how this happens (with details available in GSW). It turns out that if one calculates loop amplitudes in string theory, one finds that $e^{-\phi}$ serves as a loop expansion parameter (*i.e.*, higher-loop amplitudes come multiplied by more powers of $e^{-\phi}$). Given this observation, it is natural to identify the string coupling constant as the vacuum expectation value of the dilaton:

$$g_{\rm string} \;=\; e^{-\langle\phi\rangle} \;. \qquad (6.2.21)$$

This string coupling constant describes the strength of string interactions. Given this definition, we then find that the graviton state couples to matter with the expected gravitational strength only if we choose

$$\alpha' \;=\; \frac{G_{\rm Newton}}{g^2_{\rm string}} \qquad (6.2.22)$$

where $G_{\rm Newton}$ is Newton's constant. Substituting this result into (6.2.5), we then find

$$M_{\rm string} \;=\; g_{\rm string}\, M_{\rm Planck} \;, \qquad (6.2.23)$$

where $M_{\rm Planck} \equiv 1/\sqrt{G_{\rm Newton}}$. Thus, because it contains gravity, string theory becomes a theory whose fundamental mass scale is related to the Planck scale.

We can also construct more and more massive string states. Ultimately, these fill out an infinite tower of string states. It is clear that such additional states all have $\alpha' M^2 > 0$. Given the above value for α', this implies that these additional states all have Planck-scale masses. Such Planck-scale excited states are therefore not of direct relevance for string phenomenology. Let us note, however, one interesting fact about these states. For any given spacetime mass level M, the string state with maximum spin is achieved by exciting only the lowest vibrational modes α^μ_{-1} and $\tilde\alpha^\mu_{-1}$. We thus find that for a given spacetime mass M, the maximum spin $J_{\rm max}$ that can be realized is

$$\alpha' M^2 \;=\; 2J_{\rm max} - 4 \;. \qquad (6.2.24)$$

For example, we see that the maximum spin that can be realized for a massless state is $J = 2$ (the graviton). The relation (6.2.24) was originally observed for hadron resonances, and historically gave rise to the so-called "dual resonance models" (which eventually became modern string theory). In such dual resonance models, the relation (6.2.24) describes a so-called "Regge trajectory", with α' serving as the so-called "Regge slope". It is for this reason that in modern string theory, we continue to refer to α' as the Regge slope.

Before concluding, let us briefly mention one further important issue. In ordinary four-dimensional quantum field theory, we know that a massless spin-one state (*e.g.*, a photon) naïvely has four distinct states (corresponding to the four components of a vector field A^μ). However, the underlying gauge invariance allows us to make a unitary gauge choice wherein only two of these states (the two helicity states) are truly physical. The timelike and longitudinal states decouple, leaving only the transverse components. In the above description of the string spectrum, however, we have taken a covariant approach analogous to the description of a photon as a four-component vector. One might then wonder which of these states are truly physical. This issue is an important one in string theory, and once again we cannot here provide a proper proof. We shall therefore make recourse to another

> **Great Leap #4:** The physical string states are those which are realized by exciting the oscillator modes of only the *transverse* coordinates X^i $(i = 1, ..., 24)$.

Proving this statement requires showing that even after we have used conformal invariance to set the string worldsheet metric to $\eta_{\alpha\beta}$, there still remains sufficient freedom to make a further "gauge" choice wherein we set the oscillator modes of the timelike and longitudinal spacetime coordinates to zero. This gauge choice, which is called *light-cone gauge*, is thus the analogue of unitary gauge in quantum field theory, and essentially tells us that only the 24 transverse coordinates correspond to physical degrees of freedom in the string worldsheet action. An important by-product of this fact is that every remaining string state has a non-negative norm. This is non-trivial. For example, if our metric signature is chosen such that $\eta^{00} = -1$, then the state $\alpha_{-n}^{\mu=0}|0\rangle$ has a negative norm. However, one can demonstrate that in light-cone gauge all resulting states are physical and have non-negative norm.

6.2.4. *Summary*

Let us quickly review those features of the bosonic string that we shall need to bear in mind in subsequent lectures. We shall separate these features into worldsheet features and spacetime features.

Worldsheet: The worldsheet fields consist of D copies of the left- and right-moving spacetime coordinates X_L^μ and X_R^μ (the worldsheet bosons). The fact that these X coordinates are periodic as we traverse the closed string loop implies that they have integer modings α_n and $\tilde{\alpha}_n$, where $n \in \mathbb{Z}$. The relevant worldsheet symmetry is conformal invariance, which tells us that the number of these X^μ fields is $D = 26$ and also tells us that the vacuum energy corresponding to these fields is $a_L = a_R = -1$. As we have stated above, a useful way to think about these results is to imagine that there is a "background" conformal anomaly $c_{\text{background}} = -26$, and that each X^μ field makes a contribution $c_X = 1$. In general, the "background" conformal anomaly is only a function of the relevant worldsheet symmetry (in this case conformal invariance), and it will always remain true that $c_X = 1$. Thus, cancellation of the conformal anomaly requires $D = 26$. A similar interpretation can also be given to the vacuum energy. When calculating the vacuum energies, only the physical (*i.e.*, transverse) fields are relevant. It is a general result that each X field contributes $a_X = -1/24$ to the vacuum energy. Therefore, we find $a_L = a_R = 24a_X = -1$.

Spacetime: The above worldsheet theory leads to the following features in spacetime. We find that the *spacetime dimension* (often called the *critical* spacetime dimension) is 26. The spectrum consists of a spinless tachyon, as well as a massless gravity multiplet consisting of the graviton $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, and the dilaton ϕ. There is also an infinite tower of massive (Planck-scale) string states.

Comments: Two remarkable things have happened. First, we have a theory of quantized gravity! The graviton has emerged as the quantum excitation of a closed string. This alone is very exciting, but also somewhat mysterious. We started by assuming a closed string propagating through an external, fixed, flat spacetime. But this string itself includes a graviton mode, which implies a distortion in that background spacetime. This then acts back to change the worldsheet theory. Thus, in some sense, the string itself not only "creates" the spacetime in which it propagates, but is then affected by this change in the spacetime geometry. This coupling or interplay between the string and its spacetime is not fully understood, and is clearly at the heart of the many mysterious features of string theory as a theory of quantum gravity.

A second remarkable thing has also happened, although we have not demonstrated it explicitly. As indicated in (6.2.21), a coupling constant has been determined *not* as a free parameter, but rather *dynamically* as the vacuum expectation of a string field. It is in this sense that string theory contains no free parameters, and that all parameters such as coupling constants are determined dynamically.

There are, however, a number of drawbacks to this bosonic string theory. First, it contains a tachyonic state. We must somehow find a way to eliminate this. Second, all string excitations are spacetime *bosons* (*i.e.*, they have integer spin). We must find a way to obtain spacetime fermions. Third, there are no massless spin-one states (which we would wish to associate with gauge fields). Thus, there are no gauge symmetries. It is for these reasons that we shall go on to consider more complicated string theories.

And finally, there is another major drawback that we need to be aware of. Although it is compelling that the string coupling g_{string} is in principle determined dynamically, as the vacuum expectation value of the dilaton scalar field, in practice we do not understand how to calculate the potential of the dilaton field and thereby deduce its vacuum expectation value. In the bosonic string we are considering here, the dilaton potential $V(\phi)$ is actually divergent for all $\phi < \infty$, and so this question cannot be meaningfully addressed. However, even in the more realistic string theories to be discussed, this potential is either completely flat (as happens in a supersymmetric context), or generally takes a shape that sends $\langle\phi\rangle \to \infty$. This is the famous *dilaton runaway problem*. Solving this problem is perhaps one of the most important (unsolved) problems in string phenomenology.

How can we remedy these features? One possibility is prompted by the appearance of the tachyon. In ordinary quantum field theory, the existence of a tachyon (a state with a negative mass-squared) signals that the vacuum has been misidentified (as in the Higgs mechanism); the theory then "rolls" to a different vacuum configuration in which the tachyon is eliminated. So it is natural to speculate that perhaps the bosonic string theory also "rolls" to a new vacuum in such a way that the tachyon is no longer present and the dilaton is stabilized. Perhaps fermions and gauge fields might also appear in this new vacuum, as desired. However, as we have already indicated, it is not known how the bosonic string behaves in this context. We do not know if there exists a new ("stable") vacuum to roll to, and if so, what its properties might be. Of course, knowing the potential $V(\phi)$ would be extremely useful, yet as we indicated this potential is naïvely divergent and therefore requires some knowledge of the non-perturbative structure of string theory. So (at least for the time being) this option does not appear promising.

A second possibility, then, is simply to abandon the bosonic string and attempt to construct a new string theory altogether. And this is what we shall now do.

6.3. Lecture #3: Neutrinos are fermions: The superstring

As we saw in the last lecture, the bosonic string has two glaring failures: it contains a tachyon, and it does not give rise to spacetime fermions. Both of these features are troubling, especially since the announced goal of these lectures is to derive a neutrino from string theory, and we know that the neutrino is a fermionic object. We therefore seek to construct a new string theory which can give rise to excitations with half-integer spins.

6.3.1. *The action*

We have already seen that string theories are defined by their two-dimensional worldsheet actions. Thus, in order to construct a new string theory, we must construct a new worldsheet action. At the very least, this action should contain that of the bosonic string, since we still wish to retain the spacetime interpretatation that we had previously. Thus, our only option is to *introduce additional worldsheet fields* into the action:

$$S \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma\;\left(\partial_\alpha X^\mu \partial^\alpha X_\mu \;+\; ...\;\right)\;. \qquad (6.3.1)$$

What new fields can we add? If our goal is to produce spacetime fermions, a natural guess would be to add worldsheet fermions! These would complement the worldsheet bosonic fields X^μ that are already present. For the moment, let us denote such fermionic fields schematically as ψ. We would then attempt to consider an action of the form

$$S \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma\;\left(\partial_\alpha X^\mu \partial^\alpha X_\mu \;+\; \bar\psi i\rho^\alpha\partial_\alpha\psi\right)\;. \qquad (6.3.2)$$

Here $\psi(\sigma_1, \sigma_2)$ represents our two-dimensional fermionic fields, and ρ^α are an appropriate set of two-dimensional Dirac matrices (the analogues of the γ^μ matrices in four dimensions).

We then face a number of questions. First, how many ψ fields must we add? Second, what kinds of worldsheet fermions should these be? Should they be Dirac fermions, or Majorana fermions, or Majorana-Weyl fermions? Third, how should these two-dimensional spinors ψ transform under the (internal) $SO(D-1,1)$ spacetime Lorentz symmetry? We already know that the X^μ fields, for example, transform as vectors under this symmetry. Note that it is not obvious that the ψ fields should necessarily transform as spinors under $SO(D-1,1)$ and carry a spacetime spinor index. In particular, all we know thus far is that the ψ fields transform as spinors under worldsheet *two-dimensional* Lorentz transformations. This does not *a priori* give us any information about their *spacetime* transformation properties.

There is also another potential worry that appears if we try to add new worldsheet fields. We have already seen in the bosonic string that worldsheet conformal invariance was sufficiently powerful a symmetry to allow us to choose a light-cone gauge and thereby eliminate all negative-norm states. However, the presence of new worldsheet fields implies the existence of new quantum excitation modes in the resulting string spectrum, and some of these new states may also have negative norm. Thus, conformal symmetry may no longer be sufficient (and indeed would not be sufficient) to allow us to eliminate these states as well.

It turns out that all of these questions have a common answer: we can impose an extra symmetry beyond simple worldsheet conformal invariance. Indeed, the extra symmetry that we shall impose is nothing but worldsheet (*i.e.*, two-dimensional) *supersymmetry*. Specifically, we shall require that the ψ fields be the two-dimensional superpartners of the X fields, so that the resulting action has a manifest worldsheet (two-dimensional) supersymmetry.* This new theory will be called the *superstring*.

*We remark that this is only one possible choice, and will ultimately lead us to the so-called Ramond/Neveu-Schwarz (RNS) formalism. Another possible choice would be to demand *spacetime* supersymmetry, and to imagine that the ψ fields are the Grassmann coordinates θ of a super-spacetime. This possibility would then lead to the so-called Green-Schwarz (GS) formalism. It turns out that these two formalisms are ultimately equivalent, however, and both provide suitable descriptions of the resulting superstring theory. This equivalence is possible because the RNS superstring ultimately also has spacetime supersymmetry (as we shall discover below). In these lectures, however, we shall restrict our attention to the RNS formulation in which the ψ fields are *worldsheet* (rather than spacetime) superpartners of the X^μ fields. Aside from being more useful for string phenomenology, the RNS formalism has the philosophical advantage that it treats the *string* as the fundamental object, with the spacetime structure emerging as a *derived consequence*. The RNS formalism thus reinforces one of the central themes of these lectures, namely that we define a string theory by its worldsheet properties alone, and then deduce the spacetime effects of these properties as consequences. The GS formalism, on the other hand, has the benefit of being manifestly spacetime supersymmetric from

It is important to stress that this supersymmetry that we will be discussing is *not* the spacetime supersymmetry that might be seen in the next round of accelerator experiments. Instead, this is a *worldsheet* supersymmetry which stems directly from the worldsheet interpretation of the original Polyakov action (6.2.4), and which relates the worldsheet bosons X to worldsheet fermions ψ via a worldsheet supercurrent J.

Imposing this worldsheet supersymmetry then answers all of the questions we previously raised. How many ψ fields? The answer is D, one for each boson X^μ. What kind of ψ spinor? The answer is a Majorana (two-component) spinor. How does the ψ field transform under the $SO(D-1,1)$ spacetime Lorentz symmetry? The answer is that the ψ field must transform as a *vector* under the Lorentz symmetry, since the X^μ field (for which it is the worldsheet superpartner) also transforms as a vector. In other words, the *worldsheet* supersymmetry commutes with the *spacetime* Lorentz symmetry, and thus does not change the Lorentz index structure. Thus, the ψ fields transform as spacetime vectors, and carry a spacetime vector index: $\psi^\mu(\sigma_1, \sigma_2)$.

This last point may initially seem confusing, so we reiterate: the ψ fields are worldsheet fermions, but spacetime bosons! They transform as spinors under worldsheet Lorentz transformations, but as vectors under the spacetime Lorentz transformations.

Given this, we can now explicitly write down the superstring action:

$$S \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma\;\left(\partial_\alpha X^\mu \partial^\alpha X_\mu \;-\; i\bar{\psi}_\mu \rho^\alpha \partial_\alpha \psi^\mu\right)\;. \tag{6.3.3}$$

Our worldsheet fields are $X^\mu(\sigma_1, \sigma_2)$ and $\psi^\mu(\sigma_1, \sigma_2)$, and the μ index (with $\mu = 0, 1, 2, ..., D-1$) is a vector index with respect to the internal symmetry $SO(D-1,1)$. From the worldsheet perspective, each X^μ is a scalar field (containing one component), while each ψ^μ is a two-component spinor. The ρ^α are two-dimensional Dirac matrices satisfying the two-dimensional Clifford algebra $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$, and $\bar{\psi} \equiv \psi^\dagger \rho^0$. One can then show that the action (6.3.3) is invariant under the worldsheet supersymmetry transformations $\delta X^\mu = \bar{\epsilon}\psi^\mu$, $\delta\psi^\mu = -i\rho^\alpha \epsilon \partial_\alpha X^\mu$, where ϵ is a constant anticommuting spinor that parametrizes the "magnitude" of the supersymmetry transformation. The corresponding generator of this worldsheet supersymmetry transformation is the worldsheet supercurrent $J_\alpha = \frac{1}{2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu$.

It is convenient to choose a particular Weyl (chiral) representation for

the very beginning.

the two-dimensional ρ^α matrices:

$$\rho^0 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;, \quad \rho^1 \;=\; \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \Longrightarrow \quad \rho^0\rho^1 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;. \tag{6.3.4}$$

Here the product $\rho^0\rho^1$ plays the role of the chirality operator (the analogue of γ_5 in four dimensions), and thus in this basis we can identify the upper and lower components of the two-component Majorana spinor ψ as being left-moving and right-moving respectively. Our worldsheet action (6.3.3) then decomposes into the form

$$S \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma \;\left(\partial_\alpha X^\mu \partial^\alpha X_\mu \;-\; \psi_{\mu R}\partial_- \psi^\mu_R \;-\; \psi_{\mu L}\partial_+ \psi^\mu_L\right) \tag{6.3.5}$$

where $\partial_\pm$ are derivatives with respect to the left- and right-moving worldsheet coordinates $\sigma_1 \pm \sigma_2$. The worldsheet content of this theory therefore consists of D left-moving worldsheet bosons X_L, D right-moving worldsheet bosons X_R, D left-moving worldsheet Majorana-Weyl (one-component) fermions ψ_L, and D right-moving worldsheet Majorana-Weyl (one-component) fermions ψ_R. There are two worldsheet supercurrents in this theory:

$$J_L \;=\; \psi_{\mu L}\,\partial_+\, X^\mu_L \;, \qquad J_R \;=\; \psi_{\mu R}\,\partial_-\, X^\mu_R \;. \tag{6.3.6}$$

Note that our original goal in constructing the superstring had been to obtain spacetime fermions. However, it may seem from the above that we have failed in this regard, since we have only introduced new fields ψ which themselves are spacetime vectors. How then are we to obtain spacetime fermions? It turns out that this will happen in a surprising way.

Let us proceed to analyze this string following the same steps as we used for the bosonic string. First, we see that our worldsheet symmetry has been enlarged: rather than simply have conformal invariance, we now have conformal invariance *plus* worldsheet supersymmetry. Together, this is called *superconformal invariance*, which is a much larger symmetry than conformal invariance alone.

This enlargement of the worldsheet symmetry changes many of the features of the resulting string. The most profound is the value of the spacetime dimension D. Recall from our discussion of the bosonic string that associated with each worldsheet symmetry there is a particular "background" conformal (central charge) anomaly, and that it is necessary to choose a sufficient number of worldsheet fields so as to cancel this anomaly and ensure that conformal invariance is maintained even at the quantum level. The same argument applies here as well, except that

Great Leap #5: The "background" conformal anomaly associated with *superconformal* invariance is not $c = -26$ but rather $c = -15$. Likewise, the conformal anomaly contribution from each worldsheet Majorana fermion is $c = 1/2$.

We can understand the origin of the "background" conformal anomaly $c = -15$ as follows. Just as in the bosonic string, a certain contribution $c = -26$ is attributable to the conformal ghosts resulting from conformal gauge fixing. The new feature here is that we now have an additional contribution $+11$ which is attributable to the worldsheet *superpartners* of these ghosts. Together, this produces a background anomaly $c = -15$. What this means is that we must choose the number D of worldsheet bosons and fermions such that this "background" anomaly is cancelled. We have already seen that the anomaly contribution from each worldsheet boson X^μ is $c_X = 1$. Since the anomaly contribution from each Majorana fermion is $c_\psi = 1/2$, we must satisfy

$$D\,(1 + \tfrac{1}{2}) - 15 = 0 \qquad \Longrightarrow \qquad D = 10\ . \tag{6.3.7}$$

Thus, we see that the critical dimension of the superstring is $D = 10$ rather than $D = 26$. Moreover, just as for the bosonic string, the *super*conformal symmetry of the superstring worldsheet action again allows us to choose a light-cone gauge in which only *eight* transverse bosons and *eight* transverse fermions represent the truly physical propagating worldsheet fields.

6.3.2. *Quantizing the superstring*

Let us now quantize the superstring, just as we did for the bosonic string. The boundary conditions (6.2.6) for the X^μ fields remain valid even for the superstring, since the X^μ continue to have the interpretation of spacetime coordinates. Therefore the mode-expansions (6.2.9) continue to apply.

The only new feature, then, is the mode-expansion for the fermionic fields ψ^μ. However, unlike the bosonic fields X^μ which must be periodic because of their interpretation as spacetime coordinates, these fermionic fields ψ^μ do not have any immediate interpretation in spacetime. Therefore, the only boundary conditions that might be imposed on these fields are those that are required directly from the symmetries of the action. In particular, we must choose boundary conditions for the ψ^μ fields so as to maintain the single-valuedness of the action as we traverse the closed string (*i.e.*, as $\sigma_1 \to \sigma_1 + \pi$), and so as to maintain the worldsheet supersymmetry of the action (whose algebra includes a requirement that the supercurrent square to the Hamiltonian, *i.e.*, $J \cdot J \sim H$). It turns out that are only two choices of boundary conditions that satisfy these requirements. One possibility is that the ψ^μ fields are periodic under $\sigma_1 \to \sigma_1 + \pi$:

$$\text{Ramond:} \qquad \psi^\mu(\sigma_1 + \pi, \sigma_2) \;=\; +\,\psi^\mu(\sigma_1, \sigma_2)\;. \tag{6.3.8}$$

Such periodic boundary conditions are typically called "Ramond" (R) boundary conditions, after P. Ramond (who introduced these fermionic boundary conditions in 1971). The second possibility is that the ψ^μ fields are *anti*-periodic under $\sigma_1 \to \sigma_1 + \pi$:

$$\text{Neveu-Schwarz:} \qquad \psi^\mu(\sigma_1 + \pi, \sigma_2) \;=\; -\,\psi^\mu(\sigma_1, \sigma_2)\;. \tag{6.3.9}$$

Such periodic boundary conditions are typically called "Neveu-Schwarz" (NS) boundary conditions, after A. Neveu and J. Schwarz (who introduced these fermionic boundary conditions in 1971). As we shall see in Lecture #4, both of these boundary conditions are ultimately required for the self-consistency of the superstring.

In the case of periodic (Ramond) boundary conditions, the mode-expansion of the ψ^μ field resembles that of the X^μ field:

$$\begin{aligned}\text{Ramond:} \qquad \psi^\mu_L(\sigma_1 + \sigma_2) &= \sum_{n\in\mathbb{Z}} b^\mu_n \, e^{-2in(\sigma_1+\sigma_2)} \\ \psi^\mu_R(\sigma_1 - \sigma_2) &= \sum_{n\in\mathbb{Z}} \tilde b^\mu_n \, e^{+2in(\sigma_1-\sigma_2)}\;.\end{aligned} \tag{6.3.10}$$

Here $b^\mu_n, \tilde b^\mu_n$ are the (fermionic) creation and annihilation operators, satisfying the *anti*-commutation relations

$$\{b^\mu_m, b^\nu_n\} \;=\; \eta^{\mu\nu}\delta_{m+n} \tag{6.3.11}$$

where we recall the hermiticity condition $b^\mu_{-n} = (b^\mu_n)^\dagger$. The same relations hold for the right-moving modes as well. This hermiticity condition follows from the fact that the ψ fields are Majorana (*i.e.*, real) fields. Note that unlike the bosonic mode-expansion (6.2.9), we have joined the zero-modes together with the excited modes in (6.3.10).[†] There is also no "center-of-mass" term in the mode-expansion (a fermionic analogue of x^μ) because the ψ fields are Grassmann variables and thus lack a classical limit. Finally, also note that unlike the bosonic α^μ_n modes, which are rescaled relative to

[†] We are cheating slightly here, since the treatment of Ramond zero-modes for Majorana worldsheet fermions is actually quite subtle. In some sense, each Majorana fermion has only "half" a zero-mode. We will provide a rigorous discussion of this fact in Lecture #5. In the meantime, it will suffice to ignore this subtlety.

the usual harmonic oscillator modes by powers of the mode frequency n, the fermionic b_n^μ modes are defined without this rescaling and hence satisfy the usual harmonic-oscillator commutation relations (6.3.11) directly. This too is traditional in string theory.

In the case of anti-periodic (Neveu-Schwarz) boundary conditions, the mode-expansion of the ψ^μ field involves *half-integer* rather than integer modes:

$$\text{Neveu-Schwarz:} \qquad \begin{aligned} \psi_L^\mu(\sigma_1+\sigma_2) &= \sum_{r\in\mathbb{Z}+1/2} b_r^\mu\, e^{-2ir(\sigma_1+\sigma_2)} \\ \psi_R^\mu(\sigma_1-\sigma_2) &= \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu\, e^{+2ir(\sigma_1-\sigma_2)} \end{aligned} \tag{6.3.12}$$

Once again, $b_r^\mu, \tilde{b}_r^\mu$ are the (fermionic) creation and annihilation operators, satisfying the *anti*-commutation relations

$$\{b_r^\mu, b_s^\nu\} \;=\; \eta^{\mu\nu}\delta_{r+s} \tag{6.3.13}$$

where we have the hermiticity condition $b_{-r}^\mu = (b_r^\mu)^\dagger$.

The expressions for the total energy of a given string configuration now receive contributions from not only the bosonic oscillator modes, as in (6.2.12), but also the fermionic oscillator modes. These new contributions are given by

$$\begin{aligned} \text{R:} \qquad L_0^{(\text{osc})} &= \sum_{n=0}^{\infty} n\, b_{-n}^\mu b_{n\mu} \\ \text{NS:} \qquad L_0^{(\text{osc})} &= \sum_{r=1/2}^{\infty} r\, b_{-r}^\mu b_{r\mu}\ , \end{aligned} \tag{6.3.14}$$

with similar expressions for the right-movers.

Finally, we must consider the vacuum energies a_L and a_R for the superstring. Recall that for the bosonic string, each of the 24 transverse X^μ fields contributed $a_X = -1/24$, yielding a total of $a_L = a_R = -1$. This contribution from each bosonic field remains the same for the superstring, so we continue to have $a_X = -1/24$. It therefore only remains to determine the vacuum-energy contributions from the worldsheet Majorana fermions, and it is found that

> **Great Leap #6:** Each Ramond fermion contributes vacuum energy $a_\psi = +1/24$, whereas each Neveu-Schwarz fermion contributes vacuum energy $a_\psi = -1/48$.

We thus see that like the bosons, the Neveu-Schwarz fermions contribute negative vacuum energies, while Ramond fermions contribute positive vacuum energies.

Given these mode-expansions and commutation relations, it is instructive to consider the Fock space of an individual Ramond (R) or Neveu-Schwarz (NS) fermion. It turns out to be simplest to consider the Fock space of an individual (left- or right-moving) NS fermion first. The two lowest-lying states are

$$\begin{array}{lll} \text{vacuum:} & |0\rangle & L_0^{(\text{osc})} = 0 \\ \text{first-excited state:} & b_{-1/2}|0\rangle & L_0^{(\text{osc})} = 1/2\ . \end{array} \tag{6.3.15}$$

Note that relative to the vacuum, all further excited states are reached through only half-integer excitations. Also note that the vacuum of the NS Fock space is unique, just like that of the bosons X^μ. What this means is that from the spacetime perspective, the vacuum is spinless (and hence a spacetime bosonic state), and that all subsequent excitations of the vacuum are also spacetime bosons. Recall, in this connection, that the fermion mode operators b are only fermionic from the worldsheet perspective; they are still bosonic operators (just like the fields ψ^μ themselves) relative to *spacetime* Lorentz symmetries.

