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Convergence of Peridynamics to Classical Elasticity Theory

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Convergence of Peridynamics to Classical Elasticity Theory

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Abstract

The peridynamic model of solid mechanics is a nonlocal theory containing a length scale. It is based on direct interactions between particles separated by a finite distance. The maximum interaction distance provides a length scale for the material model. This paper addresses the question of whether the peridynamic model for an elastic material reproduces the classical local model as this length scale goes to zero. We show that if the deformation, constitutive model, and any nonhomogeneities are sufficiently smooth, then the peridynamic stress tensor converges in this limit to a Piola-Kirchhoff stress tensor that is a function only of the local deformation gradient tensor, as in the classical theory. This limiting Piola-Kirchhoff stress tensor field is differentiable and obeys the classical partial differential equation for the equation of motion. The limiting, or *collapsed,* stress-strain model is hyperelastic and obeys the conditions in the classical theory for angular momentum balance, isotropy, and objectivity, provided the original peridynamic constitutive model satisfies these conditions.

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1 Introduction

The peridynamic model of solid mechanics [1] has been proposed as a way to model deformation of bodies in which discontinuities, especially cracks, occur spontaneously. The objective is to replace the classical continuum description, which assumes a smooth deformation, so that the basic equations remain applicable even when singularites appear in the deformation. This is in contrast to the classical approach, in which the inability to evaluate the spatial derivatives on a crack leads to the need for the special techniques of fracture mechanics. The basic equations of the peridynamic model include the equation of motion,

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \mathbf{L}_{\mathbf{u}}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) \qquad \forall \mathbf{x} \in \mathcal{B}, \quad t \geq 0, \tag{1}$$

$$\mathbf{L}_{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \left\{ \underline{\mathbf{T}}[\mathbf{x},t]\langle \mathbf{x}' - \mathbf{x} \rangle - \underline{\mathbf{T}}[\mathbf{x}',t]\langle \mathbf{x} - \mathbf{x}' \rangle \right\} \, dV_{\mathbf{x}'}$$

where $\mathcal{B}$ is the reference configuration of the body, ρ is the density in the reference configuration, $\mathbf{u}$ is the displacement, and $\mathbf{b}$ is the body force density. The term $\mathbf{L}_{\mathbf{u}}(\mathbf{x},t)$ is a functional of displacement that represents the internal force density (per unit volume) that is exerted on $\mathbf{x}$ by other points in the body. $\underline{\mathbf{T}}[\mathbf{x},t]$ is the *force state* at $\mathbf{x}$ at time t. The force state is a mapping from a *bond* $\mathbf{x}' - \mathbf{x}$ to a force density (per unit volume squared) at $\mathbf{x}$. In the peridynamic theory, a material model is provided by a relation between the deformation near $\mathbf{x}$ and the force state at $\mathbf{x}$:

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})$$

where $\hat{\underline{\mathbf{T}}}$ is the constitutive model and $\underline{\mathbf{Y}}$ is the *deformation state*. $\underline{\mathbf{Y}}$ is a mapping from bonds connected to any $\mathbf{x}$ to the deformed images of these bonds:

$$\underline{\mathbf{Y}}[\mathbf{x},t]\langle \mathbf{x}' - \mathbf{x} \rangle = \mathbf{\Phi}(\mathbf{x}',t) - \mathbf{\Phi}(\mathbf{x},t) = (\mathbf{x}' + \mathbf{u}(\mathbf{x}',t)) - (\mathbf{x} + \mathbf{u}(\mathbf{x},t)) \tag{2}$$

where $\mathbf{\Phi}$ is the deformation. We assume that there is a number $\delta > 0$ called the *horizon* such that

$$|\boldsymbol{\xi}| > \delta \quad \Longrightarrow \quad \underline{\mathbf{T}}\langle \boldsymbol{\xi} \rangle = \mathbf{0}. \tag{3}$$

