Iterative Coding for Network Coding

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Outline

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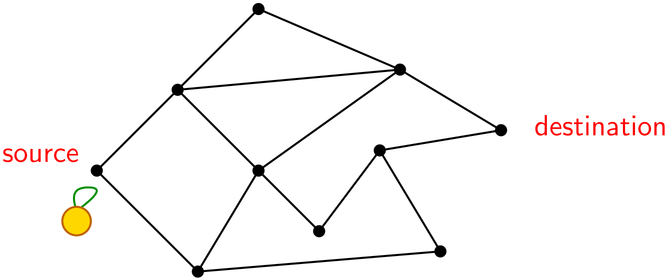
Information is not oranges

Current approach to communication networks



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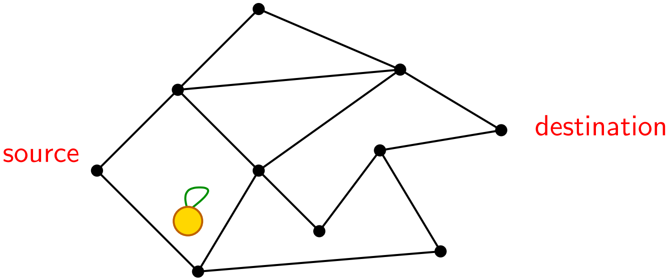

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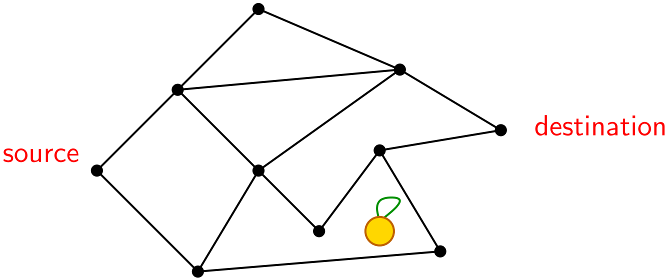

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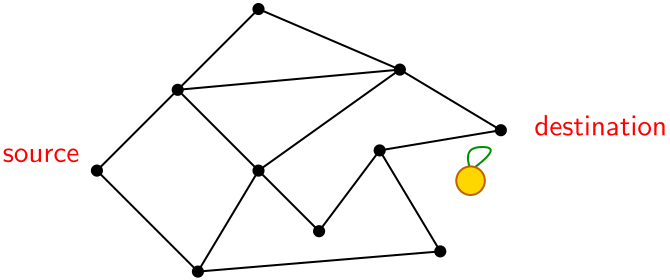
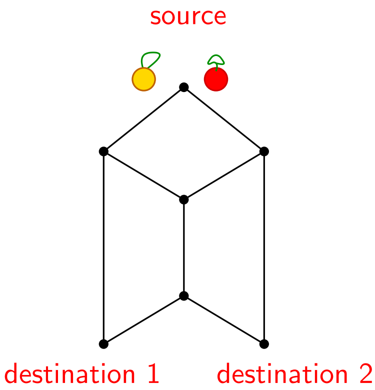
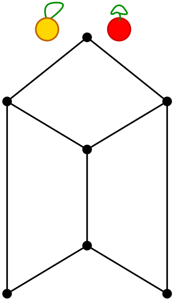


What are we losing? The butterfly example



[Ahlswede,Cai,Li,Yeung, 2000]

The butterfly example: Routing



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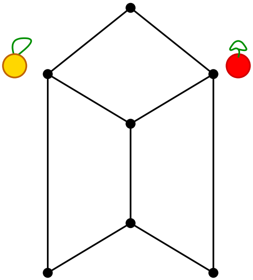

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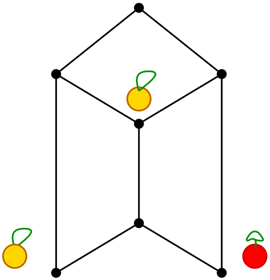

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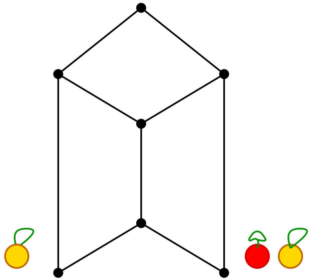