Let us now consider the corresponding Fock space for the Ramond fermions with periodic boundary conditions. Once again, we have a tower of states

$$\begin{array}{lll} \text{vacuum:} & |0\rangle & L_0^{(\text{osc})} = 0 \\ \text{first-excited state:} & b_{-1}|0\rangle & L_0^{(\text{osc})} = 1 \end{array} \tag{6.3.16}$$

which now continues upwards through integer, rather than half-integer, steps. However, in this case it is important to observe that we also have a *zero-mode* in the theory. The existence of this zero-mode means that it is possible to excite this zero-mode without increasing the overall energy of the state. We therefore have the additional tower of states

$$\begin{array}{lll} \text{vacuum:} & b_0^\dagger|0\rangle & L_0^{(\text{osc})} = 0 \\ \text{first-excited state:} & b_{-1}b_0^\dagger|0\rangle & L_0^{(\text{osc})} = 1\ . \end{array} \tag{6.3.17}$$

(Note that b_0 and $b_0^\dagger$ are equivalant.) In other words, combining (6.3.16) and (6.3.17), we see that the Ramond vacuum consists of *two degenerate states*,

$$|0\rangle \qquad \text{and} \qquad b_0^\dagger|0\rangle\ , \tag{6.3.18}$$

and that all further excitations maintain this two-fold degeneracy.

How can we interpret this two-fold degeneracy of the Ramond vacuum? It may seem, at first, that both of the states in (6.3.18) cannot be considered as the true vacuum, because the second state in (6.3.18) appears to be realized as a zero-mode excitation of the first. However, let us define the first state in (6.3.18) as $|V_0\rangle$ and let us also define $|V_1\rangle \equiv \sqrt{2}b_0^\dagger|0\rangle$, which is a rescaling of the second state in (6.3.18). Then using (6.3.11), it is easy to show that

$$|V_1\rangle = \sqrt{2}b_0^\dagger|V_0\rangle \ , \qquad |V_0\rangle = \sqrt{2}b_0^\dagger|V_1\rangle \ . \tag{6.3.19}$$

Thus, we see that neither state in (6.3.18) is more fundamental than the other, and there exists an unbroken symmetry between them — they are realized as zero-mode excitations of each other. The interpretation of this fact is that the true Ramond vacuum state is a two-component object, a spacetime spinor! It then follows that all of the excited states in the Ramond spectrum are also spacetime spinors, since they are realized as non-zero-mode excitations of a spinorial ground state.

Of course, the above discussion is only suggestive, since we have not proven that these two vacuum states actually form a Lorentz spinor representation with respect to the spacetime Lorentz algebra. However, it is easy to see that this is indeed the case. Observe from (6.3.11) that the zero-modes satisfy the algebra $\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$. Thus, if we define $\Gamma^\mu \equiv \sqrt{2}ib_0^\mu$, then we see that $\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}$, which is nothing but the spacetime Clifford algebra. In other words, the zero-modes act as spinorial gamma-matrices. This implies that all states built upon such a vacuum state will transform in spinor representations of the spacetime Lorentz symmetry group $SO(D-1,1)$, and hence will be spacetime fermions.

This is a remarkable result. Even though we have introduced worldsheet ψ^μ fields which are spacetime bosons and which carry a spacetime Lorentz *vector* index, the algebra of zero-modes in the case of Ramond boundary conditions has managed to change these vector indices into spinor indices and thereby produce spacetime fermions. Of course, this is completely analogous to what happens in the usual four-dimensional Dirac equation, where the γ^μ matrices are matrices in a spinor space but nevertheless carry vector indices. Thus, we see that by choosing Ramond boundary conditions for worldsheet fermions, string theory affords us with the same possibility. We therefore now see that string theory can indeed give rise to spacetime fermions: while excitations of worldsheet Neveu-Schwarz fermions give rise to spacetime bosons, excitations of worldsheet Ramond fermions give rise to spacetime fermions.

6.4. Lecture #4: Some famous superstrings

The next step is to determine the spectrum of the full superstring, just as we did for the bosonic string. However, the presence of two possibilities (Neveu-Schwarz and Ramond) for the modings of the fermions introduces several new complications relative to the bosonic string, and enables us to make different choices for what kind of superstring we wish to construct. These different choices are typically called different "string models", and so we are finally in a position to begin to discuss string model-building. That is the subject of the present lecture.

6.4.1. *String sectors*

Recall from the previous lecture that in light-cone gauge, the worldsheet field content of the ten-dimensional superstring consists of eight right-moving bosons X_R, eight right-moving Majorana-Weyl (one-component) fermions ψ_R, and a similar set of left-moving fields X_L and ψ_L. The bosons X_L and X_R must have periodic (integer) modings because of their interpretation as spacetime coordinates, but their worldsheet fermionic superpartners ψ_L and ψ_R can have either Ramond (periodic, integer) or Neveu-Schwarz (anti-periodic, half-integer) modings. The question then immediately arises: What rules govern the possible self-consistent choices of fermion modings? *A priori*, the appearance of 16 distinct fermions would seem to lead to 2^{16} different choices.

It is easy to see that not all possibilities are allowed, however. One quick way to see this is to realize that if some of the right-moving fermions had different periodicities than other right-moving fermions, then these different periodicities would necessarily break spacetime Lorentz invariance because these fermions carry a spacetime vector index μ. A similar situation would also hold for the left-moving fermions. This would then imply that all of the right-moving fermions should have the same periodicity as each other, and that all of the left-moving fermions should have the same periodicity as each other (though not necessarily the same as that of the right-moving fermions). However, this argument is not really satisfactory because we do not necessarily wish to preserve the full *ten*-dimensional Lorentz invariance (or even its eight-dimensional transverse subgroup); after all, our sole phenomenological requirement is that *four*-dimensional Lorentz invariance must be maintained. Moreover, it goes against the spirit of string theory (as we have been presenting it) that we should demand a certain phenomenological property of the resulting *spacetime* physics when formulating our worldsheet theory. In string theory the spacetime physics is a *consequence* of the worldsheet physics, and we would ultimately like to base our worldsheet choices directly on worldsheet symmetries.

Fortunately, it is easy to find a worldsheet argument that leads to the same constraint. Recall that the worldsheet symmetry that we must maintain is superconformal invariance. The worldsheet supersymmetry that makes up superconformal invariance is generated by the two worldsheet supercurrents given in (6.3.6). Because these two supercurrents are also worldsheet fermionic, they may also be either periodic or anti-periodic as we traverse the closed string. Indeed, each individual term $\psi^\mu \partial X_\mu$ in these supercurrents will have the periodicity property of the fermion ψ^μ. However, in order for each of these supercurrents J_R and J_L to have a unique, well-defined periodicity as we traverse the closed string, we see that it is necessary that all right-moving fermions have the same periodicity as each other, and that all left-moving fermions have the same periodicity as each other. This is required in order to preserve worldsheet supersymmetry. Thus, we have our first constraints on fermion modings:

- All right-moving fermions ψ^μ_R must have the same periodicity as each other, either Ramond or Neveu-Schwarz.
- All left-moving fermions ψ^μ_L must have the same periodicity as each other, either Ramond or Neveu-Schwarz.

Note that there is no requirement that the right- and left-moving periodicities be the same.

#	$\psi_R^{i=1,\ldots,8}$	$\psi_L^{i=1,\ldots,8}$	a_R	a_L
1	NS	NS	$-1/2$	$-1/2$
2	R	R	0	0
3	R	NS	0	$-1/2$
4	NS	R	$-1/2$	0

Given these constraints, we see that we are left with four distinct periodicity choices for our sixteen Majorana-Weyl worldsheet fermions, as shown in Table 6.1. Each individual choice is called a *sector* or *spin structure* of the superstring, so we see that the ten-dimensional superstring has four possible sectors. For future convenience, these sectors have been numbered in Table 6.1. We have also indicated the corresponding right- and left-moving vacuum energies of these sectors. Recall from the previous lecture (in particular, Great Leap #6) that the vacuum-energy contribution of each Ramond fermion is $+1/24$, while that of each Neveu-Schwarz fermion is $-1/48$ and that of each worldsheet boson is $-1/24$. Therefore, generally assuming $n_{\rm NS}$ Neveu-Schwarz fermions and $n_{\rm R}$ Ramond fermions, we can add these individual contributions to find

$$a \;=\; -\frac{1}{24}\left(8 - n_{\rm R} + \tfrac{1}{2} n_{\rm NS}\right) \;=\; -\frac{n_{\rm NS}}{16}\;. \qquad (6.4.1)$$

The second equality results from setting $n_{\rm R} = 8 - n_{\rm NS}$. Of course, as discussed above, in the ten-dimensional superstring we are restricted to the cases $n_{\rm NS} = 0, 8$ for both the right- and left-moving fermions.

6.4.2. *Modular invariance and GSO projections*

The next question that arises is whether we are free to pick any one of these sectors to construct our superstring theory, or whether we must consider all of them together, superposing the spectrum from each sector separately in order to construct the full superstring spectrum. What rules govern the choices of sectors?

Ultimately, it turns out that a special form of conformal invariance known as *modular invariance* will give us the answer. In keeping with the spirit of these lectures, we will not be able to provide a proper mathematical discussion of modular invariance. (Indeed, doing so would require a preliminary discussion of string partition functions and the modular group.) However, we can discuss the relevance and implications of modular invariance at a conceptual level.

Recall from Lecture #2 that our string actions always have a certain symmetry known as conformal invariance, which reflects the fact that the action should be invariant under *local* reparametrizations and rescalings of the coordinates (σ_1, σ_2) that parametrize the string worldsheet. For *tree*-level string interactions, demanding this *local* symmetry is sufficient to ensure that the resulting physics is indeed invariant under arbitrary coordinate reparametrizations. This is because any tree-level string interaction has the topology of a sphere (a genus-zero surface, with no handles), and on a sphere it can be shown that any possible net coordinate reparametrization can be generated or "built up" in small steps as the cumulative effect of small, local coordinate reparametrizations. Geometrically, this is equivalent to saying that any closed loop on the surface of a sphere can be continuously shrunk to a point, as illustrated in Fig. 6.7(a), by sliding the loop along the surface of the sphere towards one side. Thus, demanding invariance under *local* coordinate reparametrizations (*i.e.*, conformal invariance) by itself is sufficient to guarantee consistency for tree-level string amplitudes.

However, this situation changes drastically if we now consider one-loop amplitudes. As discussed in Lecture #1, these amplitudes have the world-sheet topology of a *torus* (a genus-*one* surface), and we see from Fig. 6.7(b) that on a torus there exist *two* types of closed loops that cannot be continuously shrunk to a point. Such loops are said to be *non-contractible*, which is indeed the defining property of such higher-genus surfaces. The presence of these non-contractible loops means that for torus diagrams, there exist possible coordinate reparametrizations that *cannot* be built up from local coordinate reparametrizations alone. Indeed, these reparametrizations non-trivially involve "large", discrete mappings around these non-contractible loops. Thus, we see that demanding conformal invariance alone is not sufficient to ensure that one-loop string amplitudes are truly invariant under worldsheet coordinate reparametrizations: we must also demand an invariance under these "large" discrete mappings around these non-contractible loops. This additional global invariance is called "modular invariance", and just like conformal invariance, it too stems from our need to maintain the overall invariance of the string under reparametrizations and rescalings of the worldsheet coordinates.

One might wonder, at this stage, why we are suddenly worrying about modular invariance, whereas we did not need to consider modular invariance in Lecture #2 when we discussed the bosonic string. The truth of the matter is that we must *always* consider modular invariance in addition to conformal invariance, regardless of the type of (closed) string we are discussing. However, in the simple case of the 26-dimensional bosonic string, it turns out that all amplitudes are trivially modular-invariant, so we did not need to make recourse to modular invariance in order to distinguish between different possibilities. However, for the superstring (and particularly for the heterotic string to be discussed later), the possible sector choices become quite numerous, and it turns out that modular invariance is the powerful tool by which we are able to narrow down the self-consistent possibilities.

What, then, are the effects of modular invariance? It turns out that at the level of string model-building, modular invariance has two primary effects:

- it forces us to consider only certain selected *sets* or *combinations* of underlying sectors, and
- it produces new *constraints* (beyond the level-matching constraint $L_0 = \bar{L}_0$) that govern which Fock-space excitations are allowed in each sector.

These new constraints are called *GSO constraints*, after F. Gliozzi, J. Scherk, and D. Olive who first imposed some of these constraints in 1977. The important point is that these conditions stem directly from modular invariance, and thus they follow from the worldsheet physics of the string and do not represent any additional arbitrary input. We will provide many explicit examples of such combinations and constraints shortly.

In order to construct a fully consistent string model, therefore, our procedure is as follows. First, we must determine which are the allowed sectors that need to be considered as part of our set. For each of these sectors in our allowed set, we then determine the corresponding Fock space of physical states by applying not only the usual level-matching constraint, but also the GSO constraints appropriate for that sector. In this way each underlying sector then gives rise to a different Fock space of states, and the full Hilbert space of states for the full string theory (*i.e.*, for the resulting string "model") is nothing but the direct sum of these different Fock spaces corresponding to each of the underlying sectors in the specified set. This then yields a fully self-consistent (and in particular, modular-invariant) theory.

This is an important point, so it is worth repeating: the full Hilbert space of string states is given by the direct sum of the different Fock spaces corresponding to different underlying boundary conditions for worldsheet fields. In order to better understand this fact, an analogy with QCD may be useful. Recall that Yang-Mills quantum field theory contains nonperturbative instanton solutions, and therefore one can imagine doing quantum field theory in an n-instanton background $|n\rangle$. Of course, as we know, the full vacuum state of QCD is not composed of any one of these $|n\rangle$ vacua by itself, but rather by an appropriately weighted *combination* of these vacua:

$$|\theta\rangle \;=\; \sum_n e^{in\theta}\,|n\rangle\ . \qquad (6.4.2)$$

This is the famous θ-vacuum of QCD. The situation that we now face in string theory is somewhat analogous. The fact that the string worldsheet fermions can have different boundary conditions (thereby giving rise to different sectors) is in some sense analogous to the fact that QCD can have different instanton backgrounds. Indeed, each underlying string sector is analogous to a different n-instanton vacuum state $|n\rangle$, and the different "combinations of sectors" that we are now being forced to consider are analogous to the different QCD θ-vacua. In this sense, then, each different "string model" that we will be constructing can be viewed as a different θ-vacuum of string theory! Of course, this analogy with the QCD θ-vacuum can take us only so far. One important difference is that whereas the θ-vacuum necessarily involves *all* of the $|n\rangle$ states regardless of the value of θ, in string theory our "vacuum" may consist of more complicated combinations of sectors which may or may not include all possible sectors. In fact, the more sectors that are included in our "combination of sectors", the more GSO constraints there are for each sector. But the important lesson that emerges from all of this is that no single sector by itself forms a consistent string vacuum; rather, we must select an appropriate combination of sectors and add together their corresponding Fock spaces in order to produce the fully self-consistent string model.

In Lecture #7, we shall provide an explicit set of rules which will enable us to quickly determine the appropriate sector combinations and GSO constraints that can be chosen in order to yield self-consistent theories. For the time being, however, we shall defer a discussion of these rules and proceed directly with the construction of actual string models in order to deduce their physical properties. Therefore, even though we shall simply assert certain sector combinations and GSO projections to be required by modular invariance, we stress that all of these features can (and ultimately will) be derived using the rules to be presented in Lecture #7.

6.4.3. *Ten-dimensional superstring models*

In the case of the ten-dimensional superstring, we have already seen that the four possible sectors are listed in Table 6.1. It then only remains to determine the particular sector combinations and GSO constraints that are required by modular invariance. In this case, it turns out that there are only two possible combinations or sets of sectors that can be considered:

- we consider the contributions from only Sectors #1 and #2, or
- we consider the contributions from *all* Sectors #1 through #4.

Moreover, for each of the above cases, it turns out that there are two possible choices of GSO projections that may be imposed in each sector. Thus, combining all of these possibilities, we see that there are four distinct possible superstring "models" that can be constructed in ten dimensions. We shall therefore now turn to a construction of these models.

6.4.3.1. *The Type 0 strings*

Let us begin by considering the first option, taking our set of sectors to consist only of Sectors #1 and #2. For each of these sectors, we need to determine the appropriate GSO constraints that must be applied in addition to the usual level-matching constraint. In order to write down these GSO constraints, let us first recall that for a given left-moving worldsheet fermion (with either Ramond or Neveu-Schwarz boundary conditions), the corresponding number operator is defined by

$$\begin{aligned} \text{R}: \qquad N^{(i)} &= \sum_{n=0}^{\infty} b^i_{-n} b_{ni} \\ \text{NS}: \qquad N^{(i)} &= \sum_{r=1/2}^{\infty} b^i_{-r} b_{ri} \ . \end{aligned} \tag{6.4.3}$$

Here the index $i = 1, ..., 8$ labels the individual fermion. For right-moving fermions, the analogous number operators $\bar{N}^{(i)}$ are constructed using the right-moving mode operators $\tilde{b}_n, \tilde{b}_r$. Let us also define N_L and N_R respectively as the total left- and right-moving number operators, *i.e.*,

$$N_L \equiv \sum_{i=1}^{8} N^{(i)} , \qquad N_R \equiv \sum_{i=1}^{8} \bar{N}^{(i)} . \tag{6.4.4}$$

Note that these number operators are defined to include only the contributions of the worldsheet *fermions*, and in particular do not include the contributions of the worldsheet bosons. It then turns out (and we shall see in Lecture #7) that if we choose our set of sectors to consist only of Sectors #1 and #2, then the appropriate GSO constraints in each sector are as follows:

$$\begin{aligned} \text{Sector \#1:} \qquad N_L - N_R &= \text{even} \\ \text{Sector \#2:} \qquad N_L - N_R &= \left\{ \begin{matrix} \text{odd} \\ \text{even} \end{matrix} \right\} . \end{aligned} \tag{6.4.5}$$

In the second line, we have used a brace notation to indicate a further choice: we can choose to impose *either* the 'odd' constraint, or the 'even' constraint. As we shall see, this is a residual choice that is not fixed by modular invariance (or by any other worldsheet symmetry), leading to two equally valid possibilities. Thus, we see that if we choose our set of sectors to consist of only Sectors #1 and #2, then this leads to *two different string models* depending on our subsequent choice of which GSO constraint we choose to impose in (6.4.5).

Let us now determine the spectra of these two models, beginning with the states that arise from Sector #1. Note that in this sector, both models have the same states (because both models have the same GSO constraint for Sector #1). As with the bosonic string, our procedure is to consider all possible excitations of the worldsheet fields (in this case, the worldsheet fermions as well as the worldsheet bosons). These excitations are subject to the level-matching constraint $L_0 = \bar{L}_0$ (which ensures that the total bosonic and fermionic worldsheet *energy* is distributed equally between left- and right-moving excitations) and the GSO constraint $N_L - N_R =$ even (which is a constraint on the worldsheet *number* operators of the worldsheet fermions only). In general, the mass-shell condition for the superstring is

$$\alpha' M^2 \;=\; 2\,(L_0 + \bar{L}_0 + a_L + a_R) \tag{6.4.6}$$

where a_L and a_R are the individual left- and right-moving vacuum energies, and where L_0 and $\bar{L}_0$ include the contributions from not only the worldsheet bosons, but also the worldsheet fermions. Note from Table 6.1 that the left- and right-moving vacuum energies in Sector #1 are $a_L = a_R = -1/2$.

We see that the tachyonic vacuum state $|0\rangle_R \otimes |0\rangle_L$ satisfies both constraints, and thus it remains in the spectrum. However, unlike the tachyon in the bosonic string (which has spacetime mass $\alpha' M^2 = -4$), we see from (6.4.6) that the tachyonic state in the superstring has spacetime mass $\alpha' M^2 = -2$. This is the result of the smaller (less negative) vacuum energy of the superstring compared to that of the bosonic string.

Because the vacuum energies in Sector #1 are $a_L = a_R = -1/2$, we see that massless states cannot be obtained by exciting the quantum modes of worldsheet bosons, for each of these excitations would add a full unit of energy. Instead, massless states can be obtained only by adding a half-unit of energy. Fortunately, this is possible in Sector #1 because in this sector, all worldsheet fermions have Neveu-Schwarz boundary conditions and therefore have half-integer modings. The first excited states in Sector #1 are therefore

$$\tilde{b}^{\mu}_{-1/2}|0\rangle_R \;\otimes\; b^{\nu}_{-1/2}|0\rangle_L \;. \tag{6.4.7}$$

Note that these states satisfy both the level-matching constraint (since $L_0 = \bar{L}_0 = 1/2$) as well as the GSO constraint (since $N_L = N_R = 1$).

The interpretation of these states is precisely the same as in the bosonic string: these states give us the gravity multiplet, consisting of the graviton $g_{\mu\nu}$, dilaton ϕ, and anti-symmetric tensor $B_{\mu\nu}$. Mathematically, this is equivalent to the tensor-product rule for Lorentz transverse $SO(8)$ vector representations:

$$\mathbf{V}_8 \otimes \mathbf{V}_8 \;=\; \mathbf{1} \;\oplus\; \mathbf{28} \;\oplus\; \mathbf{35} \tag{6.4.8}$$

where $\mathbf{V}_8$ is the eight-dimensional vector representation, and where the $\mathbf{1}$ representation is the spin-zero state, the $\mathbf{28}$ representation is the spin-one state, and the $\mathbf{35}$ representation is the spin-two state. It is indeed a general principle that *all* weakly coupled closed strings contain at least these massless states, and this is a useful cross-check of the GSO constraints.

Let us now turn to the states from Sector #2. Before concerning ourselves with the implication of the GSO constraints in (6.4.5), let us first understand the general structure of the states from this sector. In this sector, the vacuum energy (according to Table 6.1) is $(a_R, a_L) = (0, 0)$, so we see immediately that this sector contains no tachyons. Indeed, the ground state is already massless, so all that will concern us here is the nature of this ground state. As we discussed at the end of Lecture #3, the left- and right-moving ground states in this sector are each spacetime *spinors* since all worldsheet fermions in this sector have Ramond boundary conditions. Because the nature of these spinors will be important to us, let us pause to review some properties of these spinors.

Since we are considering these ten-dimensional strings in light-cone gauge, the Lorentz group that concerns us here is the transverse ("little") Lorentz group $SO(8)$. In general, the groups $SO(2n)$ share a number of properties. Their smallest representations, of course, are simply the identity representations. These are singlets, which will be denoted $\mathbf{1}$. The next representations are the vector representations, which are $(2n)$-dimensional, and which will be denoted $\mathbf{V}_{2n}$. Along with these are the spinor representations, which are (2^{n-1})-dimensional. In general, there are two types of spinor representations, $\mathbf{S}$ and $\mathbf{C}$, the so-called "spinor" and "conjugate spinor" representations. In the special case of $SO(8)$, the vector, spinor, and conjugate spinor representations are all eight-dimensional, and will be denoted $\mathbf{V}_8$, $\mathbf{S}_8$, and $\mathbf{C}_8$ respectively. The distinction between $\mathbf{S}_8$ and $\mathbf{C}_8$ is one of spacetime *chirality*, but the choice of which is to be associated with a given physical chirality is a matter of convention.

The ground state of Sector #2 has the structure

$$\{\tilde{b}_0^{\mu}\}\, |0\rangle_R \;\otimes\; \{b_0^{\nu}\}\, |0\rangle_L \tag{6.4.9}$$

where the notation $\{b_0^\mu\}$ (and similarly for the right-movers) indicates that each of the individual Ramond zero-modes can be either excited or not excited.

How can we interpret (6.4.9) physically? This issue is actually quite subtle, and we shall not have the space to give a proper discussion. Moreover, as we have already indicated, we are not giving a fully rigorous treatment of Ramond zero-modes in these lectures, since our aim is to focus more on the physics than the formalism. However, it is possible to understand the appropriate physical interpretation intuitively. First, let us *count* the number of states in (6.4.9). *A priori*, it would seem that we have 2^{16} individual states, since each Ramond fermion zero-mode can either be excited or not excited. However, this is not correct because (as we shall discuss more completely in Lecture #5, and as we have already hinted in the footnote in Sect. 3.2), one should really count only one zero-mode per *pair* of Ramond Majorana-Weyl fermions. Thus, we can imagine that there are only four independent zero-modes for the right-movers, and four for the left-movers. Therefore, (6.4.9) consists of only $2^8 = 128$ states.

All combinations of these zero-mode excitations already satisfy the level-matching constraint (since $L_0 = \bar{L}_0 = 0$). Imposing either of the GSO constraints for Sector #2 in (6.4.5) then reduces the number of allowed states by a factor of two. Specifically, if we impose the constraint $N_L - N_R =$ odd, then we can choose only an even number of right-moving zero-mode excitations together with an odd number of left-moving zero-mode excitations, or an odd number of right-moving excitations together with an even number of left-moving excitations. Choosing the constraint $N_L - N_R =$ even has the opposite effect, pairing even numbers of excitations for left- and right-movers with each other, and likewise pairing odd numbers with each other.

Interpreting these results is therefore quite simple. As we discussed at the end of Lecture #3, the left-moving states and right-moving states are spacetime spinors, and we have already seen that there are two possible spinors, $\mathbf{S}_8$ and $\mathbf{C}_8$. At this stage, the names assigned to each are arbitrary, so we shall now establish the following convention: spinors realized by an even number of zero-mode excitations will be identified with $\mathbf{C}_8$, and those realized by an odd number of zero-mode excitations will be identified with $\mathbf{S}_8$. Of course, only the relative difference between these two spinors is physically significant (having the interpretation of spacetime chirality).

Given these definitions, we see that if we choose the first GSO constraint

$N_L - N_R =$ odd, the 128 states in (6.4.9) decompose into

$$\left(\bar{\mathbf{C}}_8 \otimes \mathbf{S}_8\right) \;\oplus\; \left(\bar{\mathbf{S}}_8 \otimes \mathbf{C}_8\right) \;, \tag{6.4.10}$$

whereas if we choose the second GSO constraint $N_L - N_R =$ even, these states instead decompose into

$$\left(\bar{\mathbf{C}}_8 \otimes \mathbf{C}_8\right) \;\oplus\; \left(\bar{\mathbf{S}}_8 \otimes \mathbf{S}_8\right) \;. \tag{6.4.11}$$

If we wish to further decompose these states into representations of the Lorentz group, we can use the $SO(8)$ tensor-product relations

$$\begin{aligned} \mathbf{S}_8 \otimes \mathbf{S}_8 &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}' \\ \mathbf{C}_8 \otimes \mathbf{C}_8 &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}'' \\ \mathbf{S}_8 \otimes \mathbf{C}_8 &= \mathbf{V}_8 \oplus \mathbf{56} \;. \end{aligned} \tag{6.4.12}$$

Here the **28** representation is the anti-symmetric component of the spinor tensor product (spin-one), while the $\mathbf{35}'$ and $\mathbf{35}''$ representations are the symmetric components of the spinor tensor product (also spin-one). (These latter representations are not to be confused with the spin-two **35** graviton representation in (6.4.8).) Likewise, the **56** is a certain vectorial (spin-one) higher-dimensional representation.* However, for our present purposes it will be sufficient to think of these states in the tensor-product forms (6.4.10) and (6.4.11). Note that in each case, the tensor product of two spacetime fermionic (spinor) states produces a spacetime bosonic state. Thus, just as in Sector #1, the states emerging in Sector #2 are spacetime bosons.