The purpose of this paper is to demonstrate the convergence of the peridynamic model to the classical (local) theory of continuum mechanics. To do this, the equation of motion (1) is expressed in terms of the peridynamic stress tensor field, resulting in a partial differential equation that is formally identical to the classical equation of motion. The elastic material model is parameterized by the length scale δ in such a way that the bulk properties of the material under homogeneous deformation are independent of δ. *Subject to the assumptions of sufficient smoothness of the deformation and of the constitutive model,* it is then shown that in the limit of small δ, the peridynamic stress tensor field approaches a limit $\boldsymbol{\nu}^0$ that is a differentiable function of $\mathbf{x}$, thus supplying the Piola-Kirchhoff stress tensor field corresponding to the classical formulation of the equation of motion. This Piola-Kirchhoff stress tensor is a function of the (local) deformation gradient tensor.

We further show that the functional $\mathbf{L}_{\mathbf{u}}$ approaches $\nabla \cdot \boldsymbol{\nu}^0$, where $\nabla\cdot$ denotes the divergence operator. The Cauchy stress tensor corresponding to $\boldsymbol{\nu}^0$ is symmetric whenever the underlying peridynamic constitutive model $\hat{\underline{\mathbf{T}}}$ satisfies the appropriate condition for balance of angular momentum. Isotropy and objectivity of $\boldsymbol{\nu}^0$ also hold whenever $\hat{\underline{\mathbf{T}}}$ has these properties.

Convergence of the peridynamic equations in the limit of small δ, as well as other important results related to well-posedness and uniqueness, was established by Emmrich and Weckner [2] for the special case of a linear, isotropic material within the *bond-based* version of the peridynamic theory. This version differs from the more general *state-based* theory considered in the present paper in that in the bond-based theory, internal forces within a body occur only due to central force interactions between pairs of particles. One implication of the bond-based theory is that the bulk properties of a linear isotropic microelastic material necessarily have a Poisson ratio of 1/4. The development in [2] relies on the linearity of the problem. In contrast, the present paper takes a more direct approach that exploits the peridynamic stress tensor [3] and is more generally applicable to nonlinear constitutive models and large deformations.

2 Peridynamic states

A peridynamic state of order m is a mapping that associates with each bond a tensor of order m. A state of order 0 is called a *scalar state*, a state of order 1 is called a *vector state*, and a state of order 2 is called a *tensor state*. The set of all vector states is denoted $\mathcal{V}$. A number of notational conveniences have been introduced in [4] for manipulating states. Some of the more important notation for present purposes is summarized below. In the following, $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are vector states, $\mathbf{C}$ is a second order tensor, and $\mathbf{v}$ is a vector. The product of a tensor with a vector state is

$$(\mathbf{C}\underline{\mathbf{A}})\langle\boldsymbol{\xi}\rangle = \mathbf{C}(\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle), \tag{4}$$

the product of a tensor state with a vector is

$$(\underline{\mathbf{A}}\mathbf{v})\langle\boldsymbol{\xi}\rangle = (\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle)\mathbf{v},$$

and the dot product of two vector states is

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}} \tag{5}$$

where the symbol $\cdot$ denotes the usual Cartesian scalar product of two vectors in $\mathbb{R}^3$ and $\mathcal{H}$ is a neighborhood of $\mathbf{0}$ with radius δ. Expressed in component form, the dot product of two vector states is written as

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{A}_i\langle\boldsymbol{\xi}\rangle \underline{B}_i\langle\boldsymbol{\xi}\rangle \, dV_{\boldsymbol{\xi}}$$

where the $\underline{A}_i\langle\boldsymbol{\xi}\rangle$ are the components of $\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle$ in an orthonormal basis, and where the summation convention is in effect. The composition of two vector states is defined by

$$(\underline{\mathbf{A}} \circ \underline{\mathbf{B}})\langle\boldsymbol{\xi}\rangle = \underline{\mathbf{A}}\langle\underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle\rangle.$$

