GLOBAL GRIDS FROM RECURSIVE DIAMOND SUBDIVISIONS OF THE SURFACE OF AN OCTAHEDRON OR ICOSAHEDRON

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Abstract. In recent years a number of methods have been developed for subdividing the surface of the earth to meet the needs of applications in dynamic modeling, survey sampling, and information storage and display. One set of methods uses the surfaces of Platonic solids, or regular polyhedra, as approximations to the surface of the earth. Diamond partitions are similar to recursive subdivisions of the triangular faces of either the octahedron or icosahedron. This method views the surface as either four (octahedron) or ten (icosahedron) tessellated diamonds, where each diamond is composed of two adjacent triangular faces of the figure. The method allows for a recursive partition on each diamond, creating nested sub-diamonds, that is implementable as a quadtree, including the provision for a Peano or Morton type coding system for addressing the hierarchical pattern of diamonds and their neighborhoods, and for linearizing storage. Furthermore, diamond partitions, in an aperture-4 hierarchy, provide direct access through the addressing system to the aperture-4 hierarchy of hexagons developed on the figure. Diamond partitions provide a nested hierarchy of grid cells for applications that require nesting and diamond cells have radial symmetry for those that require this property. Finally, diamond partitions can be cross-referenced with hierarchical triangle partitions used in other methods.

Key words: global grids, geometric models, sampling grids, hierarchical spatial sampling

1. Introduction

Many methods have been designed for creating networks, or global grids, of approximately equally-spaced points or approximately equally-sized areal units on the surface of the earth to respond to needs of many applications in dynamic modeling of earth processes and in collection of observational data through census or surveys. A series of papers has looked at some of the properties of these grids (White *et al.* 1992, White *et al.* 1998, Kimerling *et al.* in press). In this paper a grid system is proposed that is based on a recursive subdivision of the faces of two of the regular polyhedra, the octahedron and the icosahedron (Figure 1).

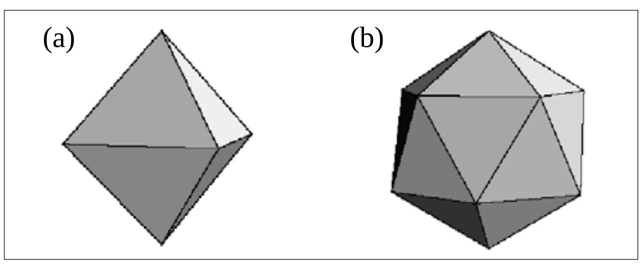

Figure 1. Octahedron (a), and Icosahedron (b).

The purpose of this paper is to present a method using diamonds as the basic tessellation units and demonstrate its properties for global grids applied to survey sampling, dynamic modeling, and storage and retrieval of global data.

2. Description of Method

This method conceives of the surfaces of the octahedron and icosahedron as composed of pairs of adjacent triangles, or diamonds, that tessellate, or cover the surface. On the polyhedra, the diamonds are bent, so to speak, across the narrow diagonal, but for data structure purposes they can be considered entire. The octahedron has four diamonds and the icosahedron ten (Figure 2).

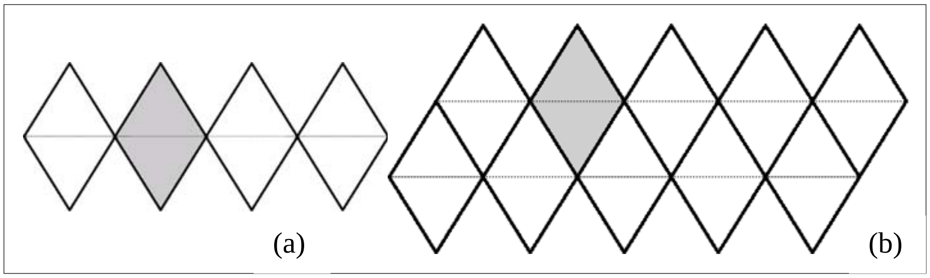

Figure 2. Unfolded diamonds for the octahedron (a), and icosahedron (b).