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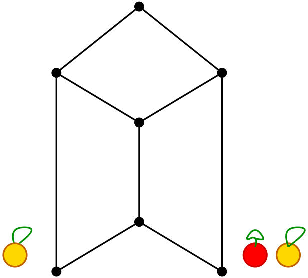
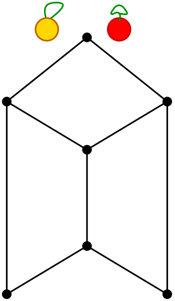


1.5 bits per cycle

The butterfly example: Network Coding



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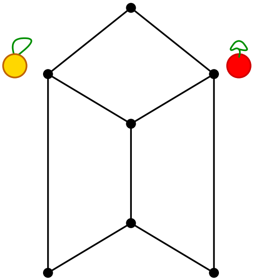

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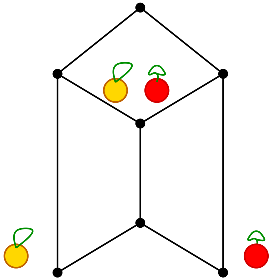

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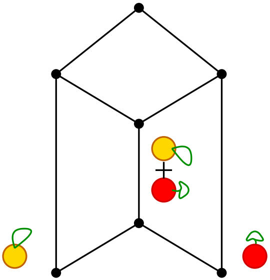

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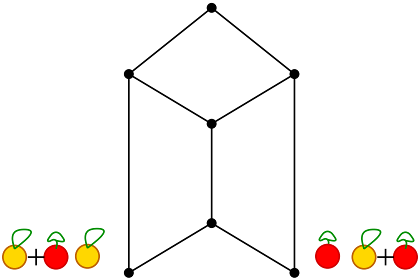

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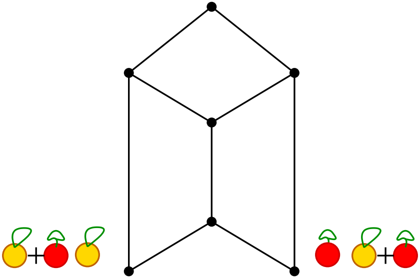

2 bits per cycle

Practical network coding

Problem

No one knows the network structure.

Source/destination do not control intermediate nodes.

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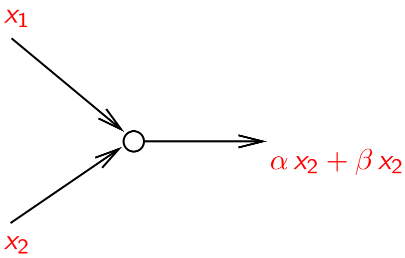
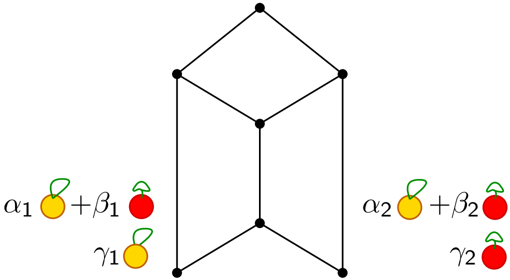
Idea



Forward random combinations

[Chou, Wu, Jain/ Ho, Kötter, Medard, Karger, Effros 2003]

How can that possibly work?



Wait a minute. . .

How am I supposed to figure out α, β, γ?

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Idea: Header

Input:

 $= [1 \quad 0 | \cdots\cdots x_1 \cdots\cdots]$

 $= [0 \quad 1 | \cdots\cdots x_2 \cdots\cdots]$

Idea: Header

Output:

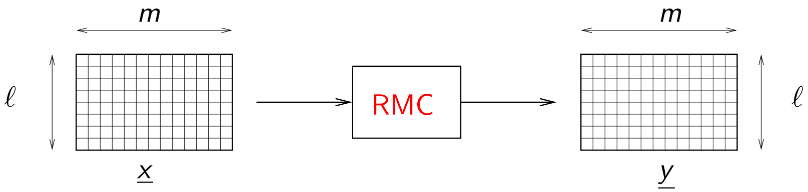
$$? \; + \; ? \; = [\alpha \quad \gamma | \cdots \alpha x_1 + \gamma x_2 \cdots]$$

$$? \; + \; ? \; = [\delta \quad \beta | \cdots \delta x_1 + \beta x_2 \cdots]$$

Noisy network coding

Rank Metric Channel (Symmetric Network Channel)