Suppose $\Psi : \mathcal{V} \to \mathbb{R}$ is a scalar valued function of a vector state $\underline{\mathbf{A}}$. For any differential $d\underline{\mathbf{A}}$, let $d\Psi$ be defined by

$$d\Psi = \Psi(\underline{\mathbf{A}} + d\underline{\mathbf{A}}) - \Psi(\underline{\mathbf{A}}).$$

If there exists a vector state valued function $\nabla\Psi(\cdot)$ such that

$$d\Psi = \nabla\Psi(\underline{\mathbf{A}}) \bullet d\underline{\mathbf{A}} \tag{6}$$

for any $\underline{\mathbf{A}}$ and $d\underline{\mathbf{A}}$, then $\nabla\Psi$ is called the Frechet derivative of Ψ, and Ψ is said to be differentiable. Geometrically, the Frechet derivative of Ψ can be thought of as the state whose "direction" results in the maximum incremental change in Ψ, thus providing an infinite-dimensional analogue of the familiar directional derivative of a function on $\mathbb{R}^3$.

If $\underline{\mathbf{\Psi}}$ is a state-valued function of a vector state $\underline{\mathbf{A}}$, then its Frechet derivative $\nabla\underline{\mathbf{\Psi}}\langle\cdot,\cdot\rangle$ is a *rank 2* state field, which means simply that it is a function of two bonds rather than one. In this case we alter the notation for the dot product so that the Frechet derivative is defined by

$$d\underline{\mathbf{\Psi}}\langle\boldsymbol{\xi}\rangle = (\nabla\underline{\mathbf{\Psi}} \bullet d\underline{\mathbf{A}})\langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}} \nabla\underline{\mathbf{\Psi}}\langle\boldsymbol{\xi},\boldsymbol{\xi}'\rangle d\underline{\mathbf{A}}\langle\boldsymbol{\xi}'\rangle \, dV_{\boldsymbol{\xi}'}$$

for any differential state $d\underline{\mathbf{A}}$. $\nabla\underline{\mathbf{\Psi}}$ has order one higher than $\underline{\mathbf{\Psi}}$; thus, if $\underline{\mathbf{\Psi}} \in \mathcal{V}$,

$$d\underline{\Psi}_i\langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}} (\nabla\underline{\mathbf{\Psi}}\langle\boldsymbol{\xi},\boldsymbol{\xi}'\rangle)_{ij} d\underline{A}_j\langle\boldsymbol{\xi}'\rangle \, dV_{\boldsymbol{\xi}'}. \tag{7}$$

3 Peridynamic stress tensor

Previous results [3] have shown that the peridynamic equation of motion (1) expressed in the form

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \mathbf{f}(\mathbf{x}',\mathbf{x}) \, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \tag{8}$$

is equivalent to the following partial differential equation:

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \nabla\cdot\boldsymbol{\nu}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) \tag{9}$$

where the symbol $\nabla\cdot$ denotes the divergence operator. Here, $\boldsymbol{\nu}$ is the *peridynamic stress tensor field* defined by

$$\boldsymbol{\nu}(\mathbf{x},t) = \frac{1}{2}\int_{\mathcal{S}}\int_0^\delta\int_0^\delta (y+z)^2 \, \mathbf{f}(\mathbf{x}+y\mathbf{m},\mathbf{x}-z\mathbf{m},t)\otimes\mathbf{m} \, dz \, dy \, d\Omega_{\mathbf{m}} \tag{10}$$

