Transverse Coupling Measurement using SVD Modes from Beam Histories*

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Abstract

In this report we investigate the measurement of local transverse coupling from turn-by-turn data measured at a large number of beam position monitors. We focus on a direct measurement of coupled lattice functions using the singular value decomposition (SVD) modes and explore the accuracy of this method.

INTRODUCTION

The coupled betatron oscillations become decoupled in the two eigenmode planes (labelled with a and b), in which beam motion can be described by the action $J_{a,b}$ and angle $\phi_{a,b}$ variables, together with the beta functions $\beta_{a,b}$ and phase advances $\psi_{a,b}$. The oscillation observed in the horizontal plane is the sum of the contributions x_a and x_b from these modes, with [1]

$$x_a = \sqrt{2J_a\beta_a}\gamma\cos(\phi_a+\psi_a), \quad (1)$$

$$x_b = \sqrt{2J_b\beta_a}c_b\cos(\phi_b+\psi_b+\Delta\psi_b) \quad (2)$$

where γ, c_b, and $\Delta\psi_b$ relate to the coupling matrix $\bar{C}$ by $\gamma = \sqrt{1-\det\bar{C}}$, $c_b = \sqrt{\bar{C}_{11}^2+\bar{C}_{12}^2}$, and $\Delta\psi_b = \arctan(\bar{C}_{12}/\bar{C}_{11})$. The oscillation in the vertical plane can be similarly expressed by switching x to y and a to b, and with $c_a = \sqrt{\bar{C}_{22}^2+\bar{C}_{12}^2}$ and $\Delta\psi_a = -\arctan(\bar{C}_{12}/\bar{C}_{22})$.

The coupling is completely determined by the coupling matrix $\bar{C}$. There are several ways to measure the elements of $\bar{C}$ using turn-by-turn beam histories at BPMs. Here we describe a method using the untangled SVD modes in Model-Independent Analysis (MIA) and explore the usefulness of this method via simulations.

THE METHOD

Extraction of Coupled Betatron Modes

Let $B_{P\times M}$ be the data matrix whose columns contain P-turn beam histories at M BPMs in a transverse plane. A singular value decomposition of B yields

$$B = \hat{U}S\hat{V}^T = \sum_{\text{modes}} \sigma_i u_i v_i^T, \quad (3)$$

where $\hat{U}_{P\times P} = [u_1,\cdots,u_P]$ and $\hat{V}_{M\times M} = [v_1,\cdots,v_M]$ are orthonormal matrices comprising the temporal and spatial eigenvectors, and $S_{P\times M}$ is a diagonal matrix with non-negative singular values σ_i along the upper diagonal. When beam motion is dominated by the coupled betatron oscillations, there are typically four SVD modes associated with this motion: two of them are dominated by the eigenmode "a" with tune ν_a and the other two by the eigenmode "b" with tune ν_b. In general, both tunes can show up in the temporal vectors of the orthonormal SVD modes, i.e., SVD modes are a mixture of the physical eigenmodes. It can be shown [2] that such a mixing can be untangled by a 4×4 rotation matrix O such that the rotated temporal vectors $\hat{U}O$ contain the normal coordinates of the "a" and the "b" eigenmodes, i.e.,

$$\hat{U}O = \frac{1}{\sqrt{P}}\begin{pmatrix} \cdots & \sqrt{2J_a/\bar{J}_a}\cos(\phi_a+\psi_a^0) & \cdots \\ \cdots & -\sqrt{2J_a/\bar{J}_a}\sin(\phi_a+\psi_a^0) & \cdots \\ \cdots & \sqrt{2J_b/\bar{J}_b}\cos(\phi_b+\tilde{\psi}_b^0) & \cdots \\ \cdots & -\sqrt{2J_b/\bar{J}_b}\sin(\phi_b+\tilde{\psi}_b^0) & \cdots \end{pmatrix}^T, \quad (4)$$

