ON THE PHYSICAL INTERPRETATION OF THE FINE-STRUCTURE CONSTANT

I will employ the usual notation: e for electron charge h for Planck's constant, $\hbar$ for the reduced Planck constant and c for the light velocity. Also I use Gaussian units.

1. The characteristic classical Coulomb energy and characteristic lengths

The definition I propose for the characteristic classical Coulomb longitudinal energy W (cccl) is

$$W(cccl) \equiv \frac{e^2}{l}, \tag{1}$$

namely, the Coulomb interaction energy of two charged particles separated by a **characteristic length *l***. This definition can be applied to both charged particles and photons, but the characteristic lengths will be different.

1.1 Characteristic length and period of charged particles

For a charged particle the characteristic length l is

$$l_{ecl} = \frac{e^2}{Mc^2}, \tag{2}$$

where M is any mass, the subscripts e and cl denote a charged classical particle. For a particle of mass equal to the rest mass of the electron, Eq. (2) obviously becomes,

$$(l_{ecl})_0 = r_0 = \frac{e^2}{m_0 c^2}, \tag{3}$$

the classical electron radius traditionally defined in the rest frame. The subscript 0 is used to denote that rest frame. For the general case of any inertial frame we denote the characteristic length of any charged particle with mass energy Mc^2 by analogy to the classical electron radius.

$$R_{ecl} \equiv \frac{e^2}{Mc^2}. \tag{4}$$

This characteristic length has an associated characteristic classical period:

$$\tau_{ecl} = \frac{R_{ecl}}{c} = \frac{e^2}{Mc^3}. \tag{5}$$

This period τ_{ecl} transforms the same as l_{ecl} under a Lorentz transformation (τ_{ecl} is familiar from the formula for relativistic Larmor radiation, and just for the record has the value $\sim 10^{-23}$ second.). Now if we perform elementary DeBroglie quantization on the mass energy we get

$$h\nu_{DB} = Mc^2 = \frac{e^2}{R_{ecl}}, \tag{6}$$

where ν_{DB} is the DeBroglie frequency. From this equation we now see that

$$R_{ecl}\,\omega_{DB} = \frac{e^2}{\hbar}. \tag{7}$$

I have now replaced the circular frequency $\boldsymbol{\nu}$ by the angular frequency ω in order to avoid a superfluity of 2π's. I will be using both of these. Eq. (7) says that the characteristic length of a charged particle multiplied by its DeBroglie angular frequency is equal to the velocity scale factor of the hydrogen Bohr orbits. If we divide both sides of Eq. (7) by c the result is, using Eq.(5) for τ_{ecl},

$$\omega_{DB}\,\tau_{ecl} = \alpha. \tag{8}$$

Therefore the product of a classical period and an angular quantum frequency for a charged particle yields the fine structure constant. Both Eq. (7) and (8) are Lorentz invariant.

1.2 Characteristic length and period of the photon

A “physical interpretation” of α is given in Wikipedia (without attribution) as the ratio of the Coulomb interaction energy of two charged particles separated by the distance λ to the photon energy, *i. e.*

$$\frac{e^2/\lambda}{\hbar\omega}=\frac{e^2}{h\nu\lambda}=\alpha \tag{9}$$

This formula becomes α if λ and ν belong to the same light wave because then, of course, $\lambda\nu=c$. Thus, while the answer is correct, it is an observation, not a derivation; its meaning is obscure because the wavelength λ does not belong to the wave of frequency ω. This is a subtle but interesting point. $\frac{e^2}{\lambda}$ is a classical expression where λ happens to have the magnitude of an electromagnetic wavelength, not necessarily that of the wave of frequency ν There is another way of proceeding that adds physical content and clarifies the meaning.

The characteristic length of the photon is the wavelength λ_{cl} of the classical electromagnetic wave that when quantized by quantum field theory produces photons. It then follows from Eq. (1) that the characteristic classical Coulomb energy of a photon is

$$\frac{e^2}{\lambda_{cl}}\ , \tag{10}$$

Where λ_{cl} as the wavelength of the classical electromagnetic wave obeys $\lambda\nu=c$.

This longitudinal Coulomb energy is equivalent to a mass energy so it does not have the same quantized frequency as the massless photon. This can be seen by equating $h\nu$, the energy of a photon to $\frac{e^2}{\lambda}$. This results in the relation

$$\frac{e^2}{\lambda} = h\nu, \text{ or } e^2 = h\nu\lambda, \tag{11}$$

which, if $\nu\lambda = c$ (as is assumed in Wikipedia) becomes $\frac{e^2}{\hbar c} = 2\pi$ which is obviously incorrect. The longitudinal Coulomb energy must be quantized by elementary DeBroglie quantization, the same way mass energy is quantized. This yields the result

$$h\nu_{DB} = \frac{e^2}{\lambda_{cl}}. \tag{12}$$

From this it follows that $\lambda_{cl}\omega_{DB} = \frac{e^2}{\hbar}$, the same as our earlier Eq. (7) for a charged particle but with λ_{cl} replacing R_{ecl}. **Now** since it is true that $\lambda_{cl}\nu_{cl} = c$ for a classical electromagnetic wave so λ_{cl} can be replaced in Eq. (12) by $\frac{c}{\nu_{cl}}$. The result then is (returning to angular frequencies),

$$\lambda\omega_{DB} = \frac{e^2}{\hbar}, \tag{13}$$

or replacing λ_{cl} by $c\tau_{cl}$ Eq. (13) becomes

$$\omega_{DB}\tau_{cl} = \alpha \tag{14}$$

Where the angular frequency ω_{DB} is a quantum frequency and τ_{cl} is the classical period of the electromagnetic wave. This is precisely the same as in the particle case.

2. Summary and Conclusion

A longitudinal characteristic Coulomb energy is defined as $\frac{e^2}{l}$ where l is the characteristic classical length $\frac{e^2}{Mc^2}$ associated with a charged particle, and is λ_{cl}, the classical electromagnetic wavelength associated with the photon. The characteristic Coulomb energies are then subjected to DeBroglie quantization. The product of the angular quantum frequencies and characteristic classical lengths for both the charged particle and the photon yields the velocity $\frac{e^2}{\hbar}$, independently of the hydrogen atom. Substituting the classical periods associated with the classical lengths yields the fine structure constant in both cases. It follows that α couples the quantum frequency and a classical period for both a charged particle and the photon--between longitudinal and transverse modes of energy as is the case in quantum electrodynamics, so these results are consistent with the historical nature of α as a coupling constant between leptons and the quantized electromagnetic field. But the explicit form of this constant as coupling a quantum frequency and a classical period does not appear to have been previously recognized. The Lorentz invariance of the coupling and the use of both quantum and classical electromagnetic theory indicate that α is determined as an intersection of these three theories.