where $\otimes$ denotes the dyadic product of two vectors: $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$. The vector $\mathbf{f}(\mathbf{p},\mathbf{q},t)$ is the force density (per unit volume squared) that point $\mathbf{p}\in\mathcal{B}$ exerts on $\mathbf{q}\in\mathcal{B}$. $\mathcal{S}$ is the unit sphere, and $d\Omega_{\mathbf{m}}$ is a differential solid angle in the direction of unit vector $\mathbf{m}$. It is required as a consequence of Newton's third law that $\mathbf{f}(\mathbf{q},\mathbf{p},t) = -\mathbf{f}(\mathbf{p},\mathbf{q},t)$. If this condition is met, then the equations (9) and (10) hold regardless of how $\mathbf{f}$ is determined, *i.e.,* regardless of the deformation and constitutive model. So, if we set

$$\mathbf{f}(\mathbf{p},\mathbf{q},t) = \underline{\mathbf{T}}[\mathbf{q},t]\langle\mathbf{p}-\mathbf{q}\rangle - \underline{\mathbf{T}}[\mathbf{p},t]\langle\mathbf{q}-\mathbf{p}\rangle \tag{11}$$

as indicated by comparing (1) with (8), then the peridynamic stress tensor defined in (10) takes the form

$$\boldsymbol{\nu}(\mathbf{x},t) = \int_{\mathcal{S}}\int_0^\delta\int_0^\delta (y+z)^2 \, \underline{\mathbf{T}}[\mathbf{x}-z\mathbf{m},t]\langle(y+z)\mathbf{m}\rangle\otimes\mathbf{m} \, dz \, dy \, d\Omega_{\mathbf{m}}. \tag{12}$$

$\underline{\mathbf{T}}$ appears only once in this expression, rather than twice as in (11), because both terms turn out to be equal after evaluating the triple integral. In the following sections, the peridynamic stress tensor field given by (12), under suitable parameterization of the constitutive model, will be shown to converge to an admissible Piola-Kirchhoff stress tensor field in the classical theory, provided the deformation and constitutive model are sufficiently smooth.

4 Parameterization of an elastic peridynamic material model

A useful constitutive model is the *elastic* peridynamic material, defined by

$$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}) = \nabla \hat{W}(\underline{\mathbf{Y}}) \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V} \tag{13}$$

where $\hat{W} : \mathcal{V} \to \mathbb{R}$ is a differentiable (in the sense of Frechet derivatives) scalar valued function called the *strain energy density function*. (In the remainder of this paper, $\underline{\mathbf{T}}$ and $\hat{\underline{\mathbf{T}}}$ represent the same force state, but the latter denotes a function of $\underline{\mathbf{Y}}$, while the former denotes particular values of this function.) Peridynamic elastic materials have many of the same properties as elastic materials in the classical theory, including the reversible storage of energy supplied by external loads.

A key consideration in the process of shrinking the horizon to zero is that the bulk properties of the material should be unchanged during this process. To ensure this, a family of strain energy density functions parameterized by the horizon will be defined such that all these functions have the same response under homogeneous deformation.

Let an elastic material model be given with horizon δ and strain energy density function $\hat{W}$; thus the force state is provided by (13). This *reference horizon* δ will be held fixed throughout the remaining discussion. Consider a family of peridynamic elastic materials parameterized by variable horizon δ', and define

$$s = \delta'/\delta,$$

so the shrinkage process means taking the limit as $s \to 0$. Let the strain energy density functions in this family of materials be given by

$$\hat{W}^s(\underline{\mathbf{Y}}) = \hat{W}(\underline{\mathbf{E}}(\underline{\mathbf{Y}})) \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V} \tag{14}$$

where $\underline{\mathbf{E}}(\underline{\mathbf{Y}})$ is the *enlarged deformation state* defined by

$$\underline{\mathbf{E}}(\underline{\mathbf{Y}})\langle\boldsymbol{\xi}\rangle = s^{-1}\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \qquad \forall \boldsymbol{\xi} \in \mathcal{H}, \quad \forall \underline{\mathbf{Y}} \in \mathcal{V} \tag{15}$$