There is a natural hierarchy of nested diamonds that allows a scale of grid appropriate for an application. The hierarchy is completely analogous to that on a square tessellation and can be composed of four subdivisions at each lower level, of nine subdivisions, or of other factors. For one of the global diamonds, the top down subdividing process resembles that of quadtrees (Figure 3).

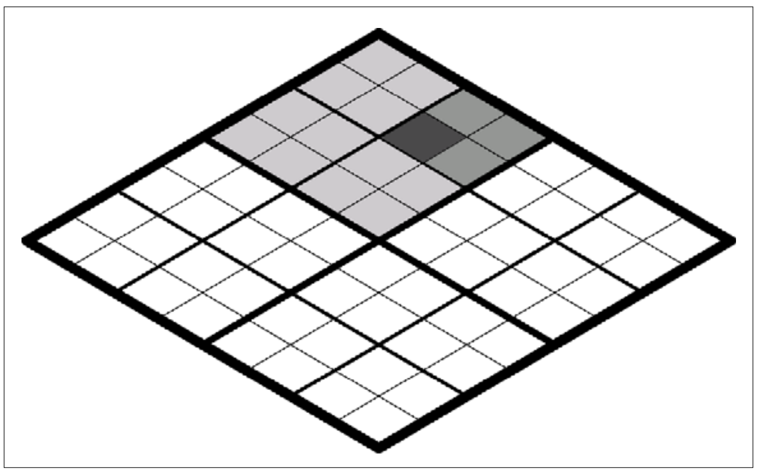

Figure 3. Hierarchical subdivision of diamonds by factors of four showing successive levels for one sub-diamond at each level in darker shades of gray, displayed in landscape aspect.

A diamond tessellation (or tiling) in the plane is an isohedral tiling, meaning a pattern with basic units that are equivalent under all of the symmetries of the tessellation, like the tessellations of squares, hexagons, or equilateral triangles (Grünbaum and Shephard 1977). Some properties of tessellations, listed in Bell *et al.* (1983), are degree of adjacency, aperture (or expansion factor between levels), circularity (the difference between smallest circumscribed and largest inscribed circles), convexity (the difference between a cell and its convex hull), orientation (the number of orientations of cells - e.g., two for triangle tessellations), limit (whether subdivisions can occur indefinitely), similarity (whether cells at all levels are the same shape), and regularity (whether cells are regular polygons). By these criteria diamond tessellations are similar to squares except that they are less circular and not regular.

An important aspect of a tessellation for many applications is ease of addressing, or numbering, the cells in a level and across levels. Finding neighbor cells in the same level or aggregating data from one level to another are two examples of processes that need a mechanism for referencing other cells. Because the diamond tessellation can be viewed, for purposes of addressing, as a skewed transformation of a square tessellation, much of the work in developing addressing systems for square tessellations is available. In particular, quadrant-recursive orderings (Mark 1990) are possible, and desirable because they simultaneously order cells in two dimensions. An example of a quadrant-recursive ordering is the Morton or Peano order (Mark 1990, Saalfeld 1998) (Figure 4a).