and the rotated spatial vectors $O^TS\hat{V}^T$ read

$$O^TS\hat{V}^T = \begin{pmatrix} \cdots & \sqrt{\bar{J}_a\beta_a}\gamma\cos(\psi_a-\psi_a^0) & \cdots \\ \cdots & \sqrt{\bar{J}_a\beta_a}\gamma\sin(\psi_a-\psi_a^0) & \cdots \\ \cdots & \sqrt{\bar{J}_b\beta_a}c_b\cos(\psi_b+\Delta\psi_b-\tilde{\psi}_b^0) & \cdots \\ \cdots & \sqrt{\bar{J}_b\beta_a}c_b\sin(\psi_b+\Delta\psi_b-\tilde{\psi}_b^0) & \cdots \end{pmatrix}. \quad (5)$$

Here $\bar{J}_{a,b}$ are the ensemble average of the turn-by-turn action $J_{a,b}$, which may not be constant due, for example, to damping. $\psi_{a,b}^0$ and $\tilde{\psi}_{a,b}^0$ are unknown phase constants to ensure orthogonality of the singular vectors. The above expressions are for the transverse plane dominated by the "a" mode; hereafter we assume it is the x-plane and label the matrices B, $\hat{U}$, S, $\hat{V}$, as well as O with a subscript x. Expressions for the y-plane can be obtained simply by replacing x with y and a with b.

Note that if $J_{a,b}$ are constants and $\phi_{a,b}$ advances by the tune $\nu_{a,b}$ every turn, the right-hand side of Eq. (5) can be obtained directly from harmonic analysis of x by computing the summation $\frac{\sqrt{2}}{P}\sum_p f_p\, x_p$ at each BPM with $f = \cos(2\pi p\,\nu_a)$, $-\sin(2\pi p\,\nu_a)$, $\cos(2\pi p\,\nu_b)$, and $-\sin(2\pi p\,\nu_b)$, respectively. This is the commonly used method.

Determination of Coupling Matrix Elements

From the spatial vectors in $O_x^TS_x\hat{V}_x^T$, it is easy to compute the amplitude from the "a"-mode $A_a^2 \equiv \bar{J}_a\beta_a\gamma^2$ and the amplitude coupled from the "b"-mode $\tilde{A}_b^2 \equiv \bar{J}_b\beta_a c_b^2$ at all BPMs by taking the square sums of the first two and the last two vectors. The phase advance $\psi_a - \psi_a^0$ can be obtained from the arctangent of the ratio of the first two vectors. The uncertainty due to the inversion of the triangular function complicates the phase determination but can usually be resolved by referencing to some estimated values of phase advance. However, such uncertainty makes it

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difficult to compute $\psi_b + \Delta\psi_b - \tilde{\psi}_b^0$ with the same technique because $\Delta\psi_b$ may take any value even with a little coupling from machine errors. From the spatial vectors in $O_y^T S_y \hat{V}_y^T$, $A_b^2 \equiv \bar{J}_b\beta_b\gamma^2$ and $\tilde{A}_a^2 \equiv \bar{J}_a\beta_b c_a^2$ can be similarly computed, and so can the phase advance $\psi_b - \psi_b^0$.

If all the BPMs can read both x and y positions, it is well known that three of the four elements of the coupling matrix $\bar{C}$ can be determined. Using the measured coupled betatron modes, this can also be done with the expressions

$$\frac{\bar{C}_{12}}{\gamma} = \mathrm{sgn}(\sin\Delta\psi_a)\sqrt{\frac{\tilde{A}_a\tilde{A}_b}{A_a A_b}\sin\Delta\psi_a\sin\Delta\psi_b} \tag{6}$$