(Gabidulin, 1985)



$$\underline{y} = \underline{x} \oplus \underline{z},$$

$\underline{z}$ uniformly random with $\mathrm{rank}(\underline{z}) = \ell\omega$

Asymptotics, Rate, Capacity

$$\ell = N\lambda,\ m = N(1-\lambda),\ N \to \infty$$

$$R = \frac{\log_2 |\mathrm{Code}|}{m\ell},$$

$$C(\lambda, \omega) = \frac{1-\lambda-\omega+\lambda\omega^2}{1-\lambda}.$$

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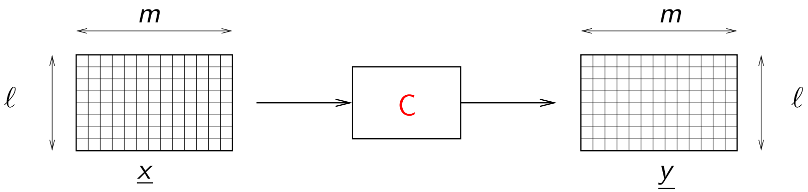
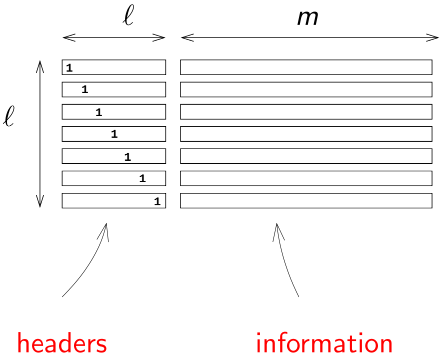
Matrix channels



$$\underline{x} = \{x_{ij}\} \in \mathbb{F}_2^{m \times \ell}$$

Network coding???

(Kötter, Kschischang, 2007)



$$\equiv \left[\ \underline{1}\ \middle|\ \ \underline{x}\ \ \right]$$

(Chou, Wu, Jain, 2003)

Reliable network

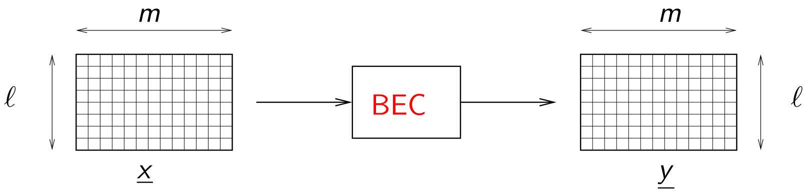
$$\left[\ \underline{1}\ \middle|\ \underline{x}\ \right] \longrightarrow \boxed{\text{NET}} \longrightarrow \left[\ G\ \middle|\ G\underline{x}\ \right] \longrightarrow \boxed{G^{-1}} \longrightarrow \left[\ \underline{1}\ \middle|\ \underline{x}\ \right]$$

Faulty network

$$\left[\ \underline{1}\ \middle|\ \underline{x}\ \right] \longrightarrow \boxed{\text{NET}} \longrightarrow \left[\ G\ \middle|\ G\underline{x}+\underline{z}'\ \right] \longrightarrow \boxed{G^{-1}} \longrightarrow \left[\ \underline{1}\ \middle|\ \underline{x}+G^{-1}\underline{z}'\ \right]$$