(Figure 1). If the deformation happens to be homogeneous with deformation gradient tensor $\mathbf{F}$, then by (2), $\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{F}\boldsymbol{\xi}$; hence from (15),

$$\underline{\mathbf{E}}(\underline{\mathbf{Y}})\langle\boldsymbol{\xi}\rangle = s^{-1}\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle = s^{-1}\mathbf{F}s\boldsymbol{\xi} = \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle.$$

To derive $\hat{\underline{\mathbf{T}}}^s$ from (14), consider a differential increment $d\underline{\mathbf{Y}}$ in the deformation state, and apply (13) and the defining relation of the Frechet derivative (6):

$$dW = \hat{\underline{\mathbf{T}}}^s(\underline{\mathbf{Y}}) \bullet d\underline{\mathbf{Y}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}})) \bullet d\underline{\mathbf{E}}(\underline{\mathbf{Y}}).$$

From this, (15), and the definition of the dot product for vector states (5),

$$\int_{\mathcal{H}^s} \underline{\mathbf{T}}^s\langle\boldsymbol{\zeta}\rangle \cdot d\underline{\mathbf{Y}}\langle\boldsymbol{\zeta}\rangle \; dV_{\boldsymbol{\zeta}} = \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot s^{-1} d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}}$$

where $\mathcal{H}^s$ is a sphere of radius $s\delta$, $\underline{\mathbf{T}}^s = \hat{\underline{\mathbf{T}}}^s(\underline{\mathbf{Y}})$, and $\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}}))$. Changing the dummy variable of integration on the left side from $\boldsymbol{\zeta}$ to $s\boldsymbol{\xi}$ results in

$$\int_{\mathcal{H}} \underline{\mathbf{T}}^s\langle s\boldsymbol{\xi}\rangle \cdot d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; (s^3 dV_{\boldsymbol{\xi}}) = \int_{\mathcal{H}} \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle \cdot s^{-1} d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}},$$

hence

$$\int_{\mathcal{H}} \left(\underline{\mathbf{T}}^s\langle s\boldsymbol{\xi}\rangle - s^{-4}\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle\right) \cdot d\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}} = 0. \tag{16}$$

Since (16) holds for any $d\underline{\mathbf{Y}}$, it follows that

$$\hat{\underline{\mathbf{T}}}^s(\underline{\mathbf{Y}})\langle s\boldsymbol{\xi}\rangle = s^{-4}\hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}}))\langle\boldsymbol{\xi}\rangle \qquad \forall\boldsymbol{\xi}\in\mathcal{H}, \forall\underline{\mathbf{Y}}\in\mathcal{V}. \tag{17}$$

To account for nonhomogeneity of a body, $\mathbf{x}$ will now be included explicitly in the constitutive model: $\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$.

5 Convergence of the peridynamic stress field

A given deformation $\boldsymbol{\Phi}$ is assumed, independent of s. The following assumptions will be made that permit a meaningful comparison between the classical and peridynamic models:

(i) The deformation $\boldsymbol{\Phi}$ is a twice continuously differentiable function of $\mathbf{x}$ and t.

(ii) The constitutive model $\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$ is a continuously differentiable function of $\underline{\mathbf{Y}}$ and $\mathbf{x}$.

Let $\mathbf{F}$ denote the usual deformation gradient tensor field,

$$\mathbf{F}(\mathbf{x}, t) = \frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{x}}(\mathbf{x}, t) \qquad \forall\mathbf{x}\in\mathcal{B}, \quad t\geq 0. \tag{18}$$

In the following discussion, the time variable t will be omitted to make the notation more concise. Assumption (i) immediately implies

$$\underline{\mathbf{Y}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle = \mathbf{F}(\mathbf{x})\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2) \qquad \forall\boldsymbol{\xi}\in\mathcal{H}. \tag{19}$$