and $\frac{\bar{C}_{11}}{\gamma} = \frac{\bar{C}_{12}}{\gamma}\cot\Delta\psi_b$ and $\frac{\bar{C}_{22}}{\gamma} = -\frac{\bar{C}_{12}}{\gamma}\cot\Delta\psi_a$, where $\gamma = 1$ to the first order of $\bar{C}$ elements. Before getting into the calculation of those $\Delta\psi$ related terms, let us make a few comments here. Everything on the right-hand sides of these expressions can be derived directly from measured SVD modes, i.e., these three elements are directly measurable. However, since $\bar{C}_{21}$ is not involved in the beam positions $x_{a,b}$ and $y_{a,b}$, it can not be measured directly unless the exact transfer matrices between BPMs are known such that one can determine the slopes in addition to the positions. One nice feature to note is that the four amplitudes depend on the BPM gains, but their ratio in Eq. (6) as well as the computed $\bar{C}$ elements do not, thanks to cancellation of the gains. Also note that, in principle, the ratio of the average action can be measured by $\bar{J}_a/\bar{J}_b = A_a\tilde{A}_a\sin\Delta\psi_a/A_b\tilde{A}_b\sin\Delta\psi_b$. Unfortunately, since it is the ratio of two small quantities, this relation is poorly conditioned for numerical computation and does not provide an accurate measure of the ratio.

To compute $\sin\Delta\psi_b$, $\cos\Delta\psi_b$, $\sin\Delta\psi_a$, and $\cos\Delta\psi_a$, we normalize all the spatial vectors in $O^T S\hat{V}^T$ (the matrix form is used for implementation in vector-oriented programs) to get

$$R_a = \begin{bmatrix} \cos(\psi_a - \psi_a^0) & \cdots \\ \sin(\psi_a - \psi_a^0) & \cdots \end{bmatrix}_{2\times M} \quad \text{and} \tag{7}$$

$$\tilde{R}_b = \begin{bmatrix} \cos(\Delta\psi_b + \psi_b - \tilde{\psi}_a^0) & \cdots \\ \sin(\Delta\psi_b + \psi_b - \tilde{\psi}_a^0) & \cdots \end{bmatrix}_{2\times M} \tag{8}$$

as well as similar expressions for R_b and $\tilde{R}_a$ (by switching a and b). From these vectors we have

$$\mathrm{diag}(R_b^T\tilde{R}_b) = \begin{bmatrix} \cos(\Delta\psi_b + \psi_b^0 - \tilde{\psi}_b^0) & \cdots \end{bmatrix}, \tag{9}$$

$$\mathrm{diag}(R_b^T J\tilde{R}_b) = \begin{bmatrix} \sin(\Delta\psi_b + \psi_b^0 - \tilde{\psi}_b^0) & \cdots \end{bmatrix}, \tag{10}$$

and similar expressions for $\Delta\psi_a$. Here $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. To further eliminate $\psi_a^0 - \tilde{\psi}_a^0$, we use the temporal vectors from both the x and the y SVD modes to obtain a 2×2 rotation matrix $R(\psi_a^0 - \tilde{\psi}_a^0)$ as

$$R(\psi_a^0 - \tilde{\psi}_a^0) = \left(\hat{U}_x O_x\right)_{1:2}^T \left(\hat{U}_y O_y\right)_{3:4}, \tag{11}$$

where the subscript $_{1:2}$ means taking the first two vectors. Note that we have assumed the "a" mode dominates x motion, thus it is associated with the first two singular vectors of $\hat{U}_x$. Similarly, $R(\psi_b^0 - \tilde{\psi}_b^0)$ can be obtained by switching a to b and x to y. Now we can compute

$$\begin{bmatrix} \cos\Delta\psi_b & \cdots \\ \sin\Delta\psi_b & \cdots \end{bmatrix} = R(\delta\psi_b^0)\begin{bmatrix} \cos(\Delta\psi_b + \delta\psi_b^0) & \cdots \\ \sin(\Delta\psi_b + \delta\psi_b^0) & \cdots \end{bmatrix} \tag{12}$$

where $\delta\psi_b^0 = \psi_b^0 - \tilde{\psi}_b^0$. Similar expression holds for $\Delta\psi_a$.