Thus, summarizing, we see that the spectra of our two resulting superstring models are as follows. First, from Sector #1, we have the tachyonic state $|0\rangle_R \otimes |0\rangle_L$. In the notation of $SO(8)$ Lorentz representations, this state may be denoted $\bar{\mathbf{1}} \otimes \mathbf{1}$; this tachyon is a Lorentz singlet. Next, we have the massless gravity multiplet. In the notation of $SO(8)$ Lorentz representations, this state takes the form $\bar{\mathbf{V}}_8 \otimes \mathbf{V}_8$. Finally, from Sector #2, we have massless states whose form depends on the particular choice of the GSO projection. In the first case, we have the states given in (6.4.10), while

*For the mathematically inclined reader, we can succinctly describe all of these states as follows. Recall that a given representation is called a p-form if it can be realized as the totally anti-symmetric combination within the tensor product of p different vector indices of $SO(8)$, with resulting dimension $8 \times 7 \times 6 \times ... \times (9-p)/p!$. Using this language, we see that singlet states are zero-forms, the **28** representations are two-forms, and the $\mathbf{35}'$ and $\mathbf{35}''$ representations are "self-dual" four-forms. (The self-duality condition eliminates exactly half of the degrees of freedom in the four-form.) Likewise, the $\mathbf{V}_8$ state is a one-form, and the **56** representation is a three-form. These different forms (and the so-called *D-branes* whose existence they imply) are important when considering the non-perturbative structure of these string theories.

in the second case, we have the states given in (6.4.11). There are then, as usual, an infinite tower of massive (Planck-scale) states above these.

The string model produced by the first GSO projection is called the *Type 0A* string model, and the second is called the *Type 0B* string model. Collectively, these are sometimes simply called the Type 0 strings. As we see, both of these strings are tachyonic, and moreover they contain only bosonic states. Furthermore, as is evident from (6.4.10) and (6.4.11), both of these strings are non-chiral. In other words, they are invariant under the transposition $\mathbf{S}_8 \leftrightarrow \mathbf{C}_8$ for the left- and right-movers. These string theories were first constructed by N. Seiberg and E. Witten in 1985. Although not relevant for phenomenology, they are currently proving to have an important role in understanding certain non-perturbative aspects of non-supersymmetric string theory.

6.4.3.2. *The Type II strings*

Let us now turn to the second choice outlined at the beginning of Sect. 4.3, namely the case in which we consider the contributions from *all* of the sectors in Table 6.1. This will result in the so-called *Type II strings.* As we discussed at the end of Sect. 4.2, it is a general property that the larger the set of sectors that we consider, the more GSO constraints there are that must be imposed in each sector. Thus, the introduction of new sectors generally leads to new GSO constraints in each of the sectors (old and new), and likewise the introduction of new GSO constraints in a given sector requires the introduction of entire new sectors to compensate.

It turns out (and we shall see explicitly in Lecture #7) that if we consider the full set of sectors in Table 6.1, then the appropriate GSO constraints in each sector are given as follows:

$$\begin{array}{ll}
\text{Sector \#1:} & N_L - N_R = \text{odd}\ , \quad N_R = \text{odd} \\
\text{Sector \#2:} & N_L - N_R = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}\ , \quad N_R = \text{odd} \\
\text{Sector \#3:} & N_L - N_R = \text{even}\ , \quad N_R = \text{odd} \\
\text{Sector \#4:} & N_L - N_R = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}\ , \quad N_R = \text{odd}\ .
\end{array} \tag{6.4.13}$$

Note that in each case where a choice is possible, these choices are correlated: we simultaneously choose either the top lines within all braces, or the bottom lines. Thus, once again there are two sets of GSO conditions that can be imposed, resulting in two distinct string models.

Before proceeding further, it is useful to note the *pattern* of these GSO projections. In the case of the Type 0 strings, we considered only Sectors #1 and #2; as shown in Table 6.1, these were the sectors for which the right-moving fermions were always identical to the left-moving fermions and shared the same boundary conditions. The corresponding GSO projections in (6.4.5) likewise did not distinguish between right- and left-moving fermions. (In this context, note that the GSO projections in (6.4.5) can equivalently be written with minus signs replaced by plus signs.) Thus, in some sense, the Type 0 strings are symmetric under exchange of left- and right-movers. However, for the Type II strings, we have now introduced two additional sectors (Sectors #3 and #4) whose structure explicitly breaks this symmetry between left- and right-movers. No longer does each sector individually exhibit this left/right symmetry. As we see from (6.4.13), the effect of this breaking is to introduce additional GSO conditions which mirror this broken symmetry by becoming sensitive to right- or left-moving number operators *by themselves*. The technical word for this breaking of symmetry is "twisting" or "orbifolding", for by including Sectors #3 and #4, we see that we have twisted the left-movers relative to the right-movers by allowing them to have oppositely moded boundary conditions. Thus, the Type II strings that will result can be viewed as twisted (or orbifolded) versions of the Type 0 strings. This twisting procedure ultimately serves as the means by which more and more complicated (and more and more phenomenologically realistic) string models may be constructed, and will be discussed more fully in Lecture #7.

Given the GSO constraints in (6.4.13), we can proceed to determine the resulting spectrum just as we did for the Type 0 strings. Let us begin with Sector #1 (this is often called the "NS-NS sector"). Because the boundary conditions of the worldsheet fermions are the same in this sector as they were for the Type 0 strings, the possible states that arise are the same as they were for the Type 0 strings, and consist of the tachyon $|0\rangle_R \otimes |0\rangle_L$ as well as the gravity multiplet (6.4.7). The only difference is that we must now impose the additional GSO constraint $N_R = \text{odd}$. It is immediately clear that the effect of this new GSO constraint is that *the tachyon is projected out of the spectrum*, while the gravity multiplet is retained. Thus, by "twisting" the Type 0 strings in just this way, we have succeeded in curing one of the major problems of the bosonic and Type 0 strings, namely the appearance of tachyons. Moreover, we have done this *without* eliminating the desirable gravity multiplet.

Let us now consider the states from Sector #2 (this is often called the

"Ramond-Ramond" sector). Once again, if we impose only the first GSO constraint in (6.4.13), we obtain the states in either (6.4.10) or (6.4.11). Imposing the additional GSO constraint in (6.4.13) then enables us to project out half of these states, so that we retain only the states

$$\bar{\mathbf{S}}_8 \otimes \left\{ \begin{matrix} \mathbf{C}_8 \\ \mathbf{S}_8 \end{matrix} \right\} . \tag{6.4.14}$$

These states are spacetime bosons.

Finally, let us consider the states that arise in the new Sectors #3 and #4. In Sector #4, the vacuum energy is $(a_R, a_L) = (-1/2, 0)$. Therefore, in order to have level-matching ($L_0 = \bar{L}_0$), we see that we are immediately forced to excite a half-unit of energy for the right-movers while not increasing the energy of the left-movers. This is the only way to produce a massless state. This also ensures that this sector does not give rise to tachyons. Fortunately, since the right-moving fermions have Neveu-Schwarz boundary conditions in this sector, these fermions have half-integer modings, and thus by exciting their lowest modes we can indeed introduce a half-unit of energy. The left-moving fermions have Ramond boundary conditions in this sector, and hence their ground state is the Ramond zero-mode state. The massless states in Sector #4 therefore take the form

$$\tilde{b}^{\mu}_{-1/2} |0\rangle_R \otimes \{b_0^{\nu}\} |0\rangle_L . \tag{6.4.15}$$

At this stage, of course, these states satisfy only the level-matching constraint. Imposing the GSO constraints then leaves us with the state in which we excite only an even (or odd) number of left-moving Ramond zero modes.

How can we interpret this state? First, we notice that this state is a spacetime *fermion* because it results from tensoring a right-moving Neveu-Schwarz state with a left-moving Ramond state. Thus, we now have a string theory that contains spacetime fermions! This is yet another benefit of performing the "twist" that takes us from the Type 0 strings to the Type II strings. However, let us examine this state a bit more closely. Clearly, it has the Lorentz structure

$$\overline{\mathbf{V}_8} \otimes \left\{ \begin{matrix} \mathbf{C}_8 \\ \mathbf{S}_8 \end{matrix} \right\} \tag{6.4.16}$$

where we have retained the spinor-labelling conventions that we employed for the Type 0 strings. The relevant tensor-product decompositions in this case are given by

$$\begin{aligned}\overline{\mathbf{V}_8} \otimes \mathbf{C}_8 &= \mathbf{S}_8 \oplus \mathbf{56}' \\ \overline{\mathbf{V}_8} \otimes \mathbf{S}_8 &= \mathbf{C}_8 \oplus \mathbf{56}''\end{aligned} \qquad (6.4.17)$$

where the $\mathbf{S}_8$ and $\mathbf{C}_8$ representations are spin-1/2 and where the $\mathbf{56}'$ and $\mathbf{56}''$ representations are spin-3/2. Thus, we see that the Type II strings contain a massless, spin-3/2 object! Just as a massless spin-two object satisfies the Einstein field equations and must be interpreted as the graviton, a massless spin-3/2 object must be interpreted as a *gravitino* — *i.e.*, a superpartner of the graviton. This implies that this string not only gives rise to spacetime bosons *and* fermions, but actually gives rise a spectrum which exhibits *spacetime supersymmetry*! This is yet another phenomenologically compelling feature.

Finally, let us now consider Sector #3. This sector has vacuum energies $(a_R, a_L) = (0, -1/2)$, so now we must excite right-moving zero-modes and left-moving $b^\mu_{-1/2}$ modes. This then leads to states of the form

$$\{\tilde{b}^\mu_0\}\, |0\rangle_R \;\otimes\; b^\nu_{-1/2}\, |0\rangle_L \;, \qquad (6.4.18)$$

and imposing the GSO projections results in states with the Lorentz structure $\bar{\mathbf{S}}_8 \otimes \mathbf{V}_8$. Once again, this also contains a gravitino!

So what do we have in the end? The first choice of GSO projections results in the so-called *Type IIA string*, while the second choice results in the *Type IIB string*. Both of these strings are tachyon-free, and their spectra contain both bosons and fermions. Moreover, these strings exhibit *spacetime* supersymmetry. This is most easily seen in the following suggestive way. Let us collect together the states from all four sectors, retaining our Lorentz-structure tensor-product notation:

$$\bar{\mathbf{V}}_8 \otimes \mathbf{V}_8 \;, \quad \bar{\mathbf{S}}_8 \otimes \left\{ \begin{matrix} \mathbf{C}_8 \\ \mathbf{S}_8 \end{matrix} \right\} \;, \quad \bar{\mathbf{V}}_8 \otimes \left\{ \begin{matrix} \mathbf{C}_8 \\ \mathbf{S}_8 \end{matrix} \right\} \;, \quad \bar{\mathbf{S}}_8 \otimes \mathbf{V}_8 \;. \qquad (6.4.19)$$

Together, this collection of states can be written in the factorized form

$$\left(\bar{\mathbf{V}}_8 \oplus \bar{\mathbf{S}}_8\right) \;\otimes\; \left(\mathbf{V}_8 \oplus \left\{ \begin{matrix} \mathbf{C}_8 \\ \mathbf{S}_8 \end{matrix} \right\}\right) \;. \qquad (6.4.20)$$

We thus see that there are *two* spacetime supersymmetries exhibited in this massless spectrum: the first exchanges $\bar{\mathbf{V}}_8 \leftrightarrow \bar{\mathbf{S}}_8$ amongst the right-movers, while the second exchanges

$$\mathbf{V}_8 \leftrightarrow \left\{ \begin{matrix} \mathbf{C}_8 \\ \mathbf{S}_8 \end{matrix} \right\} \qquad (6.4.21)$$

amongst the left-movers. Thus, the massless spectrum exhibits $N = 2$ supersymmetry. This is, of course, consistent with the appearance of two gravitinos in the massless spectrum (one from Sector #3 and one from Sector #4). Another way to understand this $N = 2$ supersymmetry is to realize that the first supersymmetry relates the bosonic states in Sector #1 to the fermionic states in Sector #3 (and the bosons in Sector #2 to the fermions in Sector #4), while the second supersymmetry relates the bosons in Sector #1 to the fermions in Sector #4 (and the bosons in Sector #2 to the fermions in Sector #3). In either case, we thus see that we have two independent spacetime supersymmetries.

It is important to note that we did not demand spacetime supersymmetry when constructing the superstring. We merely introduced *worldsheet* supersymmetry, and found that spacetime supersymmetry emerged naturally as the result of certain GSO projections. This further illustrates the fact that in string theory, spacetime properties such as supersymmetry emerge only as the consequences of deeper, more fundamental *worldsheet* symmetries. Another important point is that the same "twist" which eliminated the tachyon has introduced spacetime supersymmetry. While this is certainly an interesting phenomenon that arises for ten-dimensional superstrings, it is certainly *not* a general property that the elimination of the tachyon requires spacetime supersymmetry. In particular, we shall see in Lecture #6 that it is possible to construct string theories whose tree-level spectra lack spacetime supersymmetry but nevertheless are tachyon-free.

One might question whether we have really demonstrated the existence of $N = 2$ supersymmetry, since we have examined only the massless spectrum. However, it can be shown that any unitary theory which contains a massless spin-3/2 state necessarily exhibits supersymmetry, and hence must be supersymmetric at all mass levels (*i.e.*, for all massive, excited states as well). Of course, this is still not a proof, since we do not *a priori* know (and would therefore need to verify) that string theory is a consistent theory in this sense. However, it is possible to construct (two) explicit spacetime supercurrent operators and to demonstrate that they commute with the full (massless and massive) spectrum of the string. Another approach (as indicated in the footnote in Sect. 3.1) is to develop an alternative formulation of the superstring in which *spacetime* (rather than worldsheet) supersymmetry is manifest at the level of the string action, and to demonstrate the equivalence of the two formulations. Indeed, both approaches have been successfully carried out, thereby demonstrating that the Type II spectrum is indeed $N = 2$ supersymmetric. It is for this reason that these strings are referred to as Type II strings.

One important distinction between these two strings is their chirality. The Type IIA string, as we see, contains two supersymmetries of opposite chiralities, interchanging $\bar{\mathbf{V}}_8 \leftrightarrow \bar{\mathbf{S}}_8$ for the right-movers and $\mathbf{V}_8 \leftrightarrow \mathbf{C}_8$ for the left-movers. Equivalently, the two gravitinos associated with these supersymmetries are of opposite chiralities (because the $\mathbf{56}'$ and $\mathbf{56}''$ representations in (6.4.17) are of opposite chiralities). Because it contains supersymmetries of both chiralities, this string is ultimately non-chiral, and its low-energy (field-theoretic) limit consists of so-called *Type IIA* supergravity (whose discovery predates that of the Type IIA string). It is for this reason that this string is called the Type IIA string. The Type IIB string, by contrast, contains two supersymmetries (or two gravitinos) of the *same* chirality, exchanging $\bar{\mathbf{V}}_8 \leftrightarrow \bar{\mathbf{S}}_8$ and $\mathbf{V}_8 \leftrightarrow \mathbf{S}_8$ respectively. Thus, this string theory is *chiral*, and has a low-energy field-theoretic limit consisting of Type IIB supergravity.

We conclude, then, that by introducing a twist relative to the Type 0 strings, we have constructed a set of strings (the Type IIA and Type IIB strings) that exhibit a number of compelling features: they are tachyon-free, they contain both bosons and fermions in their spacetime spectra, they contain gravity, and they are spacetime $N = 2$ supersymmetric. Despite this success, however, there is still something that we lack: we do not, as yet, have gauge symmetries. Specifically, there are no gauge bosons (such as photons, gluons, or W and Z particles). Likewise, there are no states which carry gauge charges. Therefore, once again, we shall need to construct a new kind of string.

6.5. Lecture #5: Neutrinos have gauge charges: The heterotic string

6.5.1. *Motivation and alternative approaches*

Thus far in these lectures, we have shown how string theory can give rise to quantized gravity, spacetime bosons and fermions, spacetime supersymmetry, and tachyon-free spectra. There is, however, one important phenomenological feature that is still missing: *gauge symmetry.* In other words, we wish to have massless gauge bosons, *i.e.*, spacetime vectors that transform in the adjoint representation of some internal symmetry group. As a side issue, we would also like to find a way of breaking $N = 2$ supersymmetry to $N = 1$ supersymmetry (if our goal is to reproduce the MSSM) or even to $N = 0$ supersymmetry (if our goal is to reproduce the Standard Model).

It is worth considering why such gauge-boson states fail to appear for the ten-dimensional Type II strings discussed in the previous lecture. The problem is the following. In order to produce worldsheet bosons, we are restricted to considering only the NS-NS or Ramond-Ramond sectors (Sectors #1 and #2 in Table 6.1). In the NS-NS sector (Sector #1), the vacuum energy is $(a_R, a_L) = (-1/2, -1/2)$, so we must excite the half-energy fermionic mode oscillators $\tilde{b}^\mu_{-1/2}, b^\mu_{-1/2}$ for the both the left- and right-movers. This produces a state with *two* vector indices rather than one, and as we see from the vector-vector tensor-product decomposition in (6.4.8), this does not contain a vectorial state. In the Ramond-Ramond sector (Sector #2), by contrast, the vacuum energy is $(a_R, a_L) = (0, 0)$, which implies that our massless states comprise the tensor product of two Ramond spinors as in (6.4.10) for the Type IIA string, or as in (6.4.11) for the Type IIB string. In the case of the Type IIB string, we see from (6.4.12) that the tensor product $\bar{\mathbf{S}}_8 \otimes \mathbf{S}_8$ does not contain a vector state $\mathbf{V}_8$. Thus, the Type IIB string contains no massless vectors. In the case of the Type IIA string, we observe from (6.4.17) that indeed $\bar{\mathbf{S}}_8 \otimes \mathbf{C}_8 \supset \mathbf{V}_8$, and thus the Type IIA string does contain a massless vector. (This state is often called a "Ramond-Ramond gauge boson".) However, the $U(1)$ "gauge" symmetry associated with this state is too small to contain the Standard-Model gauge group, and moreover it can be shown that no states in the perturbative spectrum of the Type IIA string spectrum can carry this Ramond-Ramond charge.*

In each case, the fundamental obstruction that we face is that we need to generate representations of a *gauge group* (*i.e.*, an internal symmetry group) that is *different* from the Lorentz group. Until now, all of our worldsheet fields (such as $X^\mu_{L,R}$ and $\psi^\mu_{L,R}$) have carried Lorentz indices associated with the $SO(D-1, 1)$ Lorentz symmetry. In order to produce a separate gauge symmetry, we therefore need fields which do *not* carry a Lorentz index but which carry a purely internal index. (Note that these fields cannot carry a Lorentz index because we ultimately want our gauge symmetries to commute with the Lorentz symmetries.)

*Despite this fact, Ramond-Ramond charge plays a crucial role in recent developments concerning string duality. While none of the states in the *perturbative* Type IIA string spectrum carry Ramond-Ramond charge, these strings also contain non-trivial *solitonic* states (so-called *D-branes*) which do carry Ramond-Ramond charge. We shall briefly discuss D-branes in Lecture #8.

How can we do this? One idea is to *compactify* the Type II strings that we constructed in the previous lecture. Although this approach ultimately fails for phenomenological reasons, it will be instructive to briefly explain this idea. Recall that for the superstring, the critical dimension $D = 10$ emerges as the result of an anomaly cancellation argument: each worldsheet boson X contributes $c_X = 1$, each Majorana fermion ψ^μ contributes $c_\psi = 1/2$, and thus ten copies of each are necessary in order to cancel the "background" central charge associated with the worldsheet superconformal symmetry. But, even though we require ten bosons and ten fermions, there is no reason why we must endow *all* of them with Lorentz vector indices μ. Since we are ultimately interested in four-dimensional string theories, one natural idea is to consider these ten bosons and ten fermions in two groups, four with indices $\mu = 0, 1, 2, 3$, and the remaining six with purely internal indices $i = 1, ..., 6$. This internal symmetry could then be interpreted as a gauge symmetry.

This idea is in fact reminiscent of the original Kaluza-Klein idea whereby gauge symmetries are realized from higher-dimensional gravitational theories upon compactification. Moreover, this idea does succeed in producing gauge bosons (and gauge symmetries) in dimensions $D < 10$. However, the problem is that this idea fails to produce *enough* gauge symmetry. Specifically, although we obtain gauge symmetries that are large enough to contain the Standard Model gauge symmetry $SU(3) \times SU(2) \times U(1)$, we cannot obtain massless representations that simultaneously transform as triplets of $SU(3)$ and doublets of $SU(2)$. Such "quark" representations are required phenomenologically. Thus, even though this compactification idea is interesting as a way of generating certain amounts of gauge symmetry, it cannot be used in order to save the superstring.

What we require, then, is a different way of introducing worldsheet fields without Lorentz vector indices. Since we will (temporarily) abandon the idea of removing Lorentz indices from our ten worldsheet bosons and fermions, what this means is that we require a way of obtaining *even more worldsheet fields* in ten dimensions. In other words, if we want bigger gauge symmetries in $D = 4$, then we require more than six extra fields with internal indices i, which in turn means that we already want extra fields even in the original ten-dimensional interpretation.

But how can we introduce extra worldsheet fields without violating our previous conformal anomaly cancellation arguments? Just adding extra fields will reintroduce the conformal anomaly at the quantum level.

6.5.2. *The heterotic string: Constructing the action*

The idea, of course, is to abandon the Type II string and proceed to construct a new kind of string that can accomplish the goal. This string is called the *heterotic* string, and it is this string that will be our focus for the remainder of these lectures. This string was first introduced by D. Gross, J. Harvey, E. Martinec, and R. Rohm in 1985, and for more than a decade dominated (and still continues to play a pivotal role in) discussions of string phenomenology.

Let us begin by recalling the action of the bosonic string:

$$S_{\text{bosonic}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \; \left\{(\partial_- X_R^\mu)^2 + (\partial_+ X_L^\mu)^2\right\} . \tag{6.5.1}$$

Here the worldsheet symmetry is simply conformal invariance, which requires that we take $\mu = 0, 1, ..., 25$ in order to cancel the conformal anomaly. Clearly, this action contains lots of worldsheet fields. However, we saw in Lecture #2 that this string does not give rise to spacetime fermions.

Next, we considered the superstring, whose action is given by:

$$S_{\text{super}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \; \left\{(\partial_- X_R^\mu)^2 - \psi_R^\mu \partial_- \psi_{R\mu} + (\partial_+ X_L^\mu)^2 - \psi_L^\mu \partial_+ \psi_{L\mu}\right\} . \tag{6.5.2}$$

Here the worldsheet symmetry is *superconformal* invariance, which requires that we take $\mu = 0, 1, ..., 9$ in order to cancel the superconformal anomaly. Unlike the bosonic string, this string gives rise to spacetime fermions. But as we have just explained, this string does not contain enough worldsheet fields to give rise to appropriate gauge symmetries.

Clearly, each of these strings has an advantage lacked by the other. The natural solution, then, is to attempt to "weld" them together, to "crossbreed" them in such a way as to retain the desirable attributes of each. But how can this be done?

The fundamental observation is that we are always dealing with *closed* strings, and for closed strings, we have seen that the left- and right-moving modes are essentially independent of each other and form separate theories. Indeed, only the level-matching constraint $L_0 = \bar{L}_0$ serves to relate these two halves to each other, but even this constraint applies at the level of the physical Fock space rather than the level of the action. Therefore, since these two halves are essentially independent, a natural idea is to construct a new hybrid string whose left-moving half is the left-moving half of the bosonic string, but whose right-moving half is the right-moving half of the superstring. As we shall see, this fundamental idea is just what we need.

The resulting string is therefore called a *heterotic* string, where the prefex *hetero-* indicates the joining of two different things.

Given this idea, let us now see how the action for the heterotic string can be constructed. We shall do this in three successive attempts. Our first attempt would be to write an action of the form

$$S \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma\;\left\{(\partial_- X_R^\mu)^2 \;-\; \psi_R^\mu\partial_-\psi_{R\mu} \;+\; (\partial_+ X_L^\mu)^2\right\}\;. \qquad (6.5.3)$$

In this case, the worldsheet symmetry would be conformal invariance for the left-movers, but superconformal invariance for the right-movers.

But what is the spacetime dimension of such a string? If we consider the right-moving sector, then just as in the superstring we would require $D = 10$, so that $\mu = 0, 1, ..., 9$. But given this, how do we interpret the left-moving side of the heterotic string? On the left-moving side, cancellation of the *conformal* (rather than superconformal) anomaly requires that we still retain 26 X_L fields! But if only ten of these fields are spacetime coordinates, then the remaining sixteen must be mere internal scalar fields. In other words, rather than carry the μ index (which would imply that these X fields would transform as vectors under the spacetime Lorentz group $SO(9,1)$), these sixteen extra fields must instead carry a purely internal index $i = 1, ..., 16$. So our second attempt at writing a heterotic string action would result in an action of the form

$$S \;=\; -\frac{1}{4\pi\alpha'}\int d^2\sigma\;\left\{(\partial_- X_R^\mu)^2 \;-\; \psi_R^\mu\partial_-\psi_{R\mu} \;+\; (\partial_+ X_L^\mu)^2 \;+\; (\partial_+ X_L^i)^2\right\} \qquad (6.5.4)$$

where we have explicitly separated the left-moving bosons into two groups, with $\mu = 0, 1, ..., 9$ and $i = 1, ..., 16$.

But there still remains a subtlety. We cannot simply *decide* to remove the μ index from the X fields and make no other changes, because these X^i fields would continue to have a mode-expansion of the form (6.2.9) with the μ index replaced by an internal index i. While the interpretation of the oscillation exponential terms in (6.2.9) is not problematic, how would we interpret the "zero-mode" terms $x^i + \ell^2 p^i(\sigma_1 + \sigma_2)$? In the case of the spacetime coordinate fields X^μ, recall that these "zero-mode" quantities x^i and p^i are interpreted as the center-of-mass position and momentum of the string. But for purely internal fields X^i, this interpretation is problematic. To clarify this difficulty, let us consider the worldsheet energy $L_0^{(\text{com})}$ associated with these degrees of freedom, as in (6.2.13). Just as in the case of the spacetime coordinates X^μ, these worldsheet energies for the X^i fields would *a priori* take *continuous* values, thereby leading to a continuous spectrum even in $D = 10$. A continuous spectrum, of course, indicates nothing but the appearance of extra spacetime dimensions, so even though we may have replaced the index μ with the index i, we have not really solved the fundamental problem that there are too many uncompactified degrees of freedom amongst the left-movers.

Therefore, we still must find a way to replace this continuous spectrum with a discrete one. Because the following discussion is slightly technical and outside the main line of the development of the heterotic string action, we shall separate it from the main flow of the text. The reader uninterested in the following details can skip them completely and proceed directly to the resumption of the main text.

> In order to eliminate this continuous spectrum, we must compactify these extra sixteen dimensions. This is analogous to discretizing the continuous spectrum of a free particle (plane wave) by localizing it in a box. In the present case, we can choose to compactify each of these extra spacetime "coordinates" X^i on a circle of radius R_i. What this means, operationally, is that we make the following topological identification in *spacetime*:
>
> $$X^i \iff X^i + 2\pi R_i \ . \qquad (6.5.5)$$
>
> For simplicity (and as we shall see, without loss of generality), we shall take $R_i = R$ for all i. Thus, rather than demand simple periodicity of the X^i "coordinates" as in (6.2.6) as we traverse the closed string worldsheet, we must allow for the more general possibility
>
> $$X^i(\sigma_1 + \pi, \sigma_2) \ = \ X^i(\sigma_1, \sigma_2) + 2\pi n_i R \ , \qquad n_i \in \mathbb{Z} \qquad (6.5.6)$$
>
> where the integer n_i is called the "winding number". The interpretation of this condition is that as we traverse the closed string once on the *worldsheet* (*i.e.*, as $\sigma_1 \to \sigma_1 + \pi$), the spacetime "coordinate" field X^i traverses the compactified spacetime circle n_i times. In other words, the closed string "winds" around the i^{th} compactified *spacetime* circle n_i times. Because of this compactification, we see that the momentum p^i is now quantized (as we would expect for any particle in a periodic box of length R), and is restricted to take the values $p^i = m_i/R$, $m_i \in \mathbb{Z}$. Indeed, working out the most general mode-expansion consistent

with (6.5.6), we find that a given such coordinate X^i takes the form

$$X(\sigma_1, \sigma_2) \;=\; x + 2nR\sigma_1 + \ell^2 \frac{m}{R}\sigma_2 + \text{oscillators}\;, \quad (6.5.7)$$

where $\ell \equiv \sqrt{2\alpha'}$ is our fundamental length scale and where 'oscillators' generically denotes the higher frequency modes. This decomposes into left- and right-moving components

$$X_{L,R}(\sigma_1 \pm \sigma_2) \;=\; \tfrac{1}{2}x + \left(\frac{\alpha' m}{R} \pm nR\right)(\sigma_2 \pm \sigma_1) + \text{oscillators}\,. \quad (6.5.8)$$

Comparing (6.5.8) with (6.2.9) enables us to identify the left- and right-moving compactified momenta

$$p_{L,R} \;\equiv\; \frac{m}{R} \pm \frac{nR}{\alpha'}\;. \quad (6.5.9)$$

We would then simply keep X_L in our heterotic theory.