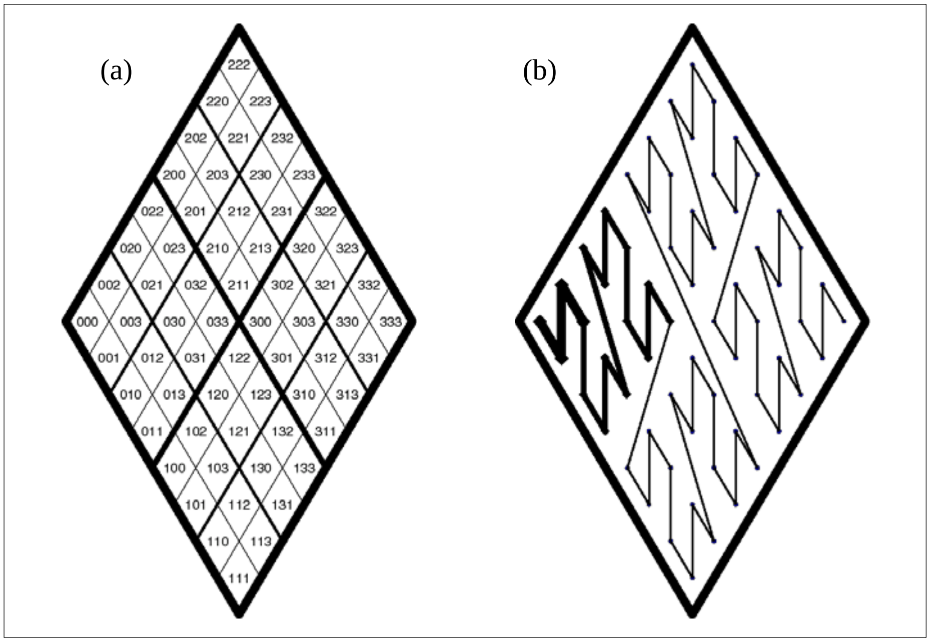

Figure 4. (a) Morton or Peano type addressing system for diamonds at the third level; (b) the linear trace of the addresses showing the recursive pattern of space filling.

The Morton order traces a zigzag course through the diamonds at any given level (Figure 4b). The binary representation of the sequence of addresses interleaves or alternates the bits for the coordinates in two dimensions placed along two adjacent sides (at an angle of 120 degrees). For the four addresses in one level, for example, the binary representations are 00, 01, 10, 11, where the second address moves one unit along one dimension (down and right), the third moves one unit along the other dimension (up and right), and the fourth moves one unit in both dimensions. If the complete set of addresses is placed in a line, the sequence represents a regular order of passing through the entire space, but more evenly moving through the two dimensions than would be the case with the equivalent of a row-major or column-major sequence.

Diamond tessellations are compatible with tessellations of triangles or hexagons. The triangular decompositions used in Fekete (1990), Dutton (1998), or Lee and Samet (1998) for example, each use different addressing systems and different projections from sphere to icosahedron or octahedron. But since diamonds are pairs of adjacent triangles, the diamond structure can be cross-referenced to any of these triangle systems by adopting the same projection. The cross-referencing of addresses would be more complicated.

There is a hexagon tessellation that corresponds to the diamond tessellation at every level of resolution (Figure 5). Hexagons, with their properties of uniform adjacency and maximal cell compactness, are desirable for applications in dynamic modeling, for example, in Murray (1967), Baumgardner and Frederickson (1985), and Thuburn (1997). The hexagon tessellation is a dual tessellation to that of the diamonds, meaning that the vertices and edge midpoints of the diamonds are the centers of the hexagons, and the vertices of the hexagons are the centers of the equilateral triangles that form the diamonds (Figure 5).