Consider the behavior of $\underline{\mathbf{Y}}$ near some $\mathbf{x}$. For any increment $\Delta\mathbf{x}$, define $\Delta\underline{\mathbf{Y}}$ by

$$\Delta\underline{\mathbf{Y}} = \underline{\mathbf{Y}}[\mathbf{x} + \Delta\mathbf{x}] - \underline{\mathbf{Y}}[\mathbf{x}]. \tag{20}$$

Suppose both of the following hold:

$$\Delta\mathbf{x} = O(s), \qquad\qquad \boldsymbol{\xi} = O(s). \tag{21}$$

Using assumption (i), equations (2), (18), (20), (21), and the first three terms of a Taylor series with remainder,

$$\Delta\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = (\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}))\Delta\mathbf{x}\boldsymbol{\xi} + O(s^3) \qquad \text{or} \qquad \Delta\underline{Y}_i\langle\boldsymbol{\xi}\rangle = F_{ij,k}(\mathbf{x})\Delta x_j\xi_k + O(s^3). \tag{22}$$

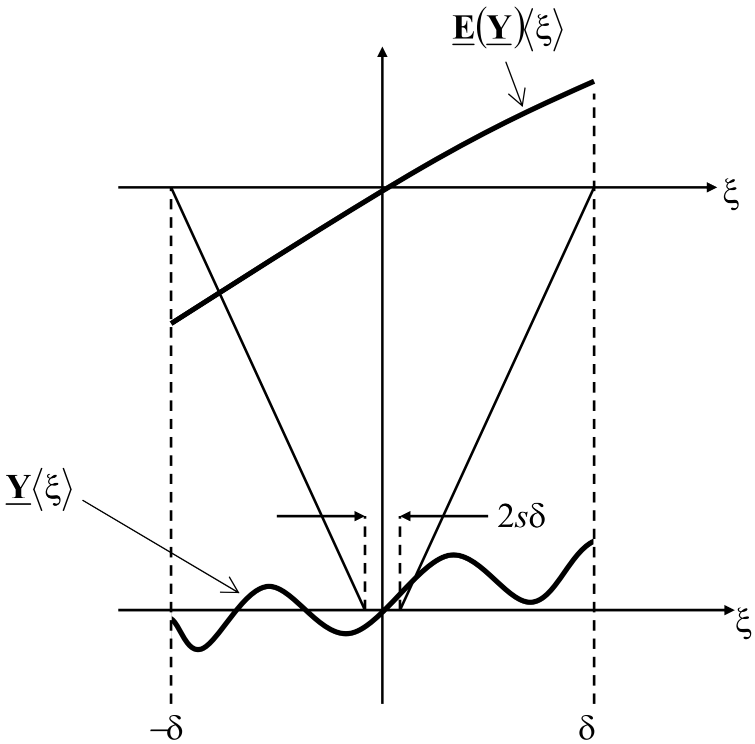

Figure 1. Enlarged deformation state $\underline{\mathbf{E}}(\underline{\mathbf{Y}})$ maps the part of the deformation state $\underline{\mathbf{Y}}$ within the small horizon $s\delta$ to the original horizon δ.

The notation $F_{ij,k} = \partial F_{ij}/\partial x_k$ is used in (22). Before proceeding further, we record the following result.

Lemma 1. Let a be a positive number, and let $g : [0,a] \to \mathbb{R}$ be an integrable function. Let

$$I(a) = \int_0^a \int_z^a g(p)\ dp\ dz. \tag{23}$$

Then

$$I(a) = \int_0^a pg(p)\ dp. \tag{24}$$

Proof. Define $k(z,a) = \int_z^a g(p)dp$; thus $I(a) = \int_0^a k(z,a)dz$. Differentiating this,

$$\frac{dI}{da} = k(a,a) + \int_0^a \frac{\partial k}{\partial a}(z,a)\ dz = 0 + \int_0^a g(a)\ dz = ag(a).$$

This is a first order differential equation whose solution under the initial condition $I(0) = 0$, which is implied by (23), is given by (24). $\square$