Let us pause here to note an interesting phenomenon: this mode-expansion is invariant under the simultaneous exchange $R \leftrightarrow \alpha'/R$, $m \leftrightarrow n$. This is a so-called *T-duality*. What this means is that unlike point particles, strings cannot distinguish between extremely large spacetime compactification radii and extremely small spacetime compactification radii. Indeed, although the usual momentum m/R is extremely small in the first case and extremely large in the second, we see from the above mode-expansions that there is another contribution to the momentum, a "winding-mode momentum" nR/α', which compensates by growing large in the first case and small in the second. Since there is no physical way of distinguishing between these two types of momenta, the string spectrum is ultimately invariant under this T-duality symmetry. This duality underlies many of the unexpected physical properties of strings relative to point particles, and has important (and still not well-understood) implications for string cosmology. More importantly, however, this duality dramatically illustrates the breakdown of the traditional (field-theoretic) view of the linearly ordered progression of length scales and energy scales as we approach the string scale.

Having succeeded in avoiding the consequences of a continuous momentum p^i, our final question is the size of the radius R. It would certainly be aesthetically undesirable if we were forced to incorporate a new, fundamental, unfixed parameter R into our string theory. Fortunately, it turns out that in $D = 10$, there are only a very restricted set of possibilities that lead to consistent theories, and these restrictions imply that we can restrict our attention to the simple case $R = \ell = \sqrt{2\alpha'}$ *without loss of generality*. Thus, we see that R can be taken to be at the string scale, and hence essentially unobservable to "low-energy" measurements.

In order to see what is special about this radius, recall that the conformal anomaly contribution for each worldsheet boson is $c_X = 1$, while the conformal anomaly contribution for each worldsheet Majorana (real) fermion is $c_\psi = 1/2$. This suggests that the spectrum of a single compactified boson X might somehow be related to the spectrum of two Majorana fermions ψ_1, ψ_2, and this is indeed the case. Such a relation is typically referred to as a "boson-fermion equivalence" (which is possible in two dimensions because the usual spin-statistics distinction between bosons and fermions does not apply in two dimensions). In general, the spectrum of a compactified boson is identical to the spectrum of two Majorana fermions which are *coupled* to each other in a radius-dependent manner, and $R = \sqrt{2\alpha'}$ is the only value of the radius for which this coupling vanishes. Thus, if X is compactified on a circle of radius $R = \sqrt{2\alpha'}$, then the spectrum of quantum excitations of X is identical to the spectrum of quantum excitations of two *free* Majorana fermions ψ_1, ψ_2 (or equivalently those of one *complex* fermion $\Psi \equiv \psi_1 + i\psi_2$).[†] In

[†]We are again cheating slightly here. The rigorous statement is that we must compactify the X boson on a so-called $\mathbb{Z}_2$ *orbifold* with this radius in order for the spectrum of X to be identical to that of two free Majorana fermions. The equivalence between these bosonic and fermionic systems can be demonstrated explicitly at the level of their full underlying left/right two-dimensional conformal field theories. By contrast, compactifying X on a *circle* of this radius yields the spectrum of a single *complex* fermion, and the full left/right conformal field theory corresponding to a single complex fermion actually differs from that corresponding to two real fermions. These distinctions between circles

fact, at a mathematical level, it turns out that this equivalence takes the form of an actual *equality* between the *product* $\psi_1\psi_2$ and the partial derivative ∂X. Note, however, that while this specific radius is special from the point of view of boson/fermion equivalence, this is *not* the self-dual radius with respect to the T-duality transformation $R \leftrightarrow \alpha'/R$.

The upshot, then, is that in the action (6.5.4), we are free to replace the worldsheet bosons X^i ($i = 1, ..., 16$) with *complex* worldsheet fermions Ψ^i ($i = 1, ..., 16$). For ten-dimensional heterotic strings, we shall see that this replacement can be made *without loss of generality*. This replacement suffices to make the center-of-mass "momenta" associated with the X^i fields discrete rather than continuous, as we require. Given this, the final action for the heterotic string takes the form:

$$S_{\text{heterotic}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \, \left\{(\partial_+ X_L^\mu)^2 - \bar{\Psi}_L^i \partial_+ \Psi_L^i + (\partial_- X_R^\mu)^2 - \psi_R^\mu \partial_- \psi_{R\mu}\right\} \tag{6.5.10}$$

where ψ_R are Majorana-Weyl (real) right-moving worldsheet fermions, where Ψ_L are complex Weyl left-moving fermions, and where $\mu = 0, 1, ..., 9$ and $i = 1, ..., 16$.

6.5.3. *Quantizing the heterotic string*

The next step, then, is to quantize the worldsheet fields of the heterotic string. The quantization of the bosonic fields X^μ and worldsheet Majorana fermions ψ_R^μ was discussed in previous lectures, and does not change in this new setting. The only new feature, then, are the mode-expansion and quantization rules for the *complex* fermions Ψ_L^i.

Once again, there are two possible mode expansions for the left-moving complex fermions Ψ, depending on whether we choose Neveu-Schwarz (anti-periodic) or Ramond (periodic) boundary conditions.[‡] In the case of anti-periodic boundary conditions, recall that our mode-expansion (6.3.12) for

and orbifolds, and likewise between a single complex fermion and two real fermions, will not be relevant for what follows.

[‡]Because there is no worldsheet supersymmetry that relates these left-moving fermions to corresponding left-moving bosons X^μ, more general boundary conditions may actually be imposed in this case. However, for heterotic strings in ten dimensions, it turns out that we can restrict our attention to periodic or anti-periodic boundary conditions without loss of generality. Fermions with generalized worldsheet boundary conditions will be discussed further in Lecture #7.

left-moving *real* (Majorana) fermions can be written in the form

$$\psi(\sigma_1+\sigma_2) \;=\; \sum_{r=1/2}^{\infty} \left[b_r e^{-ir(\sigma_1+\sigma_2)} + b_r^\dagger e^{+ir(\sigma_1+\sigma_2)}\right] \tag{6.5.11}$$

where we recall the hermiticity condition $b_{-r} = b_r^\dagger$. Thus, for a left-moving *complex* fermion, our analogous mode-expansion takes the form

$$\Psi(\sigma_1+\sigma_2) \;=\; \sum_{r=1/2}^{\infty} \left[b_r e^{-ir(\sigma_1+\sigma_2)} + d_r^\dagger e^{+ir(\sigma_1+\sigma_2)}\right] \tag{6.5.12}$$

which of course implies

$$\Psi^\dagger(\sigma_1+\sigma_2) \;=\; \sum_{r=1/2}^{\infty} \left[b_r^\dagger e^{+ir(\sigma_1+\sigma_2)} + d_r e^{-ir(\sigma_1+\sigma_2)}\right] ~. \tag{6.5.13}$$

For $r>0$, b_r destroys fermionic excitations and $b_r^\dagger$ creates them, while d_r destroys *anti-fermionic* excitations and $d_r^\dagger$ creates them. Thus, as expected, the only new feature is the presence of twice as many mode degrees of freedom, one set associated with fermionic excitations and the other with their anti-fermionic counterparts. These modes satisfy the usual anti-commutation relations

$$\{b_r^\dagger, b_s\} \;=\; \{d_r^\dagger, d_s\} \;=\; \delta_{rs} ~. \tag{6.5.14}$$

The corresponding number operator and worldsheet energy contributions are then given by

$$\begin{aligned} N &= \sum_{r=1/2}^{\infty} \left(b_r^\dagger b_r - d_r^\dagger d_r\right) \\ L_0 &= \sum_{r=1/2}^{\infty} r\left(b_r^\dagger b_r + d_r^\dagger d_r\right) ~. \end{aligned} \tag{6.5.15}$$

Note that the anti-particle excitations *subtract* from the number operator yet *add* to the total energy. Finally, as expected, the vacuum energy contribution from each complex Neveu-Schwarz fermion is twice that for each real Neveu-Schwarz fermion: $a_\Psi = 2a_\psi = -1/24$.

The Ramond case, of course, is more subtle because of the zero-mode.

It turns out that the complex-fermion mode-expansion is given by

$$\Psi(\sigma_1+\sigma_2) = \sum_{n=1}^{\infty} \left[b_n e^{-in(\sigma_1+\sigma_2)} + d_n^\dagger e^{+in(\sigma_1+\sigma_2)} \right] ~+~ b_0$$
$$\Psi^\dagger(\sigma_1+\sigma_2) = \sum_{n=1}^{\infty} \left[b_n^\dagger e^{+in(\sigma_1+\sigma_2)} + d_n e^{-in(\sigma_1+\sigma_2)} \right] ~+~ b_0^\dagger ~, \tag{6.5.16}$$

with the anti-commutation relations

$$\{b_m^\dagger, b_n\} ~=~ \{d_m^\dagger, d_n\} ~=~ \delta_{mn} ~. \tag{6.5.17}$$

In (6.5.16), we have explicitly separated out the zero-mode from the higher-frequency modes. The number operator and worldsheet energy conributions are given by

$$N = \sum_{r=1/2}^{\infty} ~ \left(b_r^\dagger b_r - d_r^\dagger d_r \right) ~+~ b_0^\dagger b_0$$
$$L_0 = \sum_{r=1/2}^{\infty} r \left(b_r^\dagger b_r + d_r^\dagger d_r \right) ~. \tag{6.5.18}$$

Note that there is no worldsheet energy contribution from the zero-modes. Finally, the vacuum energy contribution from each complex Ramond fermion is twice that for each real Ramond fermion: $a_\Psi = 2a_\psi = +1/12$.

One might wonder, at first, why there is no *anti-particle* zero-mode d_0. However, such an anti-particle zero-mode d_0 would be *equivalent* to the *particle* zero-mode b_0. The easiest way to see this is to realize that ultimately (6.5.16) represents a Fourier-decomposition of the $\Psi(\sigma_1+\sigma_2)$ into different harmonic frequencies (exponentials). By its very nature, the zero-mode is the constant term in such a decomposition (since it corresponds to zero frequency), and this constant term is nothing but b_0. However, there can only be *one* degree of freedom associated with a given constant term. Having an additional zero-mode d_0 would thus represent a redundant (non-independent) degree of freedom. Of course, whether we associate b_0 or d_0 with the constant term is purely a matter of convention.

Given this observation, we are finally in a position to explain our counting of zero-mode states in Lectures #3 and #4. Since there is only one zero-mode degree of freedom for each *complex* worldsheet fermion, there can really be only "half" a zero-mode for each *real* worldsheet fermion. This explains the footnote in Sect. 3.2, and also explains why (in the paragraph following (6.4.9)) we counted only one zero-mode excitation per *pair* of Majorana fermions. This also explains why, ultimately, the treatment of the Ramond zero-mode for a *real* worldsheet fermion is rather subtle: essentially we must take a "square root" of the complex Ramond zero-mode b_0. There does exist a consistent method for taking this square root, but this is beyond the scope of these lectures. For our purposes, it will simply be sufficient to recall that there is only one zero-mode state for each complex worldsheet fermion, or for each pair of real worldsheet fermions.

6.6. Lecture #6: Some famous heterotic strings

Our next step is to construct actual heterotic string *models*, just as we did for the superstring. This will be the subject of the present lecture.

6.6.1. *General overview*

Before plunging into details, it is worthwhile to consider the general features that will govern the construction of our heterotic string models. Recall from the previous lecture that the worldsheet fields of the heterotic string in light-cone gauge consist of eight right-moving worldsheet bosons X_R^μ, eight left-moving worldsheet bosons X_L^μ, eight right-moving Majorana (real) worldsheet fermions ψ_R^μ, and sixteen left-moving complex worldsheet fermions Ψ_L^i $(i = 1, ..., 16)$.

The role of the right-moving fermions ψ_R^μ is the same as in the superstring: if they have Neveu-Schwarz modings, the corresponding states are spacetime bosons, and if they Ramond modings, the corresponding states are spacetime fermions. Indeed, by properly stitching these sectors together, it may also be possible to obtain spacetime supersymmetry (as in the superstring). Note that unlike the superstring, however, these boson/fermion identifications hold *regardless* of the modings of the left-moving complex fermions Ψ_L^i. This is because only the right-moving fermions carry spacetime Lorentz indices μ, and hence only these fermions determine the representations of the spacetime Lorentz algebra.

The role of the left-moving complex fermions Ψ_L^i is analogous. Because they carry internal indices rather than spacetime Lorentz indices, the symmetries they carry are also internal, and as we shall see, they can be interpreted as gauge symmetries. Indeed, these Ψ_L^i fields are precisely the internal fields we were hoping to obtain in Sect. 5.1. When they have Neveu-Schwarz modings, these fermions provide "vectorial" (scalar, vector, tensor) representations of the internal gauge symmetry. When they have Ramond modings, by contrast, they provide "spinorial" representations of the internal gauge symmetry. Thus, we expect a rich gauge representation structure in these models as well.

As with the superstring, different models can be constructed depending on how the different modings are joined together to form our set of underlying sectors, and how the corresponding GSO constraints are implemented. We shall construct explicit models below. But it is already apparent that the heterotic string contains all the ingredients we require for successful phenomenology. By choosing certain combinations of right-moving fermionic modings with left-moving fermionic modings, we can control which gauge-group representations are bosonic and which are fermionic. Moreover, by choosing the relative modings *amongst* the left-moving complex fermions, we can even control the gauge group that is ultimately produced.

6.6.2. *Sectors and GSO constraints*

Just as in the superstring, we begin the process of model-building by choosing an appropriate set of underlying sectors and corresponding GSO constraints. Moreover, just as in the superstring, we know that preservation of the right-moving worldsheet supersymmetry (or equivalently spacetime Lorentz invariance) requires that we choose our eight right-moving fermions ψ_R^μ to all have the same boundary condition in each sector. This implies that we can, if we wish, combine these right-moving fermions to form four complex right-moving fermions which we can denote Ψ_R^μ. (We retain the index μ to remind ourselves that these fields carry indices with respect to the spacetime Lorentz algebra, even though strictly speaking only the real fields ψ_R^μ carry such vectorial indices.) However, unlike the superstring, there is no longer any such restriction on the boundary conditions of the left-moving fermions Ψ_L^μ. Thus, there remains substantial freedom in choosing the boundary conditions of these left-moving fermions. Ultimately this choice becomes the choice of the gauge group for the particular model in question.

In the next lecture, we shall provide a detailed discussion of the rules by which one can choose these boundary conditions and determine their associated GSO constraints. Therefore, for the time being, we shall simply restrict our attention to the sectors listed in Table 6.2. Note that the corresponding vacuum energies are also listed in Table 6.2. In order to compute these energies, we can continue to use the middle expression in (6.4.1) where we recall that $n_{\rm R}$ and $n_{\rm NS}$ count the number of *real* worldsheet fermions. Thus, for complex fermions, these numbers are doubled.

#	$\psi_R^{i=1,\ldots,8}$	$\Psi_L^{i=1,\ldots,8}$	$\Psi_L^{i=9,\ldots,16}$	a_R	a_L
1	NS	NS		$-1/2$	-1
2	R	R		0	$+1$
3	NS	R		$-1/2$	$+1$
4	R	NS		0	-1
5	NS	NS	R	$-1/2$	0
6	NS	R	NS	$-1/2$	0
7	R	NS	R	0	0
8	R	R	NS	0	0

Before proceeding further, we can immediately deduce some physical properties of the string states that would emerge in each sector. First, we see that Sector #1 is the only sector from which tachyons can possibly emerge. This is because the level-matching constraints prevent tachyons in any other sector (*i.e.*, there is no other sector which for which both a_L and a_R are negative). Second, we observe that Sectors #2 and #3 cannot give rise to massless states. This again follows from the level-matching constraints, and implies that (for phenomenological purposes) we will not need to consider the states arising in these sectors. Finally, we observe that Sectors #1,3,5,6 give rise to spacetime bosons, while Sectors #2,4,7,8 give rise to spacetime fermions.

In some sense, Sectors #1–4 are the direct analogues of the four possible sectors in Table 6.1 for the superstring. Thus, the heterotic models that result from these sectors will be the analogues of the Type 0 and Type II superstring models. However, the additional Sectors #5–8 represent new sectors that arise only for heterotic strings. We hasten to add that these sectors are not unique, and others could equally well have been chosen. We will discuss these possibilities in the next lecture.

The next issue we face is to determine which *combinations* of sectors form self-consistent sets. It turns out (following the rules to be discussed in Lecture #7) that there are three different possibilities:

- Case A: we consider Sectors #1 and #2 by themselves;
- Case B: we consider Sectors #1 through #4 by themselves; or
- Case C: we consider *all* Sectors #1 through #8.

For each of these cases, there is then a different set of GSO constraints for each sector. As we have seen in our discussion of the superstring, the more sectors we have in our model, the more GSO constraints there are in each sector. In particular, each time the number of sectors doubles, the number of GSO constraints in each sector increases by one. For completeness, Table 6.3 lists the GSO constraints that apply in each sector for each of these three cases.

Once again, observe the *pattern* of the GSO constraints. In Case A, we have only Sectors #1 and #2, for which all right-moving and left-moving boundary conditions are identical. Thus, the GSO constraints that apply in Case A combine N_L and N_R together. (Recall that since $N_{L,R} \in \mathbb{Z}$, we can just as easily write the GSO constraint for Case A as $N_L + N_R =$ odd.) When we move from Case A to Case B, we introduce two new sectors (Sectors #3 and #4 in Table 6.2) which "twist" the boundary conditions of the right-movers relative to those of the left-movers. This has the effect of introducing a new GSO constraint in each sector, one which distinguishes separately between N_L and N_R. Finally, when we move from Case B to Case C, we introduce four new sectors (Sectors #5 through #8) which introduce an additional "twist" that distinguishes between the first eight left-moving fermions $\Psi_L^{i=1,\ldots,8}$ and the second eight left-moving fermions $\Psi_L^{i=9,\ldots,16}$. The corresponding new GSO constraint in each sector is then one which is sensitive only to ${}^{(8)}N_L \equiv \sum_{i=1}^{8} N^{(i)}$. This suggests (and we shall see explicitly in Lecture #7) that the set of sectors is deeply correlated with the set of GSO constraints that are applied in each sector: each new "twist" introduces both a new set of sectors and a new GSO constraint in each sector. The fact that we are considering only Ramond or Neveu-Schwarz boundary conditions for our left-moving complex fermions Ψ_L^i means that each successive twist doubles the number of sectors and introduces one new GSO constraint in each sector. These are called $\mathbb{Z}_2$ twists. If we were to consider more general "multi-periodic" boundary conditions for the left-moving fermions (which is possible because they are not related to the left-moving worldsheet bosons by worldsheet supersymmetry), then we could introduce so-called "higher-order" twists that would result in more complicated GSO constraints. However, it turns out that in ten dimensions, we lose no generality by restricting our attention to such $\mathbb{Z}_2$ twists.

6.6.3. *Four ten-dimensional heterotic string models*

It is apparent from Table 6.3 that Case A and Case B each correspond to one heterotic string model, while Case C corresponds to two separate heterotic string models. Thus, the GSO constraints in Table 6.3 together give rise to four distinct heterotic string models. In the remainder of this lecture, we shall work out the physical properties of these four models.

6.6.3.1. *The non-supersymmetric $SO(32)$ string*

Let us begin by considering Case A, which consists of only Sectors #1 and #2. Only Sector #1 (the so-called "NS-NS sector") can contain massless states. As indicated in Table 6.1, the vacuum energy in this sector is $(a_R, a_L) = (-1/2, -1)$. Thus, at the bare minimum, the level-matching constraint $L_0 = \bar{L}_0$ forces us to excite at least a half-unit of energy on the left-moving side. This can be accomplished by exciting any of the left-moving half-unit fermionic modes, since in this sector the left-moving fermions all have Neveu-Schwarz boundary conditions and thus contain half-integer modings. This produces the 32 possible states

$$|0\rangle_R \otimes b^i_{-1/2}|0\rangle_L \quad \text{and} \quad |0\rangle_R \otimes d^i_{-1/2}|0\rangle_L \ . \tag{6.6.1}$$

Note that these states also satisfy the single applicable GSO constraint $N_L - N_R = \text{odd}$, so they remain in the spectrum. From (6.4.6), we see that these states are tachyonic with $\alpha' M^2 = -2$.

Further states are realized by exciting higher worldsheet modes. Because our worldsheet modes are quantized in minimum half-integer steps, we see that the next excited states in this model are massless. These states come in two varieties:

$$\tilde{b}^\mu_{-1/2}|0\rangle_R \otimes \alpha^\nu_{-1}|0\rangle_L \tag{6.6.2}$$

and

$$\tilde{b}^\mu_{-1/2}|0\rangle_R \otimes \begin{cases} b^i_{-1/2}\, b^j_{-1/2}\, |0\rangle_L \\ b^i_{-1/2}\, d^j_{-1/2}\, |0\rangle_L \\ d^i_{-1/2}\, b^j_{-1/2}\, |0\rangle_L \\ d^i_{-1/2}\, d^j_{-1/2}\, |0\rangle_L \end{cases} . \tag{6.6.3}$$

In (6.6.2), we have excited the lowest mode of the left-moving worldsheet boson X^μ_L, whereas in (6.6.3) we have excited two of the lowest modes of the left-moving fermions $\Psi^{i,j}_L$. Note that it is possible to excite both the particle and anti-particle modes from the same fermion Ψ^i, and thus there is no restriction that $i \neq j$. Also note that all of these states in (6.6.2) and (6.6.3) satisfy the GSO constraint $N_L - N_R = \text{odd}$. While $N_R = 1$ in all cases, we have $N_L = 0$ in (6.6.2) (since the number operators are defined not to include the contributions from worldsheet bosons), and $N_L = 2$ in (6.6.3).

How do we interpret these states? Once again, the states (6.6.2) are easily recognized as our gravity multiplet, consisting of the spin-two graviton $g_{\mu\nu}$, the spin-one anti-symmetric tensor $B_{\mu\nu}$, and the spin-zero dilaton ϕ. It is interesting to note that this state (6.6.2) is realized as a hybrid of the gravity multiplet state in the bosonic string (6.2.19) and in the superstring (6.4.7). This reflects the underlying construction of the heterotic string, and ensures that the heterotic string, like its predecessors, is also a theory of quantized gravity. Once again, the appearance of the gravity multiplet is a useful cross-check of the GSO constraints.

The states in (6.6.3) have a different interpretation, however. Clearly, their Lorentz structure indicates that they are massless Lorentz vectors. Thus, they are to be interpreted as spacetime *gauge bosons*. Thus, we see that the heterotic string has succeeded in providing us with spacetime gauge symmetry, just as we had originally hoped.

But what is the gauge group? Of course, the gauge group is ultimately determined from the i, j indices, and since (in Cases A and B) we have not destroyed the rotational symmetry in the space of the 16 complex left-moving fermions Ψ^i_L (or the 32 real left-moving fermions into which they can be decomposed), we immediately suspect that the gauge symmetry should be $SO(32)$. There are number of ways to deduce that this is correct. Perhaps the easiest way is simply to *count* the gauge boson states in (6.6.3). If we restrict our attention to the cases $i \neq j$, then there are $(2\cdot 16)(2\cdot 15)/2$ states. The first factor $(2 \cdot 16)$ reflects the fact that for each of the 16 possible choices of Ψ^i_L, we can excite either the fermion or anti-fermion mode. The second factor $(2 \cdot 15)$ reflects the same set of options for the second fermion Ψ^j_L, and we divide by two as the interchange symmetry factor. There are also the cases with $i = j$: from such cases we obtain 16 possible states, reflecting the 16 different fermions Ψ^i_L whose fermion and anti-fermion modes are jointly excited. The total number of states is then

$$\frac{(2\cdot 16)(2\cdot 15)}{2} \;+\; 16 \;=\; 496 \;=\; \dim SO(32)\;. \tag{6.6.4}$$

Of course, the above counting method for determining the gauge group is hardly precise, for there are a number of gauge groups with the same overall dimension (and we shall come across another such gauge group very soon). We therefore require a more sophisticated method which also generalizes to more complicated cases. By definition, of course, the gauge group can be determined by explicitly examining the charges of the gauge boson states and determining which Lie algebra (*i.e.*, which root system) they fill out. We therefore need a way of determining the charges of the gauge boson states. Since our gauge symmetry is ultimately associated with the left-moving worldsheet fermions Ψ_L^i, the relevant current in this case is simply the worldsheet current $J^i \equiv \bar{\Psi}_L^i \Psi_L^i$. From this, we can deduce the associated charge Q_i. It turns out that

> **Great Leap #7:** The charge associated with each worldsheet fermion Ψ_L^i for a given string state with fermionic excitation number $N^{(i)}$ is given by $Q_i \equiv N^{(i)} + q_i$. Here q_i is a "background" charge which is 0 if Ψ_L^i is a Neveu-Schwarz fermion and $-1/2$ if Ψ_L^i is a Ramond fermion.

Given this result, we can easily deduce the gauge group for the case in question. For simplicity, let us first imagine that there are only *two* left-moving fermions $\Psi_L^{i=1,2}$. In this case, (6.6.3) reduces to six states:

$$\begin{aligned} &b^1_{-1/2} b^2_{-1/2} |0\rangle_L \ , \quad b^1_{-1/2} d^2_{-1/2} |0\rangle_L \ , \quad d^1_{-1/2} b^2_{-1/2} |0\rangle_L \ , \\ &d^1_{-1/2} d^2_{-1/2} |0\rangle_L \ , \quad b^1_{-1/2} d^1_{-1/2} |0\rangle_L \ , \quad b^2_{-1/2} d^2_{-1/2} |0\rangle_L \ . \end{aligned} \tag{6.6.5}$$

For each of these states, there are two charges, Q_1 and Q_2, associated with each of the two complex fermions. If we denote these states as A through F respectively, we can plot the charges of these six states as in Fig. 6.8. The resulting diagram is easily recognized as the root system (or equivalently the weight system of the adjoint representation) of the Lie group $SO(4)$. Generalizing from two complex fermions to n complex fermions analogously yields the gauge group $SO(2n)$, provided that all n complex fermions have the same modings. Thus, in the case of 16 complex fermions, we find the gauge group $SO(32)$.

Note that this argument suffices to show that the gauge bosons fill out the adjoint representation of $SO(32)$. However, it does not demonstrate that all other string states in the model fall into representations of this gauge group. Of course, this is required for the consistency of the string. However, such a result can indeed be proven mathematically by constructing the current operators associated with the gauge group in question (as discussed above), and demonstrating that all states surviving the appropriate GSO constraints transform appropriately under these currents. For example, the 32 tachyonic states in (6.6.1) transform in the vector representation of $SO(32)$, and the gravity multiplet (6.6.2) transforms as a singlet of $SO(32)$ (as it must). However, a proof that this holds for all states in both the massless and massive string spectrum is beyond the scope of these lectures.