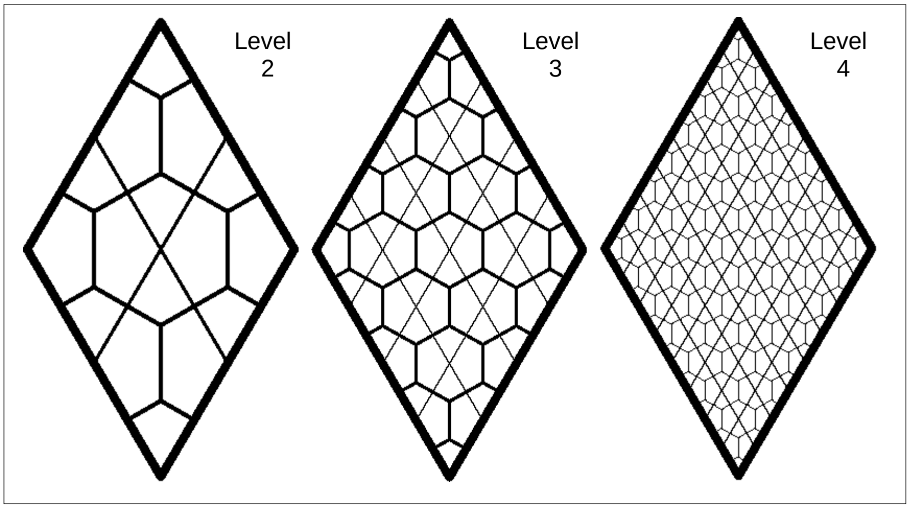

Figure 5. Hexagon tessellations corresponding to diamond tessellations at three levels.

Because of the correspondence between diamonds and hexagons, the same addressing system can be used for hexagons as for diamonds. For either the octahedron or icosahedron, there are special cases at the vertices. For the octahedron there are squares at each of the six vertices; for the icosahedron there are twelve pentagons. These squares or pentagons become increasingly small as the resolution of the grid increases. As an example, at the first level of subdivision for the octahedron there are twelve hexagons and six squares (Figure 6). An addressing assignment to correspond with the first level diamond subdivision assigns the left-most cell, a square, with zero and then proceeds to the hexagon down and right, then up, and finally down and right again in sequence as with the diamonds. If the four initial diamonds of the octahedron are numbered zero through three from left to right, and this number is prepended to the within-diamond addresses, the two-digit arrangement appears as in Figure 6. Only four hexagons, those ending in digit three, are wholly contained in a diamond; all others are shared between two diamonds. Two of the vertex squares are additional special cases and are labeled in Figure 6 as "P1" and "P2" suggesting that they play the role of "poles" in the addressing system.

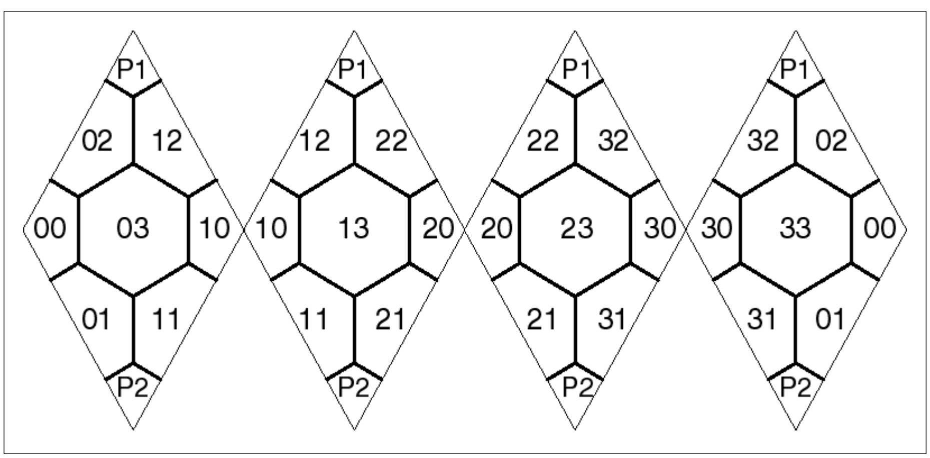

Figure 6. Addressing system for hexagons at the first level of subdivision of the octahedron.

The hexagon tessellations in Figure 5 do not nest between levels, that is, the hexagons from one level are not wholly contained in hexagons at the next higher level. Only the aperture-7 hierarchy of hexagons has the nestedness property. However at every level except the lowest (defining a "limited" tiling in the terminology of Bell *et al.* 1983), the cells of the aperture-7 hexagon hierarchy are not strict hexagons, but have an approximately hexagonal shape with a crenulated border. A fractal process can be used to generate hexagon hierarchies of aperture-7, aperture-4, and others (Arlinghaus 1993). Although the aperture-7 hierarchy would be of interest because of nesting, it is conjectured that it is not possible to tessellate the aperture-7 hierarchy of hexagons over the faces of the octahedron or icosahedron (or tetrahedron) such that the only singularities occur at the vertices. Nevertheless, the hierarchy of interest for relating to the aperture-4 diamond hierarchy is that of aperture-4. In this hierarchy a hexagon at a higher level is composed of one whole and six half hexagons from the next lower level (Figure 7a).