Binary Erasure Channel

(Elias, 1954)



$$y_{ij} = \begin{cases} x_{ij} & \text{with prob. } \epsilon, \\ * & \text{with prob. } 1-\epsilon. \end{cases}$$

independently

Asymptotics, Rate, Capacity

$\ell = N\lambda$, $m = N(1-\lambda)$, $N \to \infty$

$$R = \frac{\log_2 |\mathrm{Code}|}{m\ell},$$

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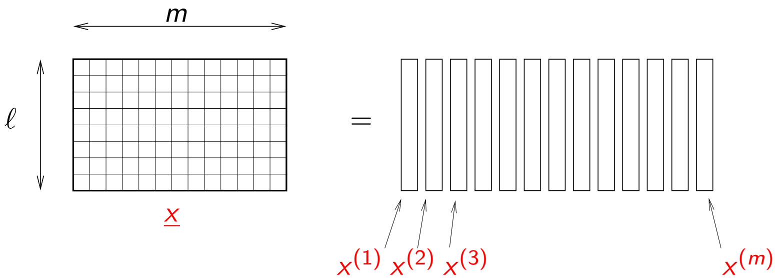
$$\begin{aligned} R &= \frac{\log_2 |\mathrm{Code}|}{m\ell}\,, \\ C(\epsilon) &= 1-\epsilon\,. \end{aligned}$$

Construction and decoding algorithm

Equivalent descripton of the channel



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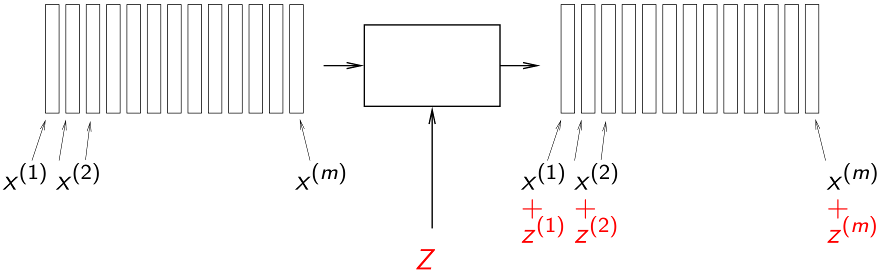
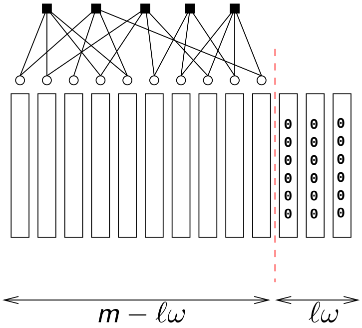


$Z \equiv$ uniformly random subspace $\subseteq \mathbb{F}_2^{\ell}$
$z^{(1)}, \ldots, z^{(m)} \in Z$

Code

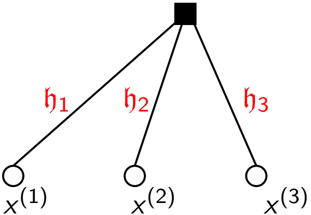


$m - \ell\omega$ 'symbols' $\longrightarrow$ LDPC

$\ell\omega$ 'symbols' $\longrightarrow$ learn the error space Z

LDPC: Check nodes

$$\mathfrak{h}_1 x^{(1)} + \mathfrak{h}_2 x^{(2)} + \mathfrak{h}_3 x^{(3)} = 0$$



'Edge labels' $\mathfrak{h}_i \in \mathbb{F}_2^{\ell \times \ell}$: $\ell \times \ell$ matrices

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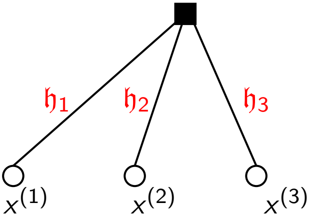

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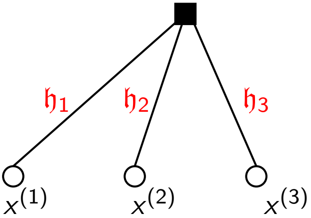

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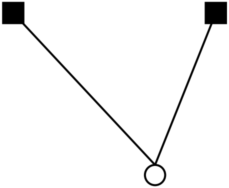
LDPC: Variable nodes



$$x^{(i)} \in \{z^{(i)} + Z\}$$

Degree $= 2$!