We should also point out that what emerges in such closed string theories is not simply the algebra associated the gauge symmetry in question, but rather an infinite-dimensional extension (or "affinization") of it. Such affine Lie algebras are discussed in Ginsparg (reference given at the end of Lecture #1), and play an important role in the consistency and phenomenology of such heterotic string theories.

To summarize, then, we see that Case A results in a tachyonic string model with quantum gravity and $SO(32)$ gauge symmetry. In addition to 32 scalar tachyons transforming in the vector representation of $SO(32)$, this model contains massless gauge bosons transforming in the adjoint representation of $SO(32)$ as well as the usual gravity multiplet. This non-supersymmetric $SO(32)$ heterotic string model is the heterotic analogue of the Type 0 string models in Lecture #4.

6.6.3.2. *The supersymmetric $SO(32)$ string*

Let us now proceed to Case B. In this case there are four sectors (#1 through #4 in Table 6.2), and we must impose the GSO constraints listed in the second column of Table 6.3.

Let us begin by considering the states from Sector #1. These are the same as those considered in Case A, except that we must now impose the additional GSO constraint $N_L =$ even. This projects out the tachyonic states (6.6.1), but preserves the gravity multiplet as well as the gauge bosons.

As we discussed previously, Sectors #2 and #3 contain no massless states. Therefore, all that remains is to consider the states from Sector #4. Here the vacuum energy is $(a_R, a_L) = (0, -1)$. The right-moving ground state in this sector is the Ramond zero-mode ground state, which we have previously denoted $\{\tilde{b}_0^\mu\}|0\rangle_R$, and thus massless states are realized only through non-zero excitations of the left-movers. The possible states are

$$\{\tilde{b}_0^\mu\}|0\rangle_R \;\otimes\; \begin{cases} \alpha_{-1}^\nu|0\rangle_L \\ b_{-1/2}^i\, b_{-1/2}^j\, |0\rangle_L \\ b_{-1/2}^i\, d_{-1/2}^j\, |0\rangle_L \\ d_{-1/2}^i\, b_{-1/2}^j\, |0\rangle_L \\ d_{-1/2}^i\, d_{-1/2}^j\, |0\rangle_L\ . \end{cases} \tag{6.6.6}$$

In each case, the GSO constraints imply that we can excite only an odd number of right-moving zero-modes. According to our previous conventions, this indicates that the right-moving ground state corresponds to the spacetime Lorentz spinor $\bar{\mathbf{S}}_8$ (rather than the conjugate spinor $\bar{\mathbf{C}}_8$).

It is, by now, easy to interpret the states in (6.6.6). The first state provides the superpartner states to the gravity multiplet, and contains a gravitino. This implies that the model has spacetime supersymmetry. Likewise, the remaining states correspond to the superpartners of the $SO(32)$ gauge bosons, and contain the $SO(32)$ gauginos. The chirality of these spinor states is fixed by the GSO constraint and the right-moving ground state $\bar{\mathbf{S}}_8$.

Summarizing, we see that this model therefore consists of the following states. We shall describe these states using the notation $\bar{R}_1 \otimes (R_2; R_3)$ where R_1, R_2 are representations of the spacetime Lorentz group, and where R_3 is a representation of the $SO(32)$ gauge group. These states consist of

$$\bar{\mathbf{V}}_8 \otimes (\mathbf{V}_8; \mathbf{1}) \ , \quad \bar{\mathbf{V}}_8 \otimes (\mathbf{1}; \mathbf{adj}) \ , \quad \bar{\mathbf{S}}_8 \otimes (\mathbf{V}_8; \mathbf{1}) \ , \quad \bar{\mathbf{S}}_8 \otimes (\mathbf{1}; \mathbf{adj}) \ , \tag{6.6.7}$$

where the first and third states form the $N = 1$ supergravity multiplet and the second and fourth states form the $SO(32)$ gauge boson supermultiplet. Together these states can be written in the factorized form

$$(\bar{\mathbf{V}}_8 \oplus \bar{\mathbf{S}}_8) \ \otimes \ \{(\mathbf{V}_8; \mathbf{1}) \oplus (\mathbf{1}; \mathbf{adj})\} \ , \tag{6.6.8}$$

thereby explicitly exhibiting the supersymmetry $\bar{\mathbf{V}}_8 \leftrightarrow \bar{\mathbf{S}}_8$.

This string is the famous supersymmetric $SO(32)$ heterotic string. Although not directly relevant for string phenomenology, this string plays a vital role in recent developments in string duality (to be discussed briefly in Lecture #8).

6.6.3.3. *The $SO(16) \times SO(16)$ and $E_8 \times E_8$ strings*

Let us now proceed to Case C. As discussed in Sect. 6.2, this case differs from Case B because we have now "twisted" the second group of eight left-moving complex worldsheet fermions relative to the first set. *A priori*, it is easy to imagine that this twist will break the gauge symmetry $SO(32) \to SO(16) \times SO(16)$. However, there a few surprises still in store for us.

We begin in Sector #1, which previously gave rise to the states given in (6.6.2) and (6.6.3). Introducing the third GSO constraint ${}^{(8)}N_L \equiv \sum_{i=1}^{8} N^{(i)}$ = even does not affect the gravity multiplet (6.6.2), but has a drastic effect on the remaining gauge boson states. We now see that we cannot excite arbitrary combinations of (i, j) fermions; instead we must choose either $(i, j) = 1, ..., 8$ or $(i, j) = 9, ..., 16$. In string-theory parlance, all of the other states have been "projected out of the spectrum". It is in this manner that we remove gauge boson states and break gauge symmetries in string theory. (There are other methods for doing this in string theory, but this is the only method at tree-level.) It is easy to see (following the arguments given above) that the remaining gauge boson states fill out the adjoint representation of two copies of $SO(16)$, and thus the gauge group is *a priori* $SO(16) \times SO(16)$. Therefore, we shall henceforth denote our string states in the notation $\bar{R}_1 \otimes (R_2; R_3, R_4)$ where $\bar{R}_1, R_2$ are the representations of the Lorentz group from the right- and left-movers, and where R_3, R_4 are the representations with respect to the two gauge group factors of $SO(16)$ respectively. Thus, we see that Sector #1 gives rise to the states

$$\bar{\mathbf{V}}_8 \otimes (\mathbf{V}_8; \mathbf{1}, \mathbf{1}) \ , \quad \bar{\mathbf{V}}_8 \otimes (\mathbf{1}; \mathbf{adj}, \mathbf{1}) \ , \quad \bar{\mathbf{V}}_8 \otimes (\mathbf{1}; \mathbf{1}, \mathbf{adj}) \ , \tag{6.6.9}$$

where the first states form the gravity multiplet and the second and third states are the $SO(16) \times SO(16)$ gauge bosons.

As before, Sectors #2 and #3 do not give rise to massless states. Let us now consider what happens in Sector #4. The states that previously emerged in Sector #4 are given in (6.6.6). We now must impose the remaining GSO constraint ${}^{(8)}N_L = \left\{ \begin{matrix} \text{odd} \\ \text{even} \end{matrix} \right\}$. Let us consider each case separately. If we impose the odd choice, then the gravitino state in (6.6.6) is projected out of the spectrum, indicating that *supersymmetry is broken*. Likewise, we find that the gaugino states are also affected: we can now excite only those states for which $i = 1, ..., 8$ and $j = 9, ..., 16$. This spinor state transforms in the $(\mathbf{16}, \mathbf{16})$ representation of $SO(16) \times SO(16)$ (*i.e.*, as the vector-vector bifundamental). By contrast, if we impose the even choice, then the gravitino state in (6.6.6) remains in the spectrum, indicating that *supersymmetry is preserved*. Likewise, the gaugino states are affected only by the new requirement that either $i, j = 1, ..., 8$ or $i, j = 9, ..., 16$. Thus, the new GSO projection projects our $SO(32)$ gauginos down to $SO(16) \times SO(16)$ gauginos, as expected. Summarizing, we find that in the "even" case, the states from Sector #4 are

$$\bar{\mathbf{S}}_8 \otimes (\mathbf{V}_8; \mathbf{1}, \mathbf{1}) \ , \quad \bar{\mathbf{S}}_8 \otimes (\mathbf{1}; \mathbf{adj}, \mathbf{1}) \ , \quad \bar{\mathbf{S}}_8 \otimes (\mathbf{1}; \mathbf{1}, \mathbf{adj}) \ . \tag{6.6.10}$$

Let us now consider Sector #5. As indicated in Table 6.2, in this sector the vacuum energy is $(a_R, a_L) = (-1/2, 0)$ and the first eight left-moving complex fermions are Neveu-Schwarz while the second eight are Ramond. Choosing the "odd" GSO constraints projects all possible massless states out of the spectrum (because there is no simultaneous solution to all three

GSO constraints in the "odd" case). By contrast, choosing the "even" GSO constraints yields the states

$$\tilde{b}^{\mu}_{-1/2}|0\rangle_R \ \otimes \ \{b^i_0\}|0\rangle_L \qquad (i=9,...,16) \tag{6.6.11}$$

where we must choose an even number of zero-mode excitations on the left-moving side. This produces a massless vector state which transforms in a (128-dimensional) *spinorial* representation of the second $SO(16)$ gauge group factor. Following our previous conventions, we shall refer to this spinor as $\mathbf{C}_{128}$ rather than its conjugate $\mathbf{S}_{128}$. This state can therefore be denoted as

$$\bar{\mathbf{V}}_8 \ \otimes \ (\mathbf{1};\mathbf{1},\mathbf{C}_{128}) \ . \tag{6.6.12}$$

We shall discuss the physical interpretation of this state shortly.

Sector #6 is similar to Sector #5, except that now the first eight left-moving complex fermions are Ramond and the second eight are Neveu-Schwarz. In a similar way we then find that there are no states in the "odd" case, while in the "even" case we find the states

$$\bar{\mathbf{V}}_8 \ \otimes \ (\mathbf{1};\mathbf{C}_{128},\mathbf{1}) \ . \tag{6.6.13}$$

We now turn to Sector #7. Here the vacuum energy is $(a_R, a_L) = (0,0)$, which implies that if we restrict our attention to massless states, we can tolerate only zero-mode excitations amongst both the left- and right-movers. In the "odd" case, we find the states

$$\{\tilde{b}^{\mu}_0\}|0\rangle_R \ \otimes \ \{b^i_0\}|0\rangle_L \qquad (i=9,...,16) \tag{6.6.14}$$

where the GSO projections restrict us to an even number of zero-mode excitations on the right-moving side and an odd number on the left-moving side. According to our conventions, this produces the state $\bar{\mathbf{C}}_8 \otimes (\mathbf{1};\mathbf{1},\mathbf{S}_{128})$. In the "even" case, by contrast, we are restricted to (6.6.14) where now we must have an even number of zero-mode excitations on the right-moving side and an odd number of the left-moving side. This produces the state

$$\bar{\mathbf{S}}_8 \ \otimes \ (\mathbf{1};\mathbf{1},\mathbf{C}_{128}) \ . \tag{6.6.15}$$

Finally, in Sector #8, we similiarly find the states $\bar{\mathbf{C}}_8 \otimes (\mathbf{1};\mathbf{S}_{128},\mathbf{1})$ in the "odd" case and

$$\bar{\mathbf{S}}_8 \ \otimes \ (\mathbf{1};\mathbf{C}_{128},\mathbf{1}) \tag{6.6.16}$$

in the "even" case.

What are we to make of these results? Collecting our states for the "odd" case, we find a string model with the following massless spectrum:

$$\begin{aligned}&\bar{\mathbf{V}}_8\otimes(\mathbf{V}_8;\mathbf{1},\mathbf{1})\ ,\quad \bar{\mathbf{V}}_8\otimes(\mathbf{1};\mathbf{adj},\mathbf{1})\ ,\quad \bar{\mathbf{V}}_8\otimes(\mathbf{1};\mathbf{1},\mathbf{adj})\\ &\bar{\mathbf{S}}_8\otimes(\mathbf{1};\mathbf{V}_{16},\mathbf{V}_{16})\ ,\quad \bar{\mathbf{C}}_8\otimes(\mathbf{1};\mathbf{S}_{128},\mathbf{1})\ ,\quad \bar{\mathbf{C}}_8\otimes(\mathbf{1};\mathbf{1},\mathbf{S}_{128})\end{aligned} \tag{6.6.17}$$

This is clearly a non-supersymmetric spectrum consisting of a gravity multiplet, vector bosons transforming of the adjoint of $SO(16)\times SO(16)$, one spinor transforming as a vector-vector bifundamental with respect to the gauge group, and two additional spinors of opposite chirality transforming in the spinor representations of the gauge group. This is the non-supersymmetric $SO(16)\times SO(16)$ heterotic string model, first constructed in 1986. Note that this spectrum configuration is anomaly-free, as required for a self-consistent string theory. Also note that this string is tachyon-free even though it is non-supersymmetric. This example thus proves that *not all non-supersymmetric strings have tachyons* (although it is certainly true that all supersymmetric strings lack tachyons). While this is the only non-supersymmetric tachyon-free heterotic string in ten dimensions, there exist a plethora of such strings in lower dimensions. We shall discuss some of the properties of such strings in Lecture #8, but this raises an interesting issue: Does string theory *predict* spacetime supersymmetry? As this example makes clear, string theory certainly does not predict spacetime supersymmetry on the basis of tachyon-avoidance. However, the general answer to this question is unknown.

Even more interesting is the model that results in the "even" case. Collecting our states from (6.6.9), (6.6.10), (6.6.12), (6.6.13), (6.6.15), and (6.6.16), we find that the total massless spectrum of this string can be written in the factorized form

$$(\bar{\mathbf{V}}_8\oplus\bar{\mathbf{S}}_8)\otimes\left\{(\mathbf{V}_8;\mathbf{1},\mathbf{1})\oplus\left(\mathbf{1};\left\{\mathbf{adj}\oplus\mathbf{C}_{128}\right\},\mathbf{1}\right)\oplus\left(\mathbf{1};\mathbf{1},\left\{\mathbf{adj}\oplus\mathbf{C}_{128}\right\}\right)\right\}. \tag{6.6.18}$$

The appearance of the right-moving factor $\bar{\mathbf{V}}_8\oplus\bar{\mathbf{S}}_8$ indicates that this model has $N=1$ supersymmetry, as expected from the appearance of a single gravitino in the massless spectrum. The left-moving factor, by contrast, contains three terms. The first term combines with the right-moving factor to produce the supergravity multiplet. The second two terms formerly gave rise to the $SO(16)\times SO(16)$ gauge supermultiplet. However, we now see that for each $SO(16)$ gauge group factor, the massless vector states transform in the $\mathbf{adj}\oplus\mathbf{C}_{128}$ representation rather than simply in the $\mathbf{adj}$

representation. While the **adj** contribution is easy to interpret (giving rise to the usual gauge bosons of $SO(16)$), the extra massless vector states transforming in the $\mathbf{C}_{128}$ representation of each gauge group factor appear to cause an inconsistency, for we know that all massless vector states must be interpreted as gauge bosons, and hence such states can only transform in the adjoint representation. Thus, the only possible way that this string can be consistent is if the massless vector states in this model somehow combine to fill out the adjoint representation of some *other* group G:

$$\mathbf{adj}_{SO(16)} \;\oplus\; \mathbf{C}_{128} \;\stackrel{?}{=}\; \mathbf{adj}_G \; . \tag{6.6.19}$$

Remarkably, this is precisely what occurs: the group G is nothing but the exceptional Lie group E_8! Indeed, the 120 states of the adjoint representation of $SO(16)$ together with the 128 states of the spinor representation of $SO(16)$ combine to produce the 248 states of the adjoint representation of E_8! In string parlance, we thus say that the presence of the "twisted" states (6.6.12), (6.6.13), (6.6.15), and (6.6.16) has *enhanced* the total gauge group from $SO(16) \times SO(16)$ to $E_8 \times E_8$. This, then, is the famous supersymmetric $E_8 \times E_8$ heterotic string.

Unlike the supersymmetric $SO(32)$ string, this string is generally considered to have excellent phenomenological prospects. It has $N = 1$ spacetime supersymmetry, quantum gravity, and an $E_8 \times E_8$ gauge symmetry. E_8 is a compelling gauge group for phenomenology because it contains E_6 as a subgroup, and E_6 is a group that contains chiral representations which can be associated with grand unification and which thereby contain all of the particle content of the Standard Model. (Of course, it is still necessary to obtain actual *matter* representations from this string, but these can arise upon compactification.) Moreover, while we can imagine the Standard Model to reside entirely within one of the E_8 gauge group factors, the other factor may be interpreted as a "hidden" sector which can also have important phenomenological uses (such as triggering supersymmetry breaking, providing dark-matter candidates, and enforcing string selection rules). Thus, historically, much of the original work in string phenomenology began with a study of the compactification of this model down to four dimensions. However, it is possible to construct heterotic string models directly in four dimensions, and to obtain models which do not necessarily have an interpretation as arising via the compactification of any particular string model in ten dimensions. Thus, as we shall see, the prospects for phenomenological heterotic string model-building are broader than merely studying the compactifications of the $E_8 \times E_8$ heterotic string.

6.6.4. *More ten-dimensional heterotic strings*

So far, we have constructed four heterotic string models in ten dimensions. Of these, two have spacetime supersymmetry, and two do not. However, it is readily apparent that further models can be constructed by introducing further "twists" which further enlarge the set of sectors in Table 6.2 and which further break the gauge group into smaller factors (or which break the original $SO(32)$ gauge group in entirely different ways). The question that arises, then, is whether there exist other ten-dimensional heterotic strings with spacetime supersymmetry, or whether there exist other non-supersymmetric strings in ten dimensions that are tachyon-free. The answer to both questions turns out to be "no". A complete list of ten-dimensional heterotic strings is given in Table 6.4.

The presence of the last string in Table 6.4 might seem surprising. After all, the rank of the gauge group for this string is only eight rather than sixteen, which implies that its construction must differ substantially from that of the previous strings. It turns out that this is indeed the case.* We briefly indicate in Lecture #7 how such strings may be constructed.

6.7. Lecture #7: Rules for string model-building

In the last several lectures, we constructed many different string models. Amongst the superstring models, we constructed the Type 0A, Type 0B, Type IIA, and Type IIB models, while amongst the heterotic string models, we constructed the non-supersymmetric $SO(32)$ model, the non-supersymmetric $SO(16) \times SO(16)$ model, and the supersymmetric $SO(32)$ and $E_8 \times E_8$ models. In each case, we simply *asserted* a set of sectors (combinations of Neveu-Schwarz and Ramond modings) and a set of GSO constraints in each sector. Of course, each of these sets of sectors and GSO constraints conspires to yield a self-consistent string model, and occasionally it is even possible to see intuitively which choices can lead to self-consistent string models. However, we ultimately wish to construct semi-realistic string models where the groups are broken down to much smaller pieces than we have been dealing with thus far (*e.g.*, $SU(3) \times SU(2) \times U(1)$,

*Unlike the other ten-dimensional heterotic strings, this string involves splitting each complex worldsheet fermion into a pair of two real worldsheet fermions and then introducing relative "twists" within each pair. In technical language, this results in a gauge group whose rank is reduced but whose so-called *affine level* is increased relative to those of the other strings. This increase in the affine level is important for string GUT model-building, and will be discussed in subsequent lectures.

or even $SU(5)$ or $SO(10)$), and this is going to require more complicated twists than we have thus far been using. Furthermore, all of our string models thus far have been in ten dimensions, yet we are ultimately going to wish to compactify our string models to four dimensions. It turns out that this will introduce even further choices for modings, twists, and their associated GSO projections. (In geometric language, these further choices amount the choice of compactification manifold.)

The question that arises, then, is to determine the minimal set of parameters that govern these choices. What we require is a way to *systematize* the whole process of string model-construction, so that we will know precisely which choices govern the construction of a string model and guarantee its internal self-consistency. In other words, we require *rules for string model-building*. This is the subject of the present lecture.

Once we learn the rules for the construction of ten-dimensional string models, it will be relatively straightforward to generalize these rules for the construction of models in four dimensions. We will then have the tools whereby we may finally construct semi-realistic four-dimensional string models.

6.7.1. *Generating the sector combinations: The 20-dimensional lattice*

The first issue we face is that of choosing the appropriate sector combinations. For example, let us recall the possible heterotic string sectors in Table 6.2. As we discussed in Sect. 6.2, this set of sectors permits only three distinct sector combinations: either we choose Sectors #1 and #2 only, or we choose Sectors #1 through #4 only, or we choose Sectors #1 through #8. How can we know which combinations are allowed, and which sectors are required in each grouping? In Sect. 6.2, we discussed how modular invariance ultimately governs these choices. Here, however, we shall develop a rule which we can use in order to deduce these sector combinations rather quickly and which can easily be generalized to more complicated situtions.

First, let us introduce some notation. Since it is rather awkward to consider left-moving complex fermions Ψ^i_L at the same time as right-moving *real* (Majorana) fermions ψ^μ_R, let us "complexify" our right-moving Majorana fermions so that *all* of our worldsheet fermions are complex. This means that instead of having eight left-moving real fermions ψ^μ_R in light-cone gauge, we have instead four complex ones Ψ^μ_R formed by pairing the left-moving real fermions in groups of two. (We retain the index μ to remind ourselves that these fields carry indices with respect to the spacetime Lorentz algebra, even though strictly speaking it is only their real component fields ψ_R^μ that carry such vectorial indices.)

We also need a more general notation for discussing the possible boundary conditions and modings that any such complex worldsheet fermion can take. In general, we can parametrize any possible worldsheet boundary condition in the form

$$\Psi(\sigma_1+\pi,\sigma_2) \;=\; -\,e^{-2\pi iv}\,\Psi(\sigma_1,\sigma_2) \tag{6.7.1}$$

where $-\frac{1}{2} \leq v < \frac{1}{2}$. Thus the quantity v parametrizes the boundary condition of the individual fermion, with

$$\begin{array}{rcc} v=0: & \text{anti-periodic} & \text{(Neveu-Schwarz)} \\ v=-1/2: & \text{periodic} & \text{(Ramond)}\;. \end{array} \tag{6.7.2}$$

General values of v correspond to so-called "multi-periodic fermions". For example, the general moding of a multi-periodic left-moving complex fermion is given by

$$\Psi_L(\sigma_1+\sigma_2) = \sum_{n=1}^{\infty}\Bigg[b_{n+v-1/2}e^{-i(n+v-1/2)(\sigma_1+\sigma_2)} \\ + d^\dagger_{n-v-1/2}e^{+i(n-v-1/2)(\sigma_1+\sigma_2)}\Bigg]\;, \tag{6.7.3}$$

and the corresponding number operator and worldsheet energy are defined accordingly. Note that these modings generalize those given in Sect. 5.3. Likewise, the vacuum energy contribution from such a fermion is given by

$$a_\Psi \;=\; \frac{1}{2}\left(v^2-\frac{1}{12}\right)\;. \tag{6.7.4}$$

This too generalizes our previous results.

In ten dimensions, it turns out that we lose no generality by considering only the specific cases $v = 0, -\frac{1}{2}$ for all worldsheet fermions. What this means is that all self-consistent ten-dimensional string models can ultimately be realized using worldsheet fermions with only Neveu-Schwarz or Ramond boundary conditions. In lower dimensions, by contrast, other choices are possible. Therefore, even though we shall primarily focus our attention on the cases $v \in \{0, -\frac{1}{2}\}$, we shall develop our formalism in such a way that it holds for arbitrary values of v.

Given this parametrization, we can describe the boundary conditions within any sector rather succinctly by specifying twenty v-values, four for the complex right-movers Ψ_R^μ and sixteen for the complex left-movers Ψ_L^i. We can group these twenty v-values to form a "boundary-condition" vector

$$\mathbf{V} \;=\; [\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \,|\, v_1, ..., v_{16}] \;, \tag{6.7.5}$$

and thus we may associate a vector with each underyling string sector. For example, the sectors in Table 6.2 now correspond to the vectors shown in Table 6.5. Note that in Table 6.5, we have used a shorthand notation in which superscripts indicate repeated components. We have also dropped the minus signs from the Ramond entries $v = -\frac{1}{2}$. We stress, however, that even though we shall no longer explicitly indicate the Ramond minus sign, it should continue to be implicitly understood for all Ramond boundary conditions. (This minus sign can play an important role for string models in lower dimensions.)

What, then, are the self-consistent combinations of sectors? Recall from the previous lecture that the first self-consistent combination of sectors comprises Sectors #1 and #2 only. Let us therefore study this simplest combination. Sector #1 (the so-called NS-NS sector) corresponds to the *zero-vector* $\mathbf{0}$, the vector whose entries all vanish. Thus, in this sense, we might associate the NS-NS sector with the *origin* in a twenty-dimensional vector space. Sector #2 (the so-called Ramond-Ramond sector) then corresponds to some other point in the vector space which is some distance away from the origin. Let us call this other location $\mathbf{V}_0 \equiv [(\frac{1}{2})^4|(\frac{1}{2})^{16}]$.

If we were to consider $\mathbf{V}_0$ to be a lattice basis vector, a natural question would be to determine the lattice that is generated by this basis vector. Because there is only one such non-zero vector, this would clearly be a one-dimensional "lattice". Since $\mathbf{V}_0 \equiv [(\frac{1}{2})^4|(\frac{1}{2})^{16}]$, the next point in the lattice would be $2\mathbf{V}_0 \equiv [(1)^4|(1)^{16}]$. How can we interpret this point? Recall from (6.7.1) that the components of such vectors (*i.e.*, the values of v) are defined only modulo 1 (*i.e.*, they are restricted to the unit interval $-\frac{1}{2} \leq v < \frac{1}{2}$). Thus, we see that $v = 1$ is physically the same as $v = 0$, once again implying a Neveu-Schwarz boundary condition. In other words, we should only add our vectors *modulo 1*. Given this, we find that $2\mathbf{V}_0 \stackrel{1}{=} \mathbf{0}$, where we have introduced the notation $\stackrel{1}{=}$ to indicate equality modulo 1. Likewise, $3\mathbf{V}_0 \stackrel{1}{=} \mathbf{V}_0$, and so forth. Thus, we see that $\mathbf{V}_0$ generates a "lattice" consisting of only two physically distinct "points":

$$\{\mathbf{0}, \mathbf{V}_0\} \;. \tag{6.7.6}$$

However, these are precisely the two "points" that comprised our first self-consistent set of sectors (Case A in Lecture #6), and which led to our first string model!