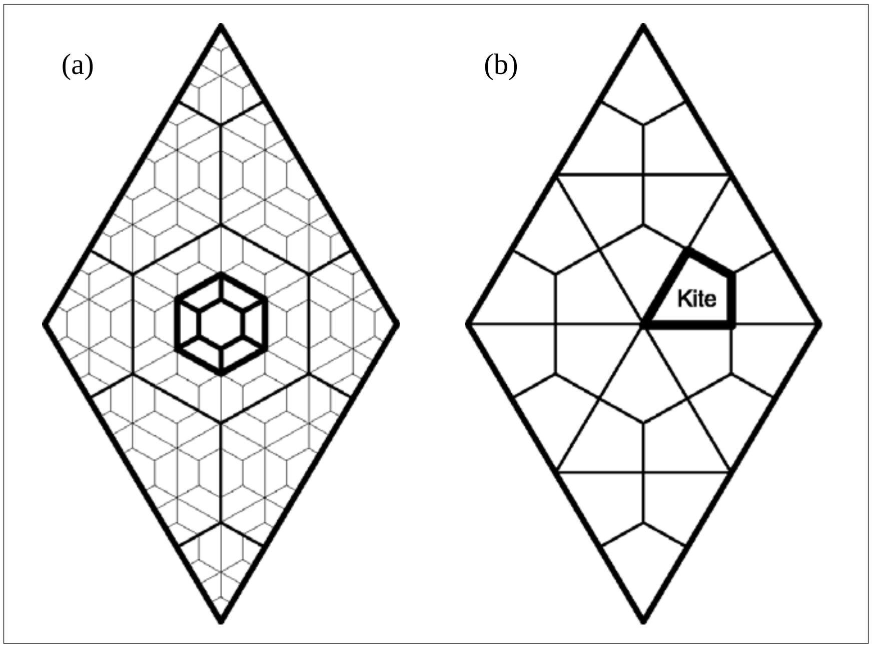

Figure 7. (a) The aperture-4 hexagon hierarchy corresponding to the diamond hierarchy, highlighting one hexagon composed of one whole and six half-hexagons. (b) The greatest common unit of intersection of hexagons, triangles, and diamonds at a given level of hierarchy is the "kite".

In addition to the hexagon hierarchy corresponding to diamonds, there is the triangle hierarchy used in several other grid systems cited above. In order to move data from one of these systems, triangles, hexagons, or diamonds, to another, one possible geometry would be that of the largest unit that is contained in all three (mislabeled "least common geographical unit" at one time, see Langran and Chrisman 1988). This unit has been called the "kite" (T. Olsen personal communication) (Figure 7b). Since kites do not nest, they would be best used for transfers of data within rather than across levels.

3. Applications

One major motivation for developments in global grids has been sampling and monitoring of natural resources over the globe (White *et al.* 1992, Olsen *et al.* 1998). Stevens (1997) described several properties of grid-based sampling designs that diamonds satisfy. Stevens showed that if the basic tessellation unit has radial symmetry about a center point and translation congruence then sampling designs can meet the criteria of the Horvitz-Thompson theorem for continuous populations, thus allowing valid design-based inference. Radial symmetry means that any point in the tessellation cell has a corresponding point at the same distance from the center in the opposite direction. Triangles, for example, do not meet this condition. Translation congruence means that the tessellation cells all have the same size, shape, and orientation.