Using the Concatenated Data Matrix

Instead of analyzing B_x and B_y separately as described above, we can also work with the concatenated data matrix $B_{P\times 2M} = [B_x, B_y]$. The temporal vectors of the SVD modes are still given by Eq. (4) but with a different set of phase constants, say, $\bar{\psi}_a^0$ and $\bar{\psi}_b^0$. (We assume the "a" eigenmode results in larger singular values and thus are the first two modes.) The spatial vectors in $O^T S\hat{V}^T$ read

$$\begin{bmatrix} A_a\cos(\psi_a - \bar{\psi}_a^0) & \vdots & \tilde{A}_a\cos(\psi_a + \Delta\psi_a - \bar{\psi}_a^0) \\ A_a\sin(\psi_a - \bar{\psi}_a^0) & \vdots & \tilde{A}_a\sin(\psi_a + \Delta\psi_a - \bar{\psi}_a^0) \\ \tilde{A}_b\cos(\psi_b + \Delta\psi_b - \bar{\psi}_b^0) & \vdots & A_b\cos(\psi_b - \bar{\psi}_b^0) \\ \tilde{A}_b\sin(\psi_b + \Delta\psi_b - \bar{\psi}_b^0) & \vdots & A_b\sin(\psi_b - \bar{\psi}_b^0) \end{bmatrix} \tag{13}$$

where the first (last) M columns correspond to x (y). Again, the square sums of the first and the last two row vectors yield A_a^2, $\tilde{A}_a^2$, $\tilde{A}_b^2$, and A_b^2. Normalizing the spatial vectors by these amplitudes gives

$$\begin{bmatrix} \cos(\psi_a - \bar{\psi}_a^0) & \vdots & \cos(\Delta\psi_a + \psi_a - \bar{\psi}_a^0) \\ \sin(\psi_a - \bar{\psi}_a^0) & \vdots & \sin(\Delta\psi_a + \psi_a - \bar{\psi}_a^0) \\ \cos(\Delta\psi_b + \psi_b - \bar{\psi}_b^0) & \vdots & \cos(\psi_b - \bar{\psi}_b^0) \\ \sin(\Delta\psi_b + \psi_b - \bar{\psi}_b^0) & \vdots & \sin(\psi_b - \bar{\psi}_b^0) \end{bmatrix}, \tag{14}$$

from which it is easy to extract the sine and cosine of $\Delta\psi_b$ and $\Delta\psi_a$ as in Eqs. (9, 10), without the complication due to the different phase constants from separate SVD analysis.

A major advantage of using a concatenated data matrix is that the small coupling oscillations become part of the large betatron modes, and thus it is easy to identify them. When analyzed separately, the weak coupling modes could be too close to the noise floor.

SIMULATIONS

To study the effectiveness of this new MIA-based method, simulations were carried out using a simple ring consisting of 80 FODO cells. The lattice is constructed and single-particle tracking was performed using MADX [3] to generate turn-by-turn data at a large number of BPM locations. Three skew quadrupoles of different strengths were introduced at arbitrary locations in the lattice to add coupling, and matching was done in MADX to get the desired

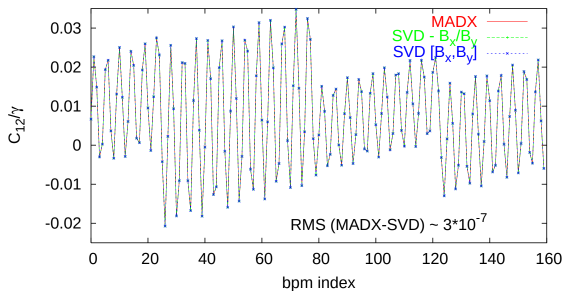

Figure 1: C_{12}/γ for FODO lattice with 80 cells and three skew quadrupoles.