Message passing decoder

Messages $\longrightarrow$ Affine subspaces $V_{i\to a} \subseteq \mathbb{F}_2^{\ell}$

Operations $\longrightarrow$ Subspace intersections/sums

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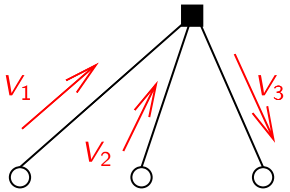

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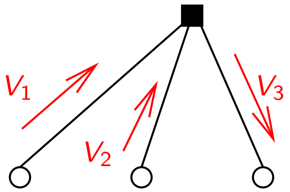

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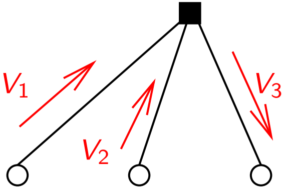

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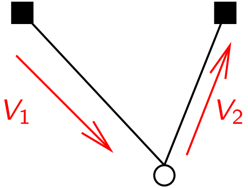

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$$V_2 = V_1 \cap \{z^{(i)} + X\}$$

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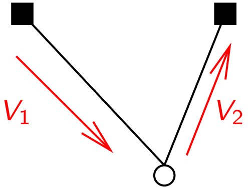

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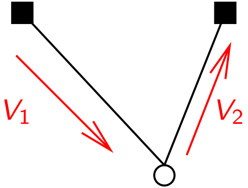

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Achieving capacity

Capacity achieving ensemble

Variable degree:

$$2$$

Check degree:

$$P_i^{(k)} = \frac{2k(k-1)}{i(i-1)(i-2)}.$$

Not the soliton!

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Theorem

If $\omega = 1/k$, then the ensemble $(2, P^{(k)})$ has rate equal to the capacity of the rank metric channel and achieves vanishing error probability under message passing decoding.

Further, it achieves vanishing error probability over the erasure channel.

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Theorem

If $\omega = 1/k$*, then the ensemble* $(2, P^{(k)})$ *has rate equal to the capacity of the rank metric channel and achieves vanishing error probability under message passing decoding.*
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Encoding complexity	$O(m\ell^2)$
Decoding complexity	$O(m\ell^2)$
Error probability	$\exp[-\Omega(\gamma^{\mathrm{iter}})]$, $\exp(-\Omega(m,\ell))$

Realistic example (from Chou et al.):

$$\ell \approx 50\,, \qquad m \approx 1700 \qquad (\text{over } \mathbb{F}_{2^8})$$

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The magic

1. Fix number of iterations.
2. For large m, message V_i uniformly random conditional on dimension.
3. For large ℓ, output dimension determined by input dimension.
4. For $\omega = 1/k$, only dimension $\ell\omega$ or 0 is possible.

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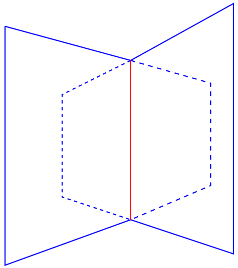

$$d_{1\cap 2} \approx \max(d_1 + d_2 - d_{\mathrm{amb}}, 0)\,.$$

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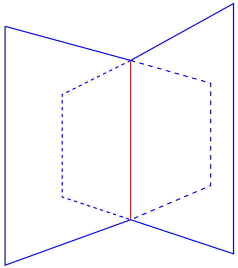

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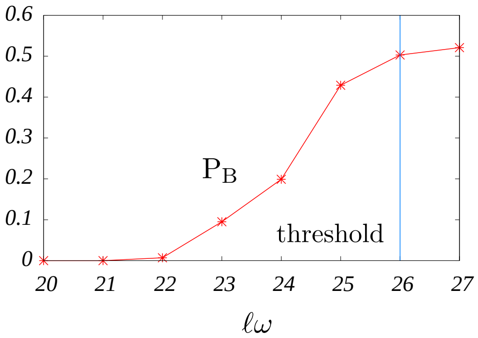
Simulations at small packet sizes

$m = 42$, $\ell = 136$ *(LP optimized degree sequence)*



Conclusion 1

q-ary LDPC's (Davey, MacKay/ Burshtein, Miller/ Rathi, Urbanke)
Mixed outcomes

Exact density evolution for large q

Capacity achieving ensembles

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Capacity achieving ensembles

Conclusion 2

LDPC codes achieve capacity (under message passing decoding) over the erasure channel (Luby et al. 97) ...

...and the rank metric channel

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