It turns out that this is a general property: *All self-consistent choices of string sectors are those that correspond to the "points" in a twenty-dimensional lattice generated by a set of basis vectors.* To illustrate this principle, let us consider the next case (Case B in Lecture #6). In this case, we included only Sectors #1 through #4. This indicates that we need a larger lattice, which in turn implies the existence of not just the single lattice-generating basis vector $\mathbf{V}_0$, but also an additional basis vector $\mathbf{V}_1$. One choice is:

$$\begin{aligned} \mathbf{V}_0 &= [(\tfrac{1}{2})^4 \,|\, (\tfrac{1}{2})^{16}] \\ \mathbf{V}_1 &= [(0)^4 \,|\, (\tfrac{1}{2})^{16}] \;. \end{aligned} \tag{6.7.7}$$

Using these choices, we can see that indeed all four of these sectors can be generated as the different "points" in the resulting lattice: Sector #1 corresponds to the origin $\mathbf{0}$, Sector #2 corresponds to $\mathbf{V}_0$ itself, Sector #3 corresponds to $\mathbf{V}_1$ itself, and Sector #4 corresponds to the remaining lattice point $\mathbf{V}_0 + \mathbf{V}_1$. Note that no other points exist in this lattice, since $2\mathbf{V}_0 \stackrel{1}{=} 2\mathbf{V}_1 \stackrel{1}{=} \mathbf{0}$. Thus, we see that the introduction of the additional basis vector $\mathbf{V}_1$ is physically equivalent to the "twist" that shifts the boundary conditions of the left-moving fermions relative to those of the right-moving fermions in Sectors #3 and #4.

Finally, let us consider the full set (Case C) consisting of Sectors #1 through #8. It is easy to see that this set is generated by the *three* basis vectors:

$$\begin{aligned} \mathbf{V}_0 &= [(\tfrac{1}{2})^4 \,|\, (\tfrac{1}{2})^{16}] \\ \mathbf{V}_1 &= [(0)^4 \,|\, (\tfrac{1}{2})^{16}] \\ \mathbf{V}_2 &= [(0)^4 \,|\, (\tfrac{1}{2})^{8}(0)^8] \;. \end{aligned} \tag{6.7.8}$$

Once again, the introduction of the new basis vector $\mathbf{V}_2$ implements the "twist" that separates the boundary conditions of the first set of eight left-moving fermions from those of the second set.

This procedure can be continued. Each additional basis vector introduces a new twist, increases the size of the resulting lattice, and leads to the introduction of new physical string sectors (so-called "twisted sectors"). For example, one further basis vector that might be introduced is $\mathbf{V}_3 \equiv [(0)^4|(\frac{1}{2})^4(0)^4(\frac{1}{2})^4(0)^4]$. This vector would have the effect of introducing a further twist amongst the left-moving fermions within each group of eight.

Clearly, given a set of N basis vectors $\mathbf{V}_i$ $(i = 0, ..., N-1)$, the procedure for generating the full set of resulting string sectors is to consider all possible lattice vectors $\sum_{i=0}^{N-1} \alpha_i \mathbf{V}_i$ where $\alpha_i \in \{0, 1\}$. Note that this restriction on the values of α_i assumes that we are considering only Neveu-Schwarz or Ramond boundary conditions for the worldsheet fermions; generalizations to multi-periodic fermions will be discussed shortly. We shall henceforth denote a given string sector as $\alpha\mathbf{V} \equiv \sum_i \alpha_i \mathbf{V}_i$. For example, the NS-NS sector (*i.e.*, Sector #1) always corresponds to $\alpha = (0, 0, ...)$ and the Ramond-Ramond sector (*i.e.*, Sector #2) corresponds to $\alpha = (1, 0, ...)$.

At this stage, we now know how to generate the full set of underlying string sectors once we are given a "primordial" set of basis vectors $\mathbf{V}_i$. The next issue that arises is to determine the rules that govern the allowed choices of these basis vectors. Of course, we have already derived one such rule: each basis vector $\mathbf{V}_i$ must take the form

$$\mathbf{V}_i \;=\; [(\bar{v})^4 \,|\, v_1, ..., v_{16}] \tag{6.7.9}$$

where the right-moving fermions all have *same* moding $\bar{v} \in \{0, -\frac{1}{2}\}$. Indeed, as we saw in Lectures #5 and #6, this requirement is necessary for the preservation of the right-moving worldsheet supersymmetry (so that the right-moving worldsheet supercurrent has a unique moding in each sector). This is also necessary for the preservation of spacetime Lorentz invariance, since the right-moving worldsheet fermions carry Lorentz spacetime indices.

As we might expect, there are still several additional conditions that our basis vectors $\mathbf{V}_i$ must satisfy. But before we can discuss these conditions, we must turn to the generation of the GSO constraints in each sector.

6.7.2. *Generating the GSO constraints*

We have already seen in previous lectures that the appearance of new string sectors is correlated with the appearance of new GSO constraints in each sector. We are now in a position to formulate this correlation more precisely: *in each string sector, there is one GSO projection for each basis vector*. Our task, then, is to find a simple way to generate the exact forms of these GSO projections.

Let us return to Case A, and consider the model consisting of only Sectors #1 and #2. As we have seen above, this model is generated by the single basis vector $\mathbf{V}_0 \equiv [(\frac{1}{2})^4 | (\frac{1}{2})^{16}]$, resulting in the two sectors $\mathbf{0}$ (Sector #1) and $\mathbf{V}_0$ (Sector #2). In each of these sectors, recall from Table 6.3 that we then had the single GSO constraint $N_L - N_R =$ odd, or equivalently

$$\sum_{i=1}^{16} N^{(i)} \;-\; \sum_{j=1}^{4} \bar N^{(j)} \;=\; \text{odd}\;. \tag{6.7.10}$$

(Here we have used the j-index to span our four complex right-moving fermions, while the i-index spans our sixteen complex left-moving fermions.) It is this GSO constraint that we now wish to write in a more transparent manner.

Given our success in using the lattice idea and modular arithmetic in order to generate the complete set of string sectors, let us attempt to write (6.7.10) in a form that makes use of both ideas. Let us first concentrate on the modular arithmetic idea. Since all of our basis vectors are defined only modulo one, let us cast (6.7.10) into the form of a modulo-one relation. Since (6.7.10) is already a modulo-two relation, this can be achieved by dividing by two:

$$\tfrac{1}{2}\sum_{i=1}^{16} N^{(i)} \;-\; \tfrac{1}{2}\sum_{j=1}^{4} N^{(j)} \;\stackrel{1}{=}\; \tfrac{1}{2} \tag{6.7.11}$$

where we have used the notation $\stackrel{1}{=}$ to indicate equality modulo 1.

Let us now try to incorporate the lattice idea. To do this, let us make a *vector* out of our twenty number operators:

$$\mathbf{N} \;\equiv\; [\bar N^{(1)}, \bar N^{(2)}, \bar N^{(3)}, \bar N^{(4)} \,|\, N^{(1)}, ..., N^{(16)}]\;. \tag{6.7.12}$$

Clearly, each different possible string state in a given sector corresponds to a different $\mathbf{N}$-vector, and the physical (surviving) string states are those satisfying (6.7.11). Let us now attempt to write (6.7.11) in a vector notation. Neglecting the minus sign in (6.7.11) for the moment, we see that (6.7.11) involves a sum of vector components, which reminds us of a vector dot product. Thus, if we define the "signature" of our twenty-dimensional lattice to be $[(-)^4 \,|\, (+)^{16}]$, we can write (6.7.11) in the form of a vector dot product:

$$[(\tfrac{1}{2})^4 \,|\, (\tfrac{1}{2})^{16}] \;\cdot\; \mathbf{N} \;\stackrel{1}{=}\; \tfrac{1}{2} \tag{6.7.13}$$

where we have introduced a vector each of whose components is equal to $\frac{1}{2}$. However, this vector is nothing but $\mathbf{V}_0$, the basis vector that generates the lattice for this model! Thus, we see that if our model is generated by the basis vector $\mathbf{V}_0$, then in each of the resulting sectors $\{\mathbf{0}, \mathbf{V}_0\}$ the GSO projections take the form

$$\mathbf{V}_0 \cdot \mathbf{N} \stackrel{1}{=} \tfrac{1}{2} \ . \tag{6.7.14}$$

This produces the non-supersymmetric $SO(32)$ string model from Lecture #6!

Let us now consider Case B, consisting of Sectors #1 through #4. As we saw in Lecture #6, this produces the *supersymmetric* $SO(32)$ heterotic string model, and is generated by the set of two basis vectors given in (6.7.7). In each of the four resulting sectors $\{\mathbf{0}, \mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_0 + \mathbf{V}_1\}$, the *two* GSO projections were $N_L - N_R =$ odd and $N_L =$ even. (Recall Table 6.3.) These now take the form

$$\mathbf{V}_0 \cdot \mathbf{N} \stackrel{1}{=} \tfrac{1}{2} \ , \qquad \mathbf{V}_1 \cdot \mathbf{N} \stackrel{1}{=} \tfrac{1}{2} \ . \tag{6.7.15}$$

Similarly, Case C is generated by the *three* basis vectors in (6.7.8), and the three GSO constraints in each sector take the general form

$$N_L - N_R = \dots \ , \qquad N_L = \dots, \qquad {}^{(8)}N_L = \dots \ . \tag{6.7.16}$$

Here ${}^{(8)}N_L \equiv \sum_{i=1}^{8} N^{(i)}$, and we shall momentarily defer a discussion of the values of the right sides of these constraint equations. We then find that these three GSO constraints take the general forms

$$\mathbf{V}_0 \cdot \mathbf{N} \stackrel{1}{=} \dots \ , \qquad \mathbf{V}_1 \cdot \mathbf{N} \stackrel{1}{=} \dots \ , \qquad \mathbf{V}_2 \cdot \mathbf{N} \stackrel{1}{=} \dots \ . \tag{6.7.17}$$

Depending on the right sides of these equations, this generates either the supersymmetric $E_8 \times E_8$ string or the non-supersymmetric $SO(16) \times SO(16)$ string.

The final question, then, is to determine what appears on the right sides of these GSO constraint equations. In general, this will be some value x which satisfies $-\frac{1}{2} \leq x < \frac{1}{2}$. This x-value is called a *GSO projection phase*, and is generally different for each sector. Thus, we know that x must itself depend on α, where (as discussed in Sect. 7.1) α parametrizes the particular sector in question. We also know from our prior experience (in particular, from Table 6.3) that x must also contain some additional *free* parameters because we occasionally still had the freedom to make choices such as $\{$ evenodd $\}$ when constructing our GSO constraints.

It turns out the final result is the following. Within any given string sector $\alpha\mathbf{V} \equiv \sum_{i=0}^{N-1} \alpha_i \mathbf{V}_i$, the states that survive are those whose number operator vectors $\mathbf{N}$ satisfy the equations

$$\mathbf{V}_i \cdot \mathbf{N} \stackrel{1}{=} \sum_{j=0}^{N-1} k_{ij}\alpha_j \ + \ s_i \ - \ \mathbf{V}_i \cdot (\alpha\mathbf{V}) \ , \qquad 0 \leq i \leq N-1 \ . \tag{6.7.18}$$

This is therefore the full set of GSO constraint equations for the sector $\alpha\mathbf{V}$. In (6.7.18), the notation is as follows. There are N different equations here, depending on the value of i. In the last term, the dot product $\mathbf{V}_i \cdot (\alpha\mathbf{V})$ is the dot product between $\mathbf{V}_i$ and the sector $\alpha\mathbf{V}$ for which the GSO constraint is being applied. In the second-to-last term, s_i is defined as the first component (*i.e.*, the first of the right-moving components) of the vector $\mathbf{V}_i$:

$$s_i \equiv \mathbf{V}_i^{(1)} . \tag{6.7.19}$$

Thus s_i parametrizes the *spacetime statistics* of the sector $\mathbf{V}_i$, with $s_i = 0$ indicating spacetime bosons and $s_i = -\frac{1}{2}$ indicating spacetime fermions. Likewise, the sum $\sum \alpha_i s_i$ (mod 1) indicates the statistics of the sector $\alpha\mathbf{V}$. In the remaining term, k_{ij} denotes a certain $N \times N$ matrix of numbers (so-called *GSO projection phases*) satisfying $-\frac{1}{2} \leq k_{ij} < \frac{1}{2}$. These are therefore the remaining degrees of freedom that enter into our GSO constraints. In the case of $\mathbb{Z}_2$ twists (for which all fermionic boundary conditions have either Neveu-Schwarz or Ramond boundary conditions), one has $k_{ij} \in \{0, -\frac{1}{2}\}$ only. The case of multi-periodic fermions will be discussed shortly.

Thus, if we are given a set of parameters $\{\mathbf{V}_i, k_{ij}\}$, we can now generate the resulting string model and the entire corresponding spectrum! These parameters are ultimately the parameters that physically describe a given string model.

6.7.3. *Self-consistency constraints*

We finally turn to the remaining question: what determines how the parameters $\{\mathbf{V}_i, k_{ij}\}$ are to be chosen? What are the rules that guarantee a self-consistent choice?

Clearly, as we have discussed earlier, modular invariance is one of many symmetries that govern these choices. Other requirements for self-consistency include proper spacetime spin-statistics relations (so that all Ramond states are indeed anti-commuting spacetime fermions, and all Neveu-Schwarz states are commuting spacetime bosons) and physically sensible GSO projections (so that unitarity is not violated, among other things). It is important to stress that these are not *additional* constraints that need to be imposed in order to guarantee the consistency of the string in spacetime; rather these constraints are intrinsic to string theory itself at the worldsheet level, emerging as string self-consistency constraints, and together imply these features in spacetime.

We have already discussed the first contraint that governs the choices of the basis vectors: they must all have the form (6.7.9), with all right-moving fermions sharing the same boundary condition. Second, these vectors must all be linearly independent with respect to addition (modulo 1); otherwise, at least one of these vectors is redundant. The third constraint also turns out to be quite simple: among our set of basis vectors, we must always start with the vector

$$\mathbf{V}_0 \;\equiv\; [(\tfrac{1}{2})^4 \,|\, (\tfrac{1}{2})^{16}] \;. \qquad (6.7.20)$$

The presence of this vector ensures that the resulting string model contains at least a Ramond-Ramond sector in addition to a NS-NS sector.

The remaining constraints serve to correlate the $\mathbf{V}_i$ vectors with the GSO projection phases k_{ij}, and take the form:

$$\begin{aligned} k_{ij} + k_{ji} &\stackrel{1}{=} \mathbf{V}_i \cdot \mathbf{V}_j \\ k_{ii} + k_{i0} &\stackrel{1}{=} \tfrac{1}{2}\mathbf{V}_i \cdot \mathbf{V}_i - s_i \;. \end{aligned} \qquad (6.7.21)$$

Note that given a set of boundary condition vectors $\mathbf{V}_i$, the constraints (6.7.21) imply that only the elements k_{ij} with $i > j$ are independent parameters. The first equation in (6.7.21) then enables us to uniquely determine k_{ij} with $i < j$, and the second equation in (6.7.21) enables us to uniquely determine the diagonal elements k_{ii}.

6.7.4. *Summary, examples, and generalizations*

Let us now summarize the rules for heterotic string model-building in $D = 10$. We begin by choosing a set of linearly independent basis vectors $\mathbf{V}_i$ ($i = 0, ..., N-1$) and a corresponding matrix of GSO projection phases k_{ij} ($i, j = 0, ..., N-1$). Our set of basis vectors may be as large as we desire; since each vector corresponds to an additional twist, larger sets of vectors lead to more complicated string models. Among our choice of basis vectors must always appear the vector $\mathbf{V}_0$ defined in (6.7.20), and every basis vector is required to have the form (6.7.9). We must also ensure that our choices of basis vectors $\mathbf{V}_i$ and GSO projection phases k_{ij} are properly *correlated* according to (6.7.21). If there does not exist a solution for k_{ij}, then our original choice of $\mathbf{V}_i$ must be discarded or repaired. These are the only constraints that govern the choices of the parameters $\{\mathbf{V}_i, k_{ij}\}$.

Given such a self-consistent choice of parameters $\{\mathbf{V}_i, k_{ij}\}$, we are then guaranteed to have a self-consistent string model. The different sectors of this model are generated as all combinations $\sum_i \alpha_i \mathbf{V}_i$ that fill out the twenty-dimensional lattice, where $\alpha_i \in \{0,1\}$. In each sector $\alpha\mathbf{V} \equiv \sum_i \alpha_i \mathbf{V}_i$, the allowed states are then those whose number operator vectors $\mathbf{N}$ simultaneously satisfy the constraints (6.7.18) for $i = 0, ..., N-1$. This is often called the *spectrum-generating formula*.

It is straightforward to see how this formalism can be applied in practice. We shall leave it as an exercise to verify that the choice

$$\mathbf{V}_0 \;\equiv\; [(\tfrac{1}{2})^4 \,|\, (\tfrac{1}{2})^{16}]\;, \qquad k_{00} = (0) \tag{6.7.22}$$

generates the non-supersymmetric $SO(32)$ heterotic string model; that the choice

$$\begin{cases} \mathbf{V}_0 \;\equiv\; [(\tfrac{1}{2})^4 \,|\, (\tfrac{1}{2})^{16}] \\ \mathbf{V}_1 \;\equiv\; [(0)^4 \,|\, (\tfrac{1}{2})^{16}] \end{cases} \qquad k_{ij} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.7.23}$$

generates the *supersymmetric* $SO(32)$ heterotic string model; and that the choices

$$\begin{cases} \mathbf{V}_0 \;\equiv\; [(\tfrac{1}{2})^4 \,|\, (\tfrac{1}{2})^{16}] \\ \mathbf{V}_1 \;\equiv\; [(0)^4 \,|\, (\tfrac{1}{2})^{16}] \\ \mathbf{V}_2 \;\equiv\; [(0)^4 \,|\, (\tfrac{1}{2})^{8} (0)^8] \end{cases} \qquad k_{ij} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k \\ 0 & k & 0 \end{pmatrix} \tag{6.7.24}$$

generate the supersymmetric $E_8 \times E_8$ string model if we choose $k = 0$, and the non-supersymmetric $SO(16) \times SO(16)$ string model if we choose $k = 1/2$. Indeed, it is a general property that if we choose our vector $\mathbf{V}_1$ as above, then spacetime supersymmetry is preserved if $k_{i0} = k_{i1}$ for all $i = 0, 1, ..., N-1$, and broken otherwise. Thus, we see that we now have a very compact notation and procedure for generating and analyzing ten-dimensional heterotic string models! We should also stress that these are not the only parameter choices of $\{\mathbf{V}_i, k_{ij}\}$ that will lead to these models. In fact, there is often a great redundancy in this procedure, so that a given physical string model can have many different representations in terms of the worldsheet parameters $\{\mathbf{V}_i, k_{ij}\}$. However, a given set of parameters always corresponds to a single, unique, self-consistent string model in spacetime.

The formalism that we have presented in this lecture is called the "free-fermionic construction", and was developed in 1986 by H. Kawai, D.C. Lewellen, and S.-H.H. Tye and by I. Antoniadis, C. Bachas, and C. Kounnas. The name stems from the fact that the fundamental degrees of freedom on the string worldsheet (in addition to the spacetime coordinate fields X^μ) are taken to be the free fermionic fields Ψ. Even though we have presented this formalism for the case of ten-dimensional heterotic strings, there also exists a straightforward generalization of this formalism to *four-dimensional* heterotic string models.

As we have indicated, this formalism also carries over directly to the case of multi-periodic complex fermions for which the boundary condition parameter v in (6.7.1) can be an arbitrary rational number in the range $-\frac{1}{2} \leq v < \frac{1}{2}$. For each resulting boundary-condition vector $\mathbf{V}_i$, let us define m_i to be the smallest integer such that if we multiply each element in $\mathbf{V}_i$ by m_i, we obtain a vector of integer entries. In general, m_i is called the "order" of the vector $\mathbf{V}_i$, and is also the order of the corresponding physical twist introduced by that vector. For example, in the case of only Neveu-Schwarz or Ramond fermions, we have $m_i = 2$ for all i, implying only $\mathbb{Z}_2$ twists. Nevertheless, even for general multi-periodic boundary conditions, the above constraints continue to apply exactly as written. Indeed, the only small change is that we now must take $\alpha_i \in \{0, 1, ..., m_i - 1\}$ when generating our lattice of corresponding string sectors. Likewise, each GSO projection phase k_{ij} must now also be chosen such that $m_j k_{ij} \in \mathbb{Z}$.

In this regard, it is important to note that the only fermions which can possibly have such generalized boundary conditions are those which are *not* the worldsheet superpartners of worldsheet bosons. This restriction arises because the structure of the worldsheet supersymmetry algebra itself restricts the corresponding fermions to have only Neveu-Schwarz or Ramond boundary conditions. For example, in the case of the ten-dimensional heterotic string, only the left-moving worldsheet fermions are *a priori* permitted to have generalized boundary conditions. By contrast, the right-moving fermions are restricted by the right-moving worldsheet supersymmetry algebra to have either Neveu-Schwarz or Ramond boundary conditions. This in turn implies that $s_i \in \{0, -\frac{1}{2}\}$, so that a given string sector continues to give rise to only spacetime bosons or spacetime fermions. Also note that although we are capable *in principle* of utilizing multi-periodic fermions while constructing ten-dimensional heterotic string models, in practice it turns out that this does not lead to new models which are physically distinct from those using only Ramond or Neveu-Schwarz fermions. It is for this reason that we can ultimately restrict ourselves to these simpler boundary conditions in ten dimensions without loss of generality. In lower dimensions, by contrast, this is no longer true, and the number of possible models grows dramatically.

This formalism can also be carried over to the case of ten-dimensional *superstrings* (rather than heterotic strings). For superstrings, the boundary-condition vectors take the simpler form

$$\mathbf{V}_i \;=\; [(\bar{v})^4 \,|\, (v)^4] \tag{6.7.25}$$

where $v, \bar{v} \in \{0, \frac{1}{2}\}$. Our mandatory vector $\mathbf{V}_0$ then takes the form $[(\frac{1}{2})^4|(\frac{1}{2})^4]$, and we define $s_i \equiv v+\bar{v}$ (mod 1) as our new spacetime statistics parameter, replacing (6.7.19). The results (6.7.21) and (6.7.18) then continue to apply directly. Of course, this formalism is fairly trivial in the case of *ten-dimensional* superstrings, for the maximal set of linearly independent basis vectors of the form (6.7.25) consists of only $\mathbf{V}_0$ and $\mathbf{V}_1 \equiv [(0)^4|(\frac{1}{2})^4]$. As we have seen in Lecture #4, this results in only four distinct superstring models in ten dimensions: omitting $\mathbf{V}_1$ from our basis set generates the Type 0 models, while including $\mathbf{V}_1$ in our basis set generates the Type II models. However, just as for the heterotic strings, this formalism can also be generalized to the case of four-dimensional superstring models where the possibilities become much richer.

It turns out that the free-fermionic formalism can be extended still further. For example, one can also extend this formalism to compactifications of the *bosonic* string. Moreover, one can even extend this formalism to special types of superstring and heterotic string models whose worldsheet actions must be represented in terms of real rather than complex fermions. Likewise, there even exist generalizations to string models involving non-free worldsheet fermions (*i.e.*, models whose worldsheet actions involve additional Thirring-type interactions between the worldsheet fermions). In fact, even though there exist alternative model-construction formalisms that do not involve free worldsheet fermions at all, the free-fermionic construction can often yield models that are physically equivalent to those that are constructed through these other means.

How general, then, is the free-fermionic construction? It turns out that for *ten-dimensional* string models, this construction is completely general. What this means is that all known physically consistent superstring and heterotic string models in ten dimensions can be realized via this construction (*i.e.*, as stemming from an underlying set of free-fermionic parameters $\{\mathbf{V}_i, k_{ij}\}$). In lower dimensions, by contrast, this construction is *not* completely general — there exist self-consistent lower-dimensional string models which cannot be written or constructed in this manner. However, the free-fermionic construction does comprise a *vast set* of semi-realistic string models. Moreover, the free-fermionic construction has the great advantage that the rules for construction are relatively simple, and that they enable one to *systematically* construct many string models and examine their phenomenological properties. Indeed, many computer programs have been written that use this formalism in order to scan the space of string models and analyze their low-energy phenomenologies. Thus, for these reasons, the free-fermionic construction has played a very useful role as the underlying method through which the majority of string model-building has historically been pursued.

6.7.5. *Assessment:*

At this point, it is perhaps useful to assess the position in which we now find ourselves. Clearly, through these constructions, we are able to produce *many* string models. In fact, as we shall see, the number of self-consistent string models in $D < 10$ is virtually infinite, and there exists a whole space of such models. This space of models is called a *moduli space*, where the so-called moduli are various continuous parameters which can be adjusted in order to yield different models. (Of course, we have seen that we have only discrete parameter choices in ten dimensions, but these parameters can become continuous in lower dimensions.) Moreover, each of these models has a completely different spacetime phenomenology. What, then, is the use of string theory as an "ultimate" theory, if it does not lead to a single, unique model with a unique low-energy phenomenology?

To answer this question, we should recall our discussion at the beginning of these lectures. Just as field theory is a language for building certain models (one of which, say, is the Standard Model), string theory is a new and deeper language by which we might also build models. The advantages of using this new language, as discussed in Lecture #1, include the fact that our resulting models incorporate quantum gravity and Planck-scale physics. Of course, in field theory, many parameters enter into the choice of model-building. These parameters include the choice of fields (for example, the choice of the gauge group, and whether or not to have spacetime supersymmetry), the number of fields (for example, the number of generations), the masses of particles, their mixing angles, and so forth. These are all *spacetime* parameters. In string theory, by contrast, we do not choose these spacetime parameters; we instead choose a set of *worldsheet* parameters. For example, in the free-fermionic construction, we choose the parameters $\{\mathbf{V}_i, k_{ij}\}$. All of the phenomenological properties in spacetime are then derived as consequences of these more fundamental choices. But still, just as in field theory, we are faced with the difficult task of model-building.

Is this progress, then? While opinions on this question may differ, one can argue that the answer is still definitely "yes". Recall that quantum gravity is automatically included in these string models. This is one of the benefits of model-building on the worldsheet rather than in spacetime. Also recall that string theory is a finite theory, and does not contain the sorts of ultraviolet divergences that plague us in field theory. This is another benefit of worldsheet, rather than spacetime, model-building. Moreover, worldsheet model-building ultimately involves choosing *fewer* parameters than we would have to choose in field theory — for example, we have seen that an entire infinite tower of string states, their gauge groups and charges and spins, are all ultimately encoded in a few underlying worldsheet parameters such as $\{\mathbf{V}_i, k_{ij}\}$. Furthermore, because of this drastic reduction in the number of free parameters, string phenomenology is in many ways more tightly constrained than ordinary field-theoretic phenomenology. Thus, it is in this way that string theory can guide our choices and expectations for physics beyond the Standard Model. Indeed, from a string perspective, we see that we should favor only those patterns of spacetime physics that can ultimately be derived from an underlying set of worldsheet parameters such as $\{\mathbf{V}_i, k_{ij}\}$. These would then serve as a "minimal set" of parameters which would govern all of spacetime physics!

Of course, at a theoretical or philosophical level, this state of affairs is still somewhat unsatisfactory. After all, we still do not know *which* self-consistent choice of string parameters ultimately corresponds to reality. However, *in principle*, string theory should be able to predict this dynamically. Indeed, even though there exists a whole moduli space of self-consistent string models, there should exist an energy or potential function in this space (*i.e.*, some function $V(\{\phi\})$ of all the moduli $\{\phi\}$) which should dynamically select a particular point in moduli space (*e.g.*, as a local or global minimum of V). This would then fix all of the moduli to specific values, or equivalently (in the language of the free-fermionic construction) tell us which choices of parameters $\{\mathbf{V}_i, k_{ij}\}$ are preferred dynamically.