tunes. The coupling matrix element C_{12}/γ was model-independently computed using Eq. (6) from the untangled SVD modes, which can be obtained either by analyzing B_x and B_y separately or by analyzing the concatenated matrix $[B_x, B_y]$ (marked as B_x/B_y or $[B_x, B_y]$ in figures, respectively). The rotation matrix O was determined using a Fourier projection. Figure 1 shows a C_{12}/γ comparison between the design (MADX) and the calculated (SVD) with both methods using the data from single-particle tracking ($x_0 = y_0 = 0.1mm$) without BPM noise. The agreement for 2000 turn tracking with $Q_x = 0.22$, $Q_y = 0.25$ is on the order of 10^{-7}. Such a small rms error is owing to the fact that 2000 turns contains an integer number of periods for both horizontal and vertical betatron oscillations, which allows a very accurate determination of the O matrix using the Fourier projection.

Turn Dependence

To investigate the dependence on turns, the rms difference between the model and the measured C_{12}/γ was calculated for the same lattice using different numbers of turns. Two cases with different number of significant digits in tunes are shown in Fig. 2, where the rms difference is plotted against the number of turns used to extract the SVD modes. There are two bands in the rms difference: the upper one is inversely proportional to the number of turns as expected from Fourier analysis used in the determination of the O matrix (the errors from SVD are much smaller) and the lower one is around 10^{-7}. This constant value shows the effect due to the number of significant digits in the tune as discussed above and is visible in both cases. However, such an effect can be removed by using more appropriate computation (employing window function for example) of Fourier coefficients.

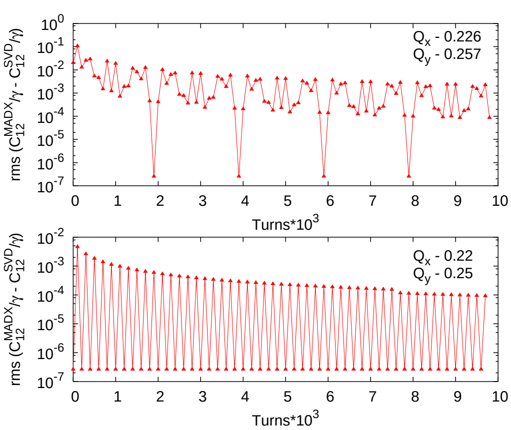

Figure 2: RMS difference between the model and measured for different number of turns. Three significant digits in tune (top) and two significant digits in tune (bottom).

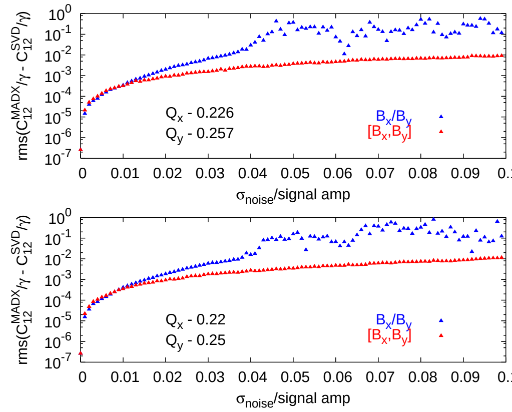

Figure 3: RMS difference between the model and measured for different levels of Gaussian noise in BPMs.

Noise Tolerance

To investigate the robustness of the SVD technique to noise in BPMs, the rms difference of C_{12}/γ was computed for different levels of Gaussian noise in turn-by-turn data. Figure 3 shows the rms errors as a function of σ_{noise}/signal amplitude. It is clear that noise degrades the C_{12}/γ measurement. The dependence on the noise roughly follows a power of $3/2$ for the concatenated case. One has to note that the choice of turns used will effect this dependence as seen in Fig. 2. It is evident that, when the coupling is close to the noise level, using the concatenated data matrix yields better results than using the x and y data separately, as explained in the previous section.

CONCLUSION

We showed how to use the SVD modes for transverse coupling measurement and demonstrated that the method works well. More simulation studies are needed to determine if this methods has any advantage over other methods.

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