Stevens also described what he called a multi-density, nested-randomized-tessellation-stratified sampling design for use in certain complex monitoring situations. An important aspect of this design is a hierarchical randomization process whereby randomization occurs at each level of a recursively hierarchical grid system. Stevens illustrated this process using the generalized balanced ternary addressing system for hexagons developed by Gibson and Lucas (1982). The quadrant-recursive addressing system for diamonds allows this type of design as well (Figure 8a). Another aspect of Stevens' design is local randomization within each tessellation cell (Figure 8b).

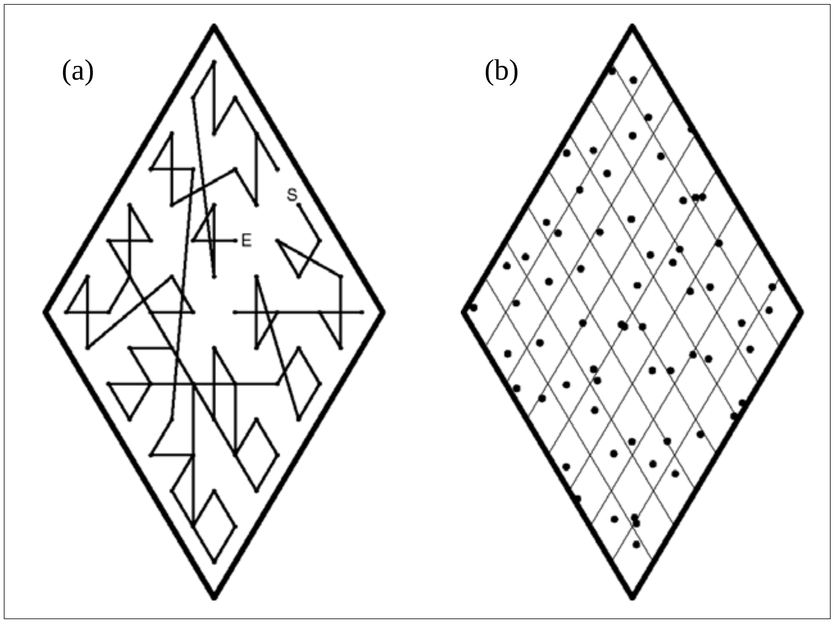

Figure 8. (a) A path through the diamonds at level three after their original order (see Figure 4b) has been hierarchically randomized, that is, the addresses for each diamond in each group of four at each level have been permuted randomly; "S" indicates the start of the path and "E" indicates the end. (b) Tessellation-stratified randomization, that is, local randomization of sampling points within each third level diamond.

Two examples illustrate the use of diamonds in surveying and analyzing natural resources. The first application is a proposed sample design for forests of North America for which the desired sample size was about 220 points across the continent (Figure 9a). The second example is a grid of 10,000 km^2 hexagons (illustrated at 100,000 km^2) for assessing species diversity of birds and mammals across Canada and North America (Figure 9b).