Unfortunately, we do not understand the dynamics of string theory well enough to carry out such an ambitious undertaking. Certainly, at the level of perturbative (weakly coupled) string theory, we have no way to distinguish amongst the possible low-energy models by calculating such a function $V(\{\phi\})$. This is particularly true for string models exhibiting spacetime supersymmetry, for which $V = 0$ exactly to all orders in perturbation theory. Even if the spacetime supersymmetry is broken, the resulting potential $V(\{\phi\})$ often turns out not to have a stable minimum. This is the so-called "runaway problem", to be discussed further in Lecture #8. Of course, one might hope that recent advances in understanding the *non-perturbative* structure of string theory will ultimately be able to provide guidance in this direction. However, as we shall discuss briefly in Lecture #8, although these non-perturbative insights (particularly those concerning string duality) have thus far changed our understanding of the size and shape of this moduli space, they have not yet succeeded in leading us to an explanation of which points in this moduli space are dynamically selected.

So where do we stand? As string phenomenologists, we can do two things. First, we can pursue *model-building*: we can search through the moduli space of self-consistent string models in order to determine how close to realistic spacetime physics we can come. This is, in some sense, a direct test of string theory as a phenomenological theory of physics. Of course, this approach to string phenomenology is ultimately limited by many factors: we have no assurance that our model-construction techniques are sufficiently powerful or general to include the "correct" string model (assuming that one exists); we have no assurance that our model-construction techniques will not lead to physically distinct models which nevertheless "agree" as far as their testable low-energy predictions are concerned; and we have no assurance that the most important phenomenological features that describe our low-energy world (such as the pattern of supersymmetry-breaking) are to be found in perturbative string theory rather than in non-perturbative string theory. For example, it may well be (and it has indeed been argued) that the true underlying string theory that describes nature is one which is intrinsically non-perturbative, and which would therefore be beyond the reach of the sorts of approaches typically followed in studies of string phenomenology.

Another option, then, is to temporarily abandon string model-building somewhat, and to seek to extract general phenomenological theorems or correlations about spacetime physics that follow directly from the general structure of string theory itself. Clearly, we would wish such information to be *model-independent*, *i.e.*, independent of our particular location in moduli space or the values of particular string parameters such as $\{\mathbf{V}_i, k_{ij}\}$. For example, if some particular configuration of spacetime physics (some pattern of low-energy phenomenology) can be shown to be inconsistent with being realized from an underlying set of $\{\mathbf{V}_i, k_{ij}\}$ parameters, and if such a demonstration can be made to transcend the particular free-fermionic construction so that it relies on only the primordial string symmetries themselves, then such patterns of phenomenology can be ruled out. In this way, one can still use string theory in order to narrow the list of possibilities for physics at higher energies, and to correlate various seemingly disconnected phenomenological features with each other. Such correlations would then be viewed as "predictions" from string theory, and we shall see many examples of this phenomenon in subsequent lectures.

In summary, then, we have seen that there exist powerful ways of constructing string models and surveying their low-energy phenomenologies, but that this leads to the problem of selecting the true model (*i.e.*, the true "ground state" or "vacuum") of string theory. Despite recent advances in understanding various non-perturbative aspects of string theory, our inability to answer the fundamental question of vacuum selection persists. Until this challenge is overcome, string phenomenology therefore must content itself with answering questions of a *relative* nature (such as questions concerning relative *patterns* of phenomenology) rather than the sorts of absolute questions (such as calculating the mass of the electron) that one would also ideally like to ask. Nevertheless, as we shall see, string theory can still provide us with considerable guidance for physics beyond the Standard Model.

6.8. Lecture #8: A final lecture

Up to this point, we have primarily discussed string *model-building* — *i.e.*, the art of building string models. Hopefully, we have given the reader some sense of the complexity of the many constraints that are involved. In this final lecture, however, we shall depart from the somewhat "linear" development we have followed thus far in order to discuss *string phenomenology* — the study of the low-energy physical attributes of these models.

In the first part of this final lecture, we shall outline some general properties of four-dimensional heterotic string models. Then, we shall contrast these with the phenomenological properties of open-string D-brane models. Finally, we shall provide general comments concerning string phenomenology as a whole, and conclude with a brief discussion of some new, recent directions in string phenomenology.

6.8.1. *General properties of perturbative $D = 4$ heterotic string models*

In previous lectures, we have discussed the construction of perturbative heterotic string models. Here, we shall now turn the general low-energy properties that emerge from these constructions.

First, such models all have big gauge groups. For perturbative heterotic strings in four dimensions, we find that

$$\text{rank}(G) \ \leq \ 22 \ . \tag{6.8.1}$$

This is the four-dimensional analogue of the observation that the maximum rank in 10 dimensions is 16, such as for the $SO(32)$ and $E_8 \times E_8$ heterotic strings. The additional six units of rank emerge from the Kaluza-Klein reduction from $D = 10$ to $D = 4$.

If the string model in question is "realistic", then typically we can write

$$G \ = \ G_1 \times G_2 \tag{6.8.2}$$

where G_1 contains $SU(3) \times SU(2) \times U(1)_{\text{hypercharge}}$. Here G_1 is called the "observable-sector" gauge group: *e.g.*, G_1 could be $SU(3) \times SU(2) \times U(1)$, $SO(6) \times SO(4)$, $SU(5)$, $SO(10)$, E_6, *etc.* By contrast, G_2 is called the "hidden-sector" gauge group.

Second, there are typically lots of massless ("observable") states! These can be classified into several categories:

- Typical representations will carry charges under both G_1 and G_2 (*i.e.*, transform as non-singlet representations of these groups). In general, we will only have spinors, vectors (*i.e.*, fundamentals), and adjoints at the massless string level. (This is indeed a theorem: the allowed representations are closely tied to something called the "affine level" of the gauge group.) These states will typically fall into two subsets. First, there may be states that can be identified as (MS)SM quarks and leptons. In such cases, all gauge symmetry groups under which the SM gauge particles transform as singlets are considered to be part of G_2, *i.e.*, the hidden sector. Second, there can be extra states *beyond* the (MS)SM. There will typically be a lot of such states as well. They may be identified as exotic quarks and leptons. They will typically have fractional electric charge. This could cause problems (see below).
- Many gauge-*singlet* states (*i.e.*, states carrying no gauge charges) will also exist in the string model. For example, such states include the graviton, antisymmetric tensor, and dilaton ϕ. Recall that $g_{\text{string}} \sim \exp(-\langle\phi\rangle)$. Thus, the dilaton must be stabilized to yield a fixed value for the string coupling, and to avoid the so-called "dilaton runaway problem" (wherein $\langle\phi\rangle \to \infty$, or $g_{\text{string}} \to 0$).
  The dilaton is just one example of a generic class of Lorentz-singlet particles called string "moduli". The effective potential for such

models is *flat* to all orders in perturbation theory. Thus, non-perturbative string effects must somehow introduce a potential for these fields, *i.e.*, lift the degeneracy of string "ground states" and select a string vacuum. But how does this happen? This is a major unsolved problem, with lots of ideas in the literature. This is critically important for string phenomenology, since the vevs of the moduli set the values for gauge couplings, particle masses, and so forth. Without knowing the values of these couplings, the best we can look for is string-constrained *patterns* (textures) in these parameters.

Third, there will be infinite towers of Planck-scale string states! These states come in increasingly larger representations of gauge groups, and likewise have higher and higher Lorentz spins. These states are the means by which string theory maintains finiteness. They propagate in all string loop diagrams, and their contributions cancel the divergences of the massless states. They are the result of conformal invariance (really its one-loop extension, called "modular invariance").

One interesting fact about these states is that the number of such massive states with spacetime mass M grows *exponentially*:

$$g_M \;\sim\; \exp(cM\sqrt{\alpha'}) \tag{6.8.3}$$

where c is a fixed positive constant. One of the implications of an exponentially growing degeneracy is as follows. Let us consider the thermodynamic partition function:

$$Z \;\equiv\; \sum_M g_M \, \exp(-M/kT) \tag{6.8.4}$$

where T is the temperature and k is Boltzmann's constant. This gives:

$$Z \;=\; \sum_M \exp[M(c\sqrt{\alpha'} - 1/kT)] \;. \tag{6.8.5}$$

Thus, if T is bigger than a critical value

$$T_c \;\equiv\; (kc\sqrt{\alpha'})^{-1} \;, \tag{6.8.6}$$

then the thermodynamic partition function *diverges*! This is the so-called "Hagedorn" phenomenon.

Does this signal a phase transition? Or is there instead a limiting ("Hagedorn") temperature for string theory beyond which one cannot go? What happens to a box of strings (*i.e.*, the "universe") if we pump in lots of energy and try to raise the temperature?

The answers to these questions are really not known. The current belief is that we have a phase transition in which all extra energy gets dumped into long string modes. But the nature of this phase transition is generally unclear. Indeed, this Hagedorn phenomenon is one of the central hallmarks of the the subject of *string thermodynamics*. As might be imagined, this subject is of critical importance for string cosmology and for string-based studies of the early universe.

Fourth, such heterotic string models will typically give rise to a single "pseudo-anomalous" $U(1)$ gauge group! Recall that in field theory, given a set of states with $U(1)$ charges Q_i, we must have $\sum_i Q_i = 0$ in order to cancel axial (triangle) anomalies. In particular, certainly the hypercharge $U(1)$ must be anomaly-free.

However, in (many/most) heterotic string models, gauge groups are big and there can be *extra* $U(1)$ gauge groups. One finds, upon summing over massless spectrum, that *one* of these gauge groups, typically denoted $U(1)_X$, has corresponding states with charges Q_X such that $\sum Q_X \neq 0$. Thus, from the field-theory point of view, this $U(1)_X$ appears to be anomalous!

In fact, however, this gauge group is not anomalous (since string theory is always anomaly-free); there are extra contributions to the apparent anomaly which come from anomalous transformations of the string "axion field" (related to the antisymmetric tensor $B_{\mu\nu}$) which cancel this anomaly. This is an intrinsically "stringy" mechanism (called the Green-Schwarz mechanism) for cancelling an anomaly.

Fifth, such models typically give rise to automatic gauge coupling unification (regardless of existence of any GUT symmetry in the string model). In fact, the gauge couplings are even unified with the gravitational coupling!

It is easy to understand why this is the case. Recall that our original "untwisted" four-dimensional heterotic string model has a "unified" gauge group $SO(44)$, with one gauge gauge coupling whose value is set by the dilaton vev. (This is the analogue of $SO(32)$ in ten dimensions.) When we break subsequently break the gauge symmetry by introducing twists ("orbifolding"), this does not affect the gauge couplings. They are still all set by the same dilaton vev (ultimately because there is only *one* dilaton to which all gauge groups can couple). Thus, gauge coupling unification is automatic in heterotic string theory.

One important question is the *scale* of the unification. Clearly, by dimensional analysis, this can be nothing but the string scale. At tree-level, we have already seen in Lecture #1 that the string scale is given by

$M_{\rm string} = g_{\rm string} M_{\rm Planck}$. However, work by Kaplunovsky has shown that at one-loop order, and with the usual GUT assumption of $g_{\rm string} \approx 0.7$, this result is shifted down to an approximate value, $M_{\rm string} \approx 5.27 \times 10^{17}$ GeV. This is generally a problem, since the expected GUT value for the unification scale is $M_{\rm GUT} = 2 \times 10^{16}$ GeV. How then do we explain this factor-of-20 discrepancy between $M_{\rm string}$ and $M_{\rm GUT}$? This is currently an open question, with many potential solutions. A comprehensive review of this subject can be found in K.R. Dienes, Phys Reports 287 (1997) 447 = hep-th/9602045.

Sixth, such models typically give rise to states with fractional electric charge. We already referred to this above. Indeed, extra states beyond the MSSM will typically have $SU(2)\times U(1)$ quantum numbers which imply non-integer values for the electric charge. In fact, one can prove (see theorem by A.N. Schellekens) that if the model has a gauge symmetry $SU(3)\times SU(2)\times U(1)$ rather than a GUT, then the string will *necessarily* give rise to such fractionally charged states. This is a result of conformal invariance and modular invariance.

One possible resolution to this problem is that such fractionally charged states might be able to *confine* to form integer-charged states under the influence of non-abelian gauge symmetries beyond the SM. However, if this is not possible in a given string model, then that model is generally considered to be phenomenologically inconsistent.

General theorems exist which enable one to classify the different types of fractional charges one can expect to find in a given string model and which can be "confined" away. (We refer the reader to papers by Schellekens; also by Dienes, Faraggi, March-Russell.)

Seventh, it turns out that such string models *cannot* contain any exact global symmetries! For example, in heterotic string theory, baryon- and lepton-number conservation, as well as other discrete symmetries, must all be parts of *local* symmetries (gauge symmetries) or be only approximate symmetries (*i.e.*, accidental).

Eighth, such heterotic string models will either exhibit spacetime supersymmetry, or they will be non-supersymmetric. If non-supersymmetric, however, they nevertheless have a hidden symmetry called a "misaligned supersymmetry" which governs how the bosons and fermions are arranged at all mass levels so that finiteness is preserved, even without SUSY. Even the supertraces, when evaluated over the entire Fock space of string states, continue to vanish. [References include K.R. Dienes, Nucl. Phys. B429 (1994) 533; K.R. Dienes, M. Moshe, and R.C. Myers, Phys. Rev. Lett. 74

(1995) 4767.]

However, it is not known whether such non-supersymmetric strings can ever be stable beyond tree level. This is an important open question in string theory. However, if such stable non-SUSY strings exist, then this could provide a whole new framework for thinking about the gauge hierarchy problem, SUSY-breaking, questions of finiteness, the role of effective field theories and in particular the massive Planck-scale states, and gauge coupling unification. This may even provide an alternative, "stringy" approach towards the hierarchy problem which does not involve either supersymmetry or extra spacetime dimensions. [For some speculative ideas along this direction, see K.R. Dienes, hep-th/0104274.]

Ninth, the spacetime string spectrum can exhibit certain dualities. Indeed, there are several kinds of duality which, taken together, form an interconnected web of relations between different kinds of string theories.

- One kind of duality is called "T-duality". Consider string #1, compactified on a circle of radius R, and string #2, compactified on a circle of radius $\sqrt{\alpha'}/R$. It turns out that these strings are indistinguishable, in the sense that they have exactly the same spacetime spectrum! What would be considered a momentum state in string #1 would be considered a winding-mode state in string #2, and vice versa. This is clearly a very "stringy" symmetry! In fact, this symmetry transcends the mere tree-level spectrum, and holds to all orders. It also applies for all correlation functions, scattering amplitudes, both perturbatively and even non-perturbatively. This is an exact symmetry of closed string theories.
  One important implication of T-duality is that closed string theory (unlike point-particle field theory) cannot distinguish between large and small compactification radii! An interesting question is what this might imply about string cosmology. Likewise, what are the implications about our ultimate ability to derive *effective field theories* from the string?
- There are also other kinds of dualities which exist amongst the different string theories. For example, there is a duality called "S-duality" which flips the sign of the dilaton and thus relates theories at weak coupling to theories at strong coupling! Under such a mapping, perturbative string states (such as the ones we have been considering all along) are exchanged with non-perturbative string states (which have not considered at all, but which are "solitons"

= D-branes in the theory). Under this mapping, for example, the $SO(32)$ heterotic (closed) string theory is mapped into the $SO(32)$ Type I (open) string theory. Combined with T-duality, one finds that all the different kinds of ten-dimensional strings are ultimately related to each other, becoming part of a larger superstructure called “M-theory”.

The study of string dualities is a vast subject which easily deserves its own lectures, and which comprises the so-called ”second” superstring revolution, dating from 1995. Indeed, insights have enabled us, in many cases, to “solve” for the strong-behavior of string theory!

I cannot give a proper introduction to M-theory here, but I will simply give some general comments. M-theory is a conjectured eleven-dimensional theory (of strings? of membranes? – we don’t know) which can be *defined* through its three fundamental properties:

- The low-energy limit of M-theory is eleven-dimensional SUGRA (recall that $D = 11$ is the maximum dimension for SUGRA).
- Compactifying M-theory on a circle of radius R yields the Type IIA string with a coupling that is a growing function of R. So, at strong coupling, the Type IIA string begins to ”see” an extra dimension and become eleven-dimensional.
- Compactifying M-theory on a line segment of length L yields the $E_8 \times E_8$ heterotic string with a coupling that grows with L. This is why one does not see this 11th dimension in studies of the perturbative heterotic string: the very act of taking the string coupling to be small reduces the 11th dimension to zero size!

Studying M-theory and its compactifications (and its phenomenological properties, such as how SUSY-breaking may be realized in this framework) has been a hot topic in the string literature. In particular, one may ask whether it is possible to compactify M-theory to four dimensions in ways that do *not* pass through an intermediate realization in terms of a $D = 10$ heterotic string, thereby constructing new classes of four-dimensional string models? The answer is to this question is ‘yes’. Thus, even without knowing the precise nature of M-theory, it has already been possible to use insights gleaned from the mere existence of such a theory in order to generate new classes of string models.

Taken together, these developments have led to the realization that many of our cherished “fundamental” string symmetries (such as conformal invariance, modular invariance, etc.) are only *effective* weak-coupling symmetries, applicable only for closed strings. Thus, as the string coupling grows in closed string theories, we expect to see *deviations* from the constraints that come from these symmetries. This could be very useful in "freeing up" certain undesirable predictions of string phenomenology, even within closed strings.

6.8.2. *General properties of $D = 4$ open-string models*

Many of the above phenomenological features of heterotic strings change when one deals with Type I theories (*i.e.*, theories which include *open* strings). Some of the most viable models in this class that have chiral spectra include so-called "intersecting D-brane Models" as well as models with D-branes at singularities. Unfortunately, we do not have the space here to discuss such constructions. However, there are excellent reviews available, In particular, we refer the reader to R. Blumenhagen, M. Cvetic, P. Langacker, and G. Shiu, hep-th/0502005 and to M. Grana, hep-th/0509003.

There are many reasons to examine such Type I theories. Of course, they are interesting in their own right since they are among the possible allowed string constructions. However, as a result of the various string dualities discussed above, such strings often represent the strong-coupling limits of the heterotic models (this is "heterotic/Type I duality", a component of S-duality). Thus, by studying Type I string models, one is often really analyzing the strong-coupling limit of a closed heterotic string model.

We cannot provide a complete discussion of such Type I models here. However, the basic ideas are simple. Unlike heterotic string models, which realize their gauge symmetries along the closed strings through Kaluza-Klein reductions from 10 or 26 dimensions (as discussed above), gauge symmetries are realized in open strings through so-called "Chan-Paton factors" which reside at the endpoints of the open strings. These are the analogues of "quarks" at the ends of the open strings, and they carry the gauge charges associated with the string states.

Nowadays these Chan-Paton factors are reinterpreted as the labels associated with D-branes, so that open strings are considered to have endpoints which are restricted to lie on D-branes. Indeed, one definition of a D-brane is that it represents a solitonic membrane-like object on which an open strings can end. A single D-brane corresponds to a $U(1)$ gauge symmetry (the corresponding photon being represented by an open string which starts and ends on the brane), while non-abelian $U(N)$ gauge symmetries are realized through stacks of N coincident D-branes. In such configurations, the non-abelian gauge bosons are realized as strings which start and end on different D-branes within the stack. The Higgs mechanism (by which certain gauge symmetries can be broken and certain corresponding gauge bosons get heavy) can be realized in this framework by separating branes within the stack; those strings which start and end on different D-branes get stretched as a result of this separation, and thus become massive as a result of the tension involved in that stretching.

In general, within such constructions, one might realize the Standard Model through an $SU(3)$ stack of D-branes and an $SU(2)$ stack of D-branes. In such a scenario, quarks (which carrying non-trivial $SU(3)$ and $U(2)$ gauge charges) would be represented by strings stretching from the $SU(3)$ stack to the $SU(2)$ stack; such states can indeed be light (or massless) if these stacks of branes intersect, and the strings lie near that intersection. Of course, gravitational physics continues to be represented by closed strings which, having no endpoints, are not tied to particular branes and can therefore propagate freely in the "bulk". In general, only those states which are neutral with respect to all gauge symmetries (such as gravitons) are permitted to wander freely in the entire volume both within and transverse to the branes. In certain constructions, other possible closed-string states might include right-handed neutrinos (which are also completely neutral with respect to all Standard-Model gauge symmetries).

In general, the requirements of spacetime supersymmetry imply that the theory contain combinations of D-branes of only certain dimensionalities; likewise, the relative positions and/or geometric intersections of these D-branes are highly constrained. There are also generally other extended objects in these theories (beyond D-branes): these include anti-Dbranes, orientifold planes, and other types of branes (such as NS branes). Anomaly cancellation considerations end up playing a huge role in determining which configurations of all of these objects are required to form self-consistent string models.

Other than these constraints, however, one has tremendous freedom in designing D-brane configurations, compactifying the theory, wrapping the D-branes around the compactification manifolds, and so forth. This is then the art of Type I model building. Because of the tremendous range of allowed D-brane configurations and dimensionalities, and because the closed-string and open-string sectors have very different properties, Type I string phenomenology turns out to be *very rich* and *unconstrained* compared to heterotic string phenomenology.

In particular, even without providing details concerning such constructions, it is possible to summarize some of the major phenomenological differences between these string models and the closed (heterotic) strings discussed above.

First, the rank of the gauge group no longer restricted to 22! Indeed, non-perturbative effecs can give rise to new gauge interactions that can increase the total rank beyond 22, and there is no bound to how large these gauge groups can become! (You can decide for yourself whether you consider this to be a good thing...)

Second, the fundamental scale of the theory ($M_{\rm string}$) is no longer tied to $M_{\rm Planck}$. The usual heterotic relation $M_{\rm string} = g_{\rm string} M_{\rm Planck}$ no longer applies to open strings. The reason is that for closed strings, both gauge forces and the gravitational force emerge together. However, for Type I strings, the gravitational force emerges from the closed-string sector, while the gauge forces typically emerge from the open-string sectors. This difference introduces an undetermined "rescaling" factor between the different sectors, and therefore allows one to "dial" $M_{\rm string}$ as we wish in such theories. [For more details, see Chapter 10 of K.R. Dienes, Phys Reports 287 (1997) 447 = hep-th/9602045, which summarizes the original proposal of Witten: E. Witten, Nucl Phys B 471, 135 (1996)]. One could conceivably dial the Type I string scale all the way down to the TeV range – see, *e.g.*, J. Lykken, PRD 54, 3693 (1996); K.R. Dienes, E. Dudas, T. Gherghetta, Nucl. Phys. B537, 47 (1999); G. Shiu, S.-H.H. Tye, Nucl. Phys. B548, 180 (1999).

This freedom to adjust the string scale and realize the Standard Model as an open string living on a brane while gravitational fields correspond to closed strings living in the bulk is the primary reason why Type I strings provide the natural realization (and inspiration) for extra-dimensional "brane-world" scenarios.

Third, for weakly coupled heterotic strings, there is only one dilaton-like field which couples to all gauge groups and matter fields in a universal way. However, in Type I theories there can generally be *multiple* dilaton-like fields.

Fourth, for Type I theories, gauge coupling unification is no longer automatic. This is a consequence of the existence of multiple dilaton-like fields. Each gauge coupling can be determined by the vev of a different dilaton field, and likewise the gauge theories living on different D-branes can experience different transverse volumes which also affect the values of their respective gauge couplings.

Fifth, in heterotic strings, there was only one anomalous $U(1)$ because there was only one dilaton to cancel this anomaly through the Green-Scwharz mechanism. However, in Type I theories there can be multiple anomalous $U(1)$'s because the presence of multiple dilatons in Type I theories implies that there can be a generalized Green-Schwarz mechanism which cancels multiple $U(1)$ anomalies.

Sixth, it turns out that whole new types of spacetime compactifications are possible. In heterotic strings, one must compactify on a so-called "Calabi-Yau" manifold if one wishes to preserve N=1 spacetime supersymmetry. (See Polchinski's textbook for a complete discussion: technically CY manifolds are six-dimensional complex manifolds with $SU(3)$ holonomy or equivalently vanishing first Chern class.) While the simple cases of tori (and orbifolds thereof) are well understood, the general full class of CY manifolds is not well understood (not even classified by mathematicians) and it is hard to perform detailed calculations of the resulting low-energy phenomenologies that emerge when heterotic strings are compactified on such spaces.

Type I string models are different. Because the matter arises locally (on branes) rather than globally (in the bulk), the compactification geometry is less constrained. For example, chirality no longer requires compactification on an orbifold, since chirality can instead emerge directly from D-brane intersections even when the compactification space is a smooth manifold.

For further discussions of these differences between the phenomenologies of open and closed strings, good references are: L.E. Ibanez, hep-th/9804236; F. Quevedo, Trieste String School Lectures, March 2002.

6.8.3. *String model-building and string phenomenology: General practice and goals*

Having discussed the different types of phenomenological features of these different types of string models, we now outline the basic way in which the string model-building game is played. Of course, the following steps are merely caricatures, with many details omitted. Nevertheless, they do indicate the rough methodology that a string phenomenologist must follow in order to claim to have a realistic string model.

The first step, as always, is to build the candidate string model itself. We have discussed how to do this in great detail in previous lectures. This is the string "model-building" aspect of string phenomenology. How one goes about doing this will depend on the particular string framework one has in mind, whether closed or open strings are involved, whether one is dealing with perturbative or non-perturbative constructions, and so forth. Each construction will carry with it its own constraints, its own techniques, and its own unique advantages and difficulties.

Once one has a particular string model in hand, one then extracts the gauge symmetry, the particle content (massless spectrum only, if one cares only about questions pertaining to observable low-energy states), and all associated charges and couplings.

The next step, if necessary (such as in heterotic strings), is to do a so-called "string vacuum shift". This is a technical step. Recall that there often exists a pseudo-anomalous $U(1)_X$ gauge symmetry. Although this is not really anomalous, it leads to an effective Fayet-Iliopoulos D-term which can break spacetime SUSY and destabilize the string vacuum. So, in order to "fix" this problem, one shifts the ground state slightly: one assigns a vev to certain moduli in the theory in order to break the $U(1)_X$ gauge symmetry and cancel the D-term. This makes the model stable again. This vev may often also break other gauge symmetries in addition to $U(1)_X$. It also can generate intermediate mass scales for various light states in the string model.

The third step is to write down an effective Lagrangian of these light fields that are derived from the string. Typically we will write something of the form

$$\mathcal{L} \;=\; \mathcal{L}_{\rm SUGRA} \;+\; \mathcal{L}_{\rm matter} \;+\; \mathcal{L}_{\rm couplings}\;. \qquad (6.8.7)$$

These different pieces are as follows.

- $\mathcal{L}_{\rm SUGRA}$: This must be appropriate for the given model in question, *e.g.*, $N = 0$ (non-susy), or $N = 1$, or Type IIA or IIB SUGRA, *etc.*
- $\mathcal{L}_{\rm matter}$: This will consist of the kinetic terms for all light fields (including an appropriate dilaton dependence).
- $\mathcal{L}_{\rm couplings}$: Here, we must include all couplings allowed by string symmetries, given the charges that these states have under both observable and hidden-sector gauge symmetries (*i.e.*, selection rules). These will include renormalizable *and* non-renormalizable couplings, where the non-renormalizable ones are suppressed by powers of the string scale $M_{\rm string}$. In principle, one should calculate the coefficients that pre-multiply these terms by explicitly evaluating the appropriate corresponding string diagrams.

All together, this is the "effective Lagrangian from the string model".

One must make sure that it is consistent with all string symmetries (*e.g.*, T-duality, S-duality, others) if you are going to ask physics questions for which those symmetries are likely to be important.

The final step is to proceed to analyze the physics of the string model by analyzing the effective field theory of the effective Lagrangian derived from the string. We treat this Lagrangian as describing the physics at the string scale, and use RGE's to pass to lower energy scales (as we would in ordinary field theory). Along the way (*i.e.*, at intermediate scales), various new features can arise. For example, although Standard Model gauge groups will hopefully stay perturbative, the *hidden-sector* gauge couplings *may*, depending on the particle content, become strong and non-perturbative at some intermediate scale. This can trigger the corresponding gauginos to condense ("gaugino condensation"). which in turn can trigger SUSY-breaking. This is indeed an elegant string-inspired but field-theoretic means of breaking SUSY at intermediate energy scales. Likewise, extra matter beyond the MSSM (with masses determined by vacuum shifting, as discussed) can decouple. Clearly, the analysis for this sep is generally very model-dependent!

Ultimately, we seek to reproduce the low-energy world at the TeV-scale — *i.e.*, we wish to reproduce the Standard Model, and then study the phenomenological implications of the extra string-inspired particles or interactions that are predicted at higher scales. For example, one might construct string GUT models (realizing standard field-theory GUT scenarios from string theory), or realize the Standard Model directly at the string scale without an intervening GUT, or...

Given this procedure as outlined, one might wonder what the goals of string phenomenology ultimately are. Is it sufficient to try to construct semi-realistic string models, or are there are other goals as well? While a conversation on this topic can easily yield as many opinions as there are people in the conversation, the following represent the personal opinions of your humble lecturer. Therefore, the reader is forewarned about the potential bias of the lecturer.

Clearly, one important and undeniable goal must be to try to construct realistic string models, *i.e.*, to see how far one can really "push" the embedding of the low-energy world into string theory, to test the extent to which one can really make string theory consistent with the real world.

Unfortunately, this is very model-dependent. Also, given the large (infinite) "moduli" space of all possible string models, it is hard (impossible?) to believe that we would really be lucky enough to stumble across the right string model (assuming one exists).

Also, although we discussed one particular method of model-construction in these lectures (the so-called "free-fermionic construction" for closed strings), its applications and scope are limited (it essentially only hits discrete points in moduli space). They are points of enhanced symmetry, so they may indeed be special, but we don't know the structure of moduli space well enough to have a feeling for whether other, more compelling points might exist. And for open strings, we have seen the possibilities are even more varied!

Therefore, an alternative goal might be to try to uncover *model-independent* phenomenological truths from string theory. For example, one might ask questions such as

- What "patterns" of low-energy phenomenology are consistent with coming from or being realized from an underlying string theory?
- What "patterns" of low-energy phenomenology can be excluded?
- What sorts of "correlations" does string theory predict between phenomenological features that would otherwise appear to be completely independent from a field theory point of view? As an example, string theory predicts correlations between gauge groups and fractional charges, *etc*. These correlations are ultimately the reflections of the deeper string symmetries (*i.e.*, worldsheet symmetries) from which all spacetime physics is ultimately derived. For further editorializing along these lines, see the comments about the string landscape at the end of this lecture. In this way, we can then ask the question:
- What *guidance* does string theory provide for answering or addressing questions of physics beyond the Standard Model?

Of course, string theory also has the potential to provide insights of a completely different nature. For example, just as field theory provides certain mechanisms for addressing long-standing questions of particle physics, string theory (viewed as a general theory of extended objects) has the potential to provide new, additional, intrinsically geometric mechanisms for solving some of these same problems. Moreover, these mechanisms may also be able to generate new approaches to solving long-statnding problems that ordinary field theories based on point particles cannot reach. Thus, string phenomenology may be able to enlarge the domain of problems that a particle physicist might hope to address, and provide new tools for this endeavor.

But finally, perhaps the most important unresolved problem within string phenomenology is to understand what selects the string vacuum. Clearly, in order to make full progress in our understanding of string theory and its low-energy phenomenological predictions, we must eventually uncover the dynamics (presumably non-perturbative or semi-perturbative, or involving a mix of perturbative and non-perturbative physics) which ultimately pushes the universe towards the true ground state of string theory, the one in which we live.

Progress along these lines will be very hard, but is very important. Perhaps insight will come (or even may be coming) from recent developments in string duality. This problem seems tied up with the whole issue of how SUSY is broken, and the cosmological constant problem, so it is likely to take some time.

6.8.4. *New/current directions in string phenomenology*

We close this final lecture with a brief discussion of three new/current directions in string phenomenology. Once again, the following list is hardly complete. However, it does capture several of the main thrusts of string phenomenology research over the past few years, and the directions which are likely to hold the attention of string phenonenologists for the forseeable future.

Large-radius compactifications / TeV-scale strings

Many of you probably consider higher-dimensional "brane world" scenarios as something separate from string theory. But in truth, much of this work is really a branch of string phenomenology: one is studying the properties of string theories in a corner of the parameter space where the compactification radii are large, or where the Standard Model is restricted to a brane (stack) as in Type I models! Indeed, the whole setup of much of this work (SM restricted to a brane, gravity propagating in the bulk, and so forth) really emerges from Type I string theories where the SM is realized through open strings (whose endpoints therefore must lie on D-branes) and the graviton is realized through closed strings (which have no endpoints and which are therefore free to wander throughout the full higher-dimensional spacetime).

Thus, when one studies issues of flavor physics or develops new higher-dimensional mechanisms for understanding hierarchies, supersymmetry breaking, proton stability, *etc*, one is really developing an understanding of the phenomenology of open strings in a particular corner of compactification parameter space! In other words, the *brane world* is nothing but a branch of string phenomenology, studied through an effective field theory approach which might ultimately emerge from an underlying (Type I) string.

In addition to Kaustubh Agashe's excellent lectures on this subject at this year's TASI school, I recommend (of course) my own TASI 2002 lectures on the brane world: K.R. Dienes, 2002 TASI Lectures: New Directions for New Dimensions: An Introduction to Kaluza-Klein Theory, Large Extra Dimensions, and the Brane World", available at http://scipp.ucsc.edu/haber/tasi_proceedings/dienes.ps.

Flux compactifications

Another line of intense research in recent years concerns the possibility of so-called *flux compactifications*. There are compactifications in which various background fluxes associated with different p-form gauge fields in the theory are actually turned on. (Previous work had always assumed that such fluxes were zero.) It turns out that turning on such fluxes has a number of important effects. For example, the constraints on the allowed compactification geometries are modified, and the extra flux contributions allow us to go beyond the simple class of Calabi-Yau compactifications.

However, the most important phenomenological aspect of such flux compactifications is that they provide a framework leading to new methods of moduli stabilization. Indeed, within the framework of flux compactifications, it has been been possible to build semi-realistic string models in which the vast majority of complex and Kähler moduli are completely frozen!

Flux compactifications thus provide a new arena in which to address the all-important issues of moduli stabilization and vacuum selection. Indeed, work of Kachru, Kallosh, Linde, and Trivedi (KKLT) has even provided a framework in which it might be possible to realize meta-stable string vacua with deSitter (dS) geometries. This is of critical importance if string theory is to make contact with cosmological evolution.

The string theory "landscape"

Finally, as we have seen repeatedly throughout these lectures, one of the most serious problems faced by practitioners of string phenomenology is the multitude of possible, self-consistent string vacua. That there exist large numbers of potential string solutions has been known since the earliest days of string theory; these result from the large numbers of possible ways in which one may choose an appropriate compactification manifold (or orbifold), an appropriate set of background fields and fluxes, and appropriate expectation values for the plethora of additional moduli to which string theories generically give rise. Although historically these string solutions were not completely stabilized, it was tacitly anticipated for many years that some unknown vacuum stabilization mechanism would ultimately lead to a unique vacuum state. Unfortunately, recent developments suggest that there continue to exist huge numbers of self-consistent string solutions (*i.e.*, string "models" or "vacua") even after stabilization. Thus, a picture emerges in which there exist huge numbers of possible string vacua, all potentially stable (or sufficiently metastable), with apparently no dynamical principle to select amongst them. Indeed, each of these potential vacua can be viewed as sitting at the local minimum of a complex terrain of possible string solutions dominated by hills and valleys. This terrain has come to be known as the "string-theory landscape".

The existence of such a landscape has tremendous practical significance because, as we have seen, the specific low-energy phenomenology that can be expected to emerge from string theory depends critically on the particular choice of vacuum state. Detailed quantities such as particle masses and mixings, and even more general quantities and structures such as the choice of gauge group, number of chiral particle generations, magnitude of the supersymmetry-breaking scale, and even the cosmological constant can be expected to vary significantly from one vacuum solution to the next. Thus, in the absence of some sort of vacuum selection principle, it is natural to tackle a secondary but perhaps more tractible question concerning whether there might exist generic string-derived correlations between different phenomenological features. In this way, one can still hope to extract phenomenological predictions from string theory.

Over the past two years, this idea has triggered a surge of activity concerning the *statistical* properties of the landscape. Investigations along these lines have focused on diverse phenomenological issues including the value of the supersymmetry-breaking scale, the value of the cosmological constant, and the preferred rank of the corresponding gauge groups, the prevalence of the Standard-Model gauge group, and possible numbers of chiral generations. Discussions of the landscape have also led to various theoretical paradigm shifts, ranging from alternative landscape-based notions of naturalness and novel cosmological inflationary scenarios to the use of anthropic arguments to constrain the set of viable string vacua. There have even been proposals for field-theoretic analogues of the string-theory landscape, as well as discussions concerning whether a landscape of sufficiently stable string vacua actually exists.

The implications of a landscape (if it exists) have been hotly debated in the string community. Undoubtedly, if the string landscape exists, it is a very rich place, full of unanticipated properties and characteristics. Nevertheless, at the very least, the possible existence of such a landscape has focused the attention of the string community on the fundamental question which has plagued string theory over the past twenty years, namely the issue of vacuum selection.

One might argue that the landscape is simply too large to permit any reasonable analysis. Indeed, one might even argue that if such a landscape exists, string theory is doomed as a predictive theory of physics, and that the answers to some of the most fundamental questions in physics might find their answers in random environmental selection (or as the result of cosmological chance).

However, it is also true that the direct examination of actual string models uncovers features and behaviors that might not otherwise be expected. Moreover, through direct enumeration, we gain valuable experience in the construction and analysis of phenomenologically viable string vacua. Finally, as string phenomenologists, we must ultimately come to terms with the landscape (if it exists). Just as in other fields ranging from astrophysics and botany all the way to zoology, the first step in the analysis of a large data set is enumeration and classification. Indeed, this is how science begins. Thus, properly interpreted, statistical landscape studies might be useful and relevant in this overall endeavor of connecting string theory to the real world.

Acknowledgments

I would like to thank the organizers of the 2006 TASI school, especially Sally Dawson, Rabi Mohapatra, and K.T. Mahanthappa, for their invitation to visit Boulder, Colorado, and deliver these lectures in such a pleasant and stimulating environment. I would also like to thank Sally and Rabi for their extreme patience waiting for the written version of these lectures to appear. But most importantly, I also wish to thank the TASI students themselves for their questions and sustained interest in these lectures. There is nothing more pleasant for a lecturer than an enthusiastic and inquisitive audience. This work was supported in part by the U.S. National Science Foundation under Grant PHY/0301998, by the U.S. Department of Energy under

Grant DE-FG02-04ER-41298, and by a Research Innovation Award from Research Corporation.

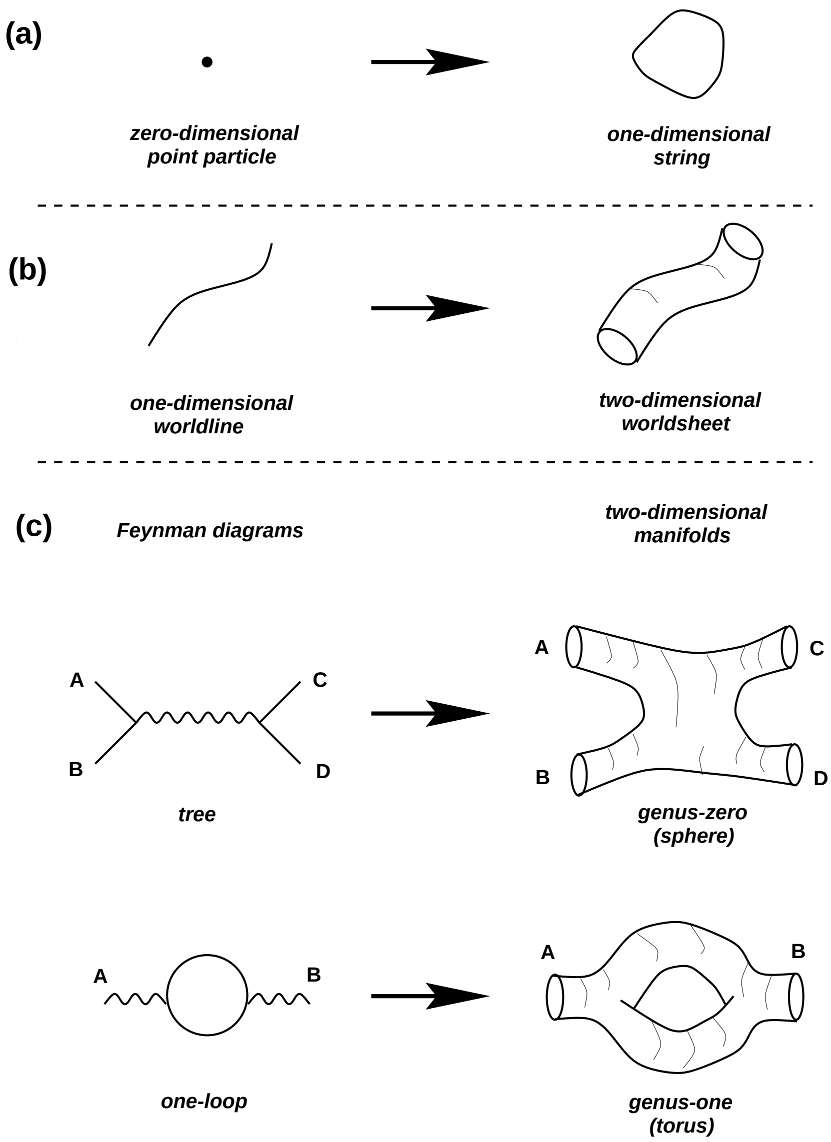

Fig. 6.3. In string theory, we replace (a) zero-dimensional elementary particles with one-dimensional strings; (b) one-dimensional worldlines with two-dimensional worldsheets; and (c) Feynman diagrams with two-dimensional manifolds. For example, tree diagrams correspond to genus-zero manifolds (spheres), and one-loop diagrams correspond to genus-one manifolds (tori).

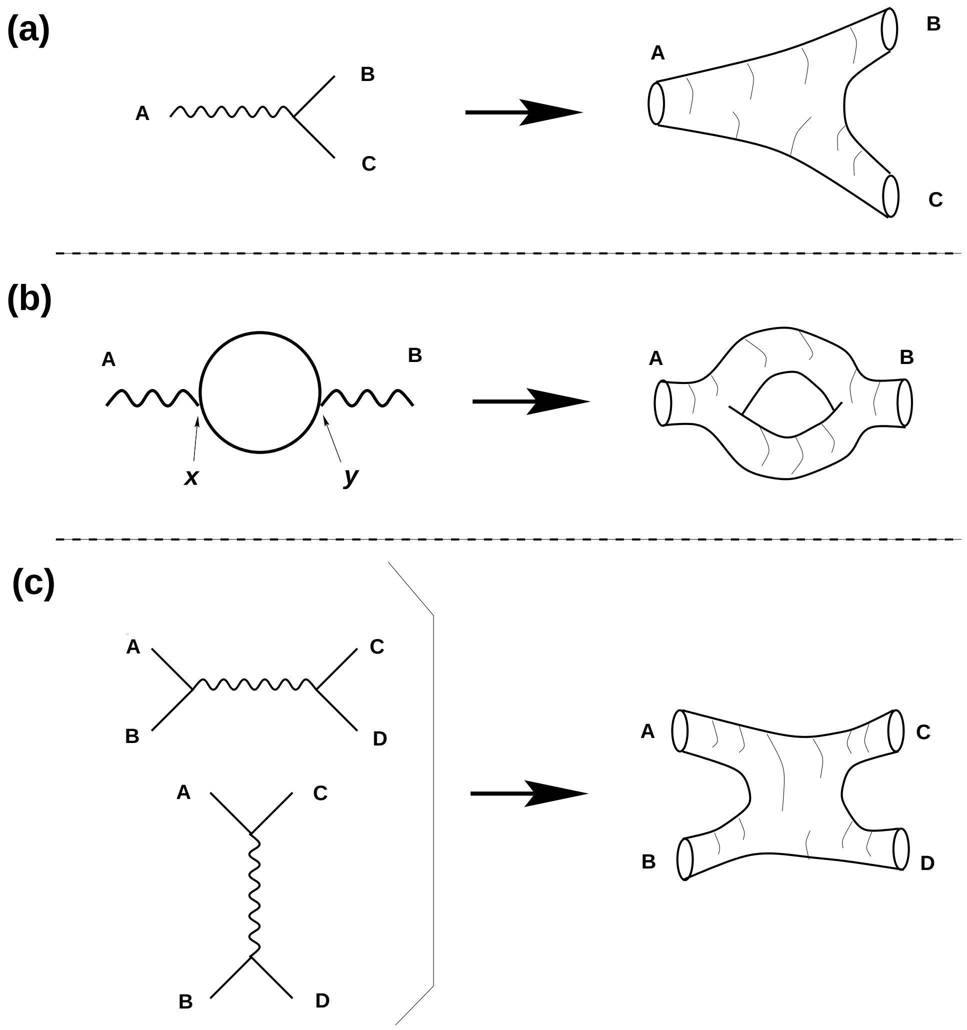

Fig. 6.4. (a) Illustration of the fact that string propagators and string vertices are not independent. (b) Illustration of the fact that string theory lacks many of the ultraviolet divergences that arise in field theory from the short-distance limit $x \to y$. (c) Illustration of the fact that one string diagram often comprises many field-theoretic diagrams.

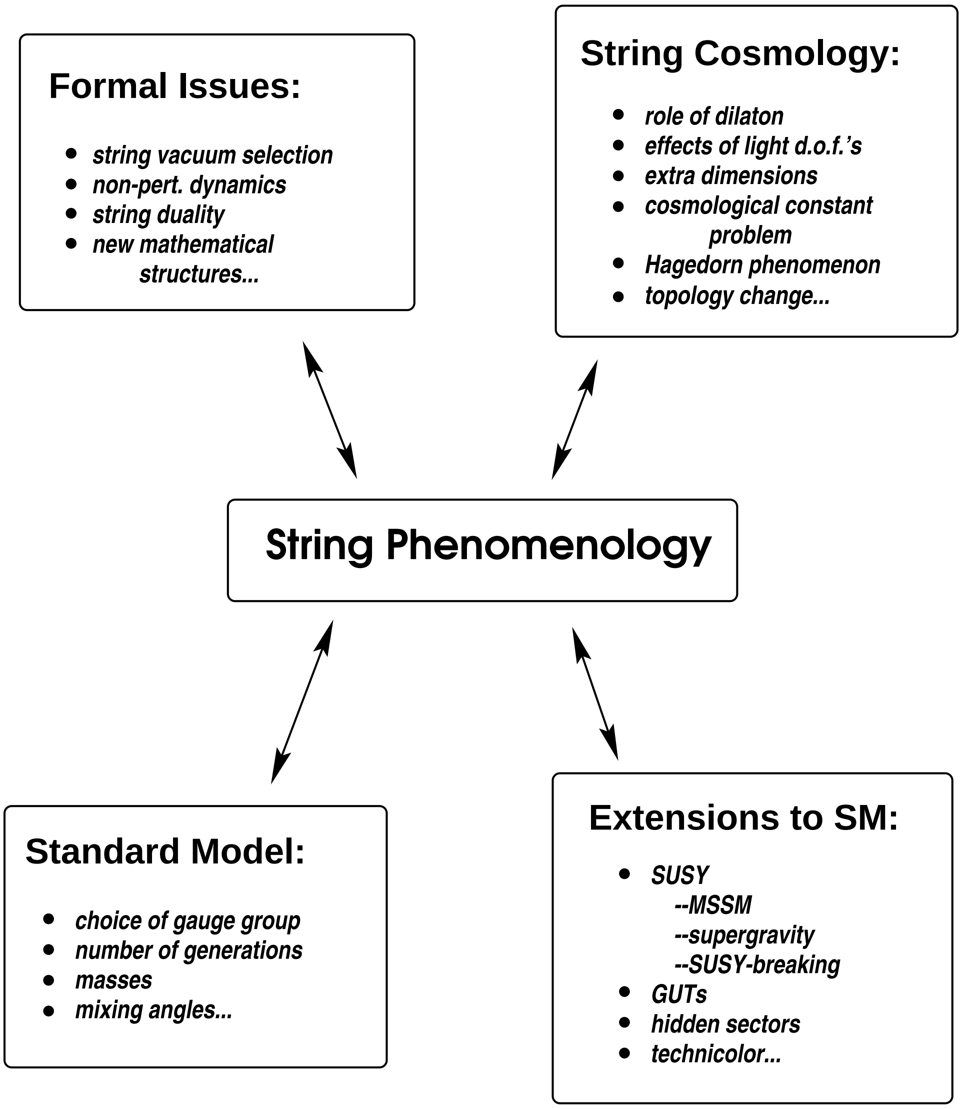

Fig. 6.5. String phenomenology is the central "meeting-ground" between Standard-Model physics, extensions to the Standard Model, formal string issues, and string cosmology.

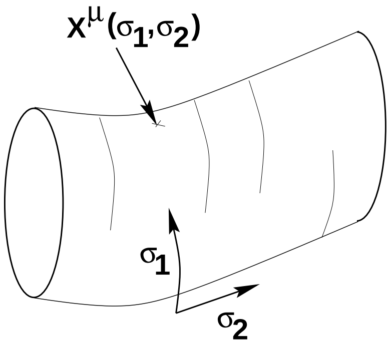

Fig. 6.6. The string worldsheet can be parametrized by two worldsheet coordinates (σ_1, σ_2). Thus, the location in the external spacetime of any point on the string worldsheet is described by a set of functions $X^\mu(\sigma_1, \sigma_2)$. It is convenient to think of σ_1 as a spacelike worldsheet coordinate, and σ_2 as a timelike worldsheet coordinate.

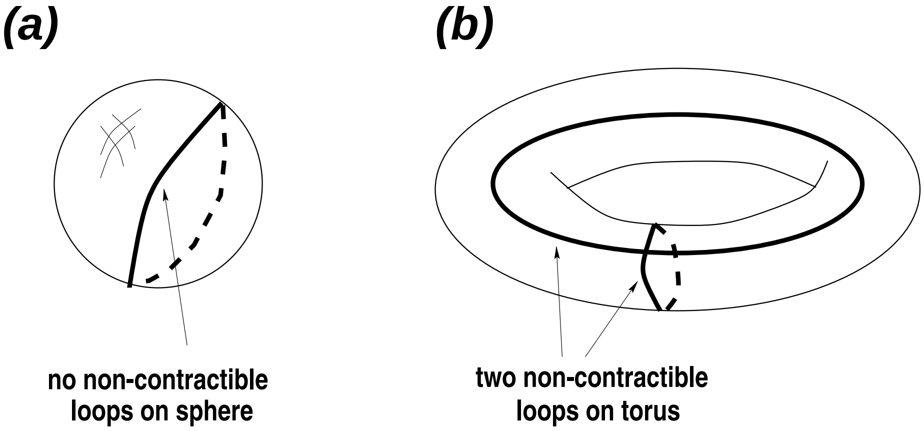

Fig. 6.7. (a) On a sphere, all closed loops can be continuously shrunk to a point. (b) On a torus, there exist two topologically distinct non-contractible loops.

Sector #	Case A	Case B	Case C
1	$N_L - N_R = \text{odd}$	$N_L - N_R = \text{odd}$ $N_L = \text{even}$	$N_L - N_R = \text{odd}$ $N_L = \text{even}$ ${}^{(8)}N_L = \text{even}$
2	$N_L - N_R = \text{odd}$	$N_L - N_R = \text{odd}$ $N_L = \text{even}$	$N_L - N_R = \text{odd}$ $N_L = \text{even}$ ${}^{(8)}N_L = \text{even}$
3	—	$N_L - N_R = \text{odd}$ $N_L = \text{even}$	$N_L - N_R = \text{odd}$ $N_L = \text{even}$ ${}^{(8)}N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$
4	—	$N_L - N_R = \text{odd}$ $N_L = \text{even}$	$N_L - N_R = \text{odd}$ $N_L = \text{even}$ ${}^{(8)}N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$
5	—	—	$N_L - N_R = \text{odd}$ $N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$ ${}^{(8)}N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$
6	—	—	$N_L - N_R = \text{odd}$ $N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$ ${}^{(8)}N_L = \text{even}$
7	—	—	$N_L - N_R = \text{odd}$ $N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$ ${}^{(8)}N_L = \text{even}$
8	—	—	$N_L - N_R = \text{odd}$ $N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$ ${}^{(8)}N_L = \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right\}$

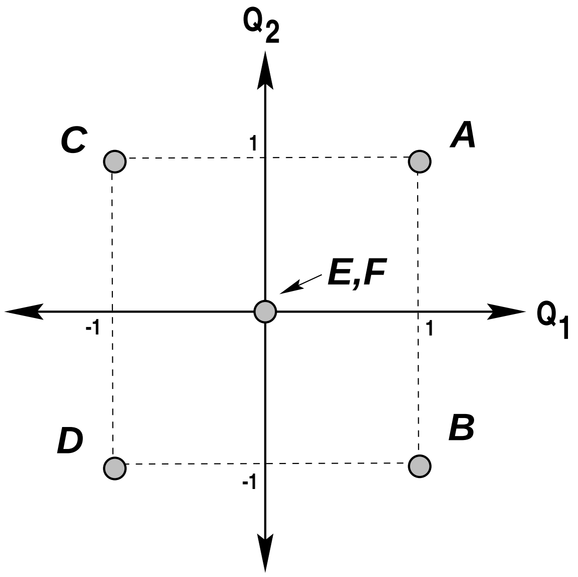

Fig. 6.8. The two-dimensional "charge lattice" associated with the six string states A through F in (6.6.5). Note that the two states E and F fill out the Cartan subalgebra of the root system. For a ten-dimensional heterotic string, the charge lattice is always sixteen-dimensional (generally implying a gauge group of rank 16), with a Cartan subalgebra consisting of sixteen gauge boson states.

gauge group	spacetime SUSY?	tachyon-free?
$SO(32)$	yes	yes
$E_8 \times E_8$	yes	yes
$SO(16) \times SO(16)$	no	yes
$SO(32)$	no	no
$SO(16) \times E_8$	no	no
$SO(8) \times SO(24)$	no	no
$(E_7)^2 \times [SU(2)]^2$	no	no
$U(16)$	no	no
E_8	no	no

Sector #	$\mathbf{V}$
1	$[(0)^4 \,\vert\, (0)^{16}]$
2	$[(\frac{1}{2})^4 \,\vert\, (\frac{1}{2})^{16}]$
3	$[(0)^4 \,\vert\, (\frac{1}{2})^{16}]$
4	$[(\frac{1}{2})^4 \,\vert\, (0)^{16}]$
5	$[(0)^4 \,\vert\, (0)^8(\frac{1}{2})^8]$
6	$[(0)^4 \,\vert\, (\frac{1}{2})^8(0)^8]$
7	$[(\frac{1}{2})^4 \,\vert\, (0)^8(\frac{1}{2})^8]$
8	$[(\frac{1}{2})^4 \,\vert\, (\frac{1}{2})^8(0)^8]$