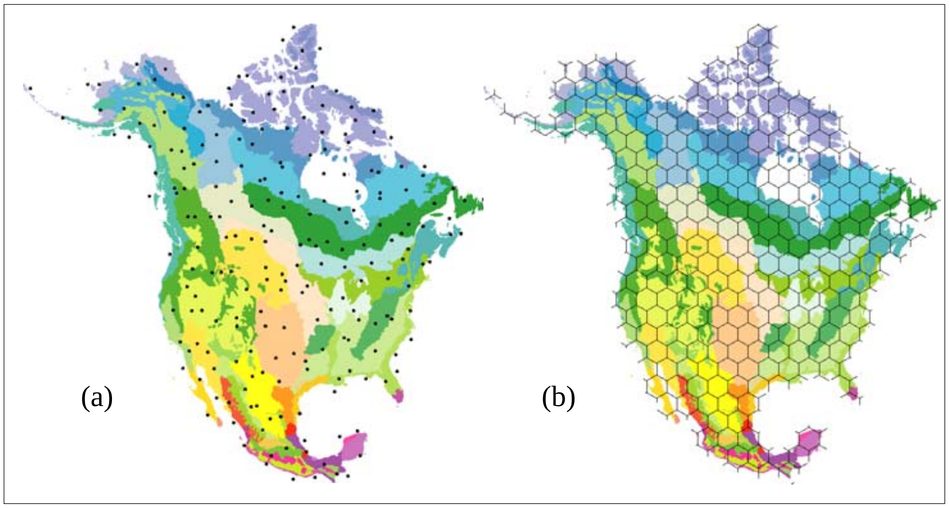

Figure 9. (a) Proposed sample design of about 220 points for a North American forest assessment. (b) Hexagon grid (here 100,000 km^2 cells) for assessing bird and mammal diversity in Canada and North America. Background for both maps are the level two ecoregions published by the Commission for Environmental Cooperation (a NAFTA organization) (1997).

Global grids based on the hexagonal tessellations that are dual to triangular decompositions of a spherical icosahedron have been used in modeling fluid dynamics of the atmosphere (Thuburn 1997). Heikes and Randall (1995) noted that simulations using icosahedron-based hexagon grids in polar orientation evolved dynamics that were asymmetric across the equator, possibly because the grids are not so symmetric. Better performance was obtained from a symmetric grid. To meet the symmetry criterion, Sahr and others (Sahr and White 1998, Kimerling *et al.* in press) have proposed the ISEA (Icosahedral Snyder Equal Area) grid whose orientation achieves equatorial symmetry by placing the midpoints of two edges at the North and South Poles. The ISEA grid can be subdivided with diamonds since it is derived from the icosahedron.

Diamond subdivisions of the icosahedron or octahedron are not dependent on the method of mapping from the surface of the polyhedral solid to the sphere or spheroid. Therefore this method may be used with an equal area grid or one optimized for performance based on other criteria (White *et al.* 1998). Furthermore, because they are nested, diamond subdivisions provide for a Kalman-filter type of prediction model for data aggregation and disaggregation between levels, as proposed by Huang and Cressie (1997).

Since diamond tessellations can be regarded as skewed square tessellations, many methods developed for quadtrees may be applied (see Samet 1984 for a review, and Lee and Samet 1998 for an application of quadtrees to triangle decompositions of an icosahedron). However, any method that depends on the metric distances between neighbors may not work, of course. Further work on diamond subdivisions should develop methods for neighbor searching, distance finding, and range searches in order to provide storage, retrieval, and geometric operations for data stored in a system based on diamonds.

4. Conclusions

The octahedron and icosahedron have been used in a number of global grid applications because they provide convenient models for the translation to the sphere of grids developed in the planes of their faces. The octahedron has the advantage that the faces can be oriented on the sphere to correspond with the octants of the spherical coordinates of latitude and longitude, and thus provides a familiar orientation. For example, one face can be placed with vertices at (1) the North Pole, (2) 0° longitude, 0° latitude, and (3) 90° E longitude, 0° latitude. The icosahedron has the advantage that it has the smallest size faces of the regular polyhedra and that, therefore, the shape and area distortions induced by mapping the faces to the sphere will, in general, be minimized (White *et al.* 1998).

Conceiving the faces of these geometric models as joined in pairs to form diamonds results in a recursively defined hierarchical grid system that has several advantages. First, diamond geometry is simpler than either hexagons or triangles, the two cell shapes that have been used in most grids based on the octahedron or icosahedron. The diamond hierarchy is nested, unlike the aperture-3 or aperture-4 hexagon hierarchies, and tessellates the spherical model, unlike the aperture-7 hexagon hierarchy. The diamond hierarchy has cells with radial symmetry and translation congruence, unlike triangles. Finally, data associated with diamonds may, in any case, be transferred to the hexagon or triangle tessellations corresponding to diamonds by using statistical prediction models operating on the mutual units of intersection, the kites. Thus the diamond system should perhaps be best seen as a multi-tessellation system where any of the three cell shapes can be used as appropriate and data can then be transferred to the other tessellations for other applications.

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