Now consider the limiting behavior of the peridynamic stress tensor $\boldsymbol{\nu}^s(\mathbf{x})$ as s becomes small. From the definition of the peridynamic stress tensor (12),

$$\boldsymbol{\nu}^s(\mathbf{x}) = \int_{\mathcal{S}} \int_0^{s\delta} \int_0^{s\delta} (y+z)^2\ \underline{\hat{\mathbf{T}}}^s(\underline{\mathbf{Y}}[\mathbf{x}-z\mathbf{m}], \mathbf{x}-z\mathbf{m})\langle (y+z)\mathbf{m}\rangle \otimes \mathbf{m}\ dz\ dy\ d\Omega_{\mathbf{m}}. \tag{25}$$

Using (17) and the change of dummy variables $y \to sy$ and $z \to sz$,

$$\begin{aligned}
\boldsymbol{\nu}^s(\mathbf{x}) &= \int_{\mathcal{S}} \int_0^{\delta} \int_0^{\delta} (sy+sz)^2 \left(s^{-4}\underline{\hat{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}-sz\mathbf{m}]), \mathbf{x}-sz\mathbf{m})\langle (y+z)\mathbf{m}\rangle \right) \\
&\qquad \otimes \mathbf{m}\ (sdz)\ (sdy)\ d\Omega_{\mathbf{m}} \\
&= \int_{\mathcal{S}} \int_0^{\delta} \int_0^{\delta} (y+z)^2 \underline{\hat{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}-sz\mathbf{m}]), \mathbf{x}-sz\mathbf{m})\langle (y+z)\mathbf{m}\rangle \\
&\qquad \otimes \mathbf{m}\ dz\ dy\ d\Omega_{\mathbf{m}}.
\end{aligned} \tag{26}$$

Now observe that by assumption (i) and equations (15), (19), (20), and (22) with $\Delta\mathbf{x} = -sz\mathbf{m}$, we have that for any $\boldsymbol{\xi} \in \mathcal{H}$,

$$\begin{aligned}
\underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}-sz\mathbf{m}])\langle \boldsymbol{\xi}\rangle &= s^{-1}\underline{\mathbf{Y}}[\mathbf{x}-sz\mathbf{m}]\langle s\boldsymbol{\xi}\rangle \\
&= s^{-1} \left(\underline{\mathbf{Y}}[\mathbf{x}]\langle s\boldsymbol{\xi}\rangle + \Delta\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}\rangle \right) \\
&= s^{-1} \left(\underline{\mathbf{Y}}[\mathbf{x}]\langle s\boldsymbol{\xi}\rangle + (\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{x}))(-sz\mathbf{m})(s\boldsymbol{\xi}) + O(s^3) \right) \\
&= s^{-1} \left(\underline{\mathbf{Y}}[\mathbf{x}]\langle s\boldsymbol{\xi}\rangle + O(s^2) \right) \\
&= s^{-1} \left(\mathbf{F}(\mathbf{x})s\boldsymbol{\xi} + O(s^2) \right) \\
&= \mathbf{F}(\mathbf{x})\boldsymbol{\xi} + O(s).
\end{aligned} \tag{27}$$

From (27),

$$\underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}-sz\mathbf{m}]) = \mathbf{F}(\mathbf{x})\underline{\mathbf{X}} + O(s) \tag{28}$$

where $\underline{\mathbf{X}}$ is the identity vector state defined by

$$\underline{\mathbf{X}}\langle\boldsymbol{\xi}\rangle = \boldsymbol{\xi} \qquad \forall\boldsymbol{\xi}. \tag{29}$$

To further simplify the integrand in (26), use (28) and assumption (ii) to yield

$$\begin{aligned}
&\hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}-sz\mathbf{m}]),\mathbf{x}-sz\mathbf{m})\langle(y+z)\mathbf{m}\rangle \\
&\qquad\qquad = \hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}+O(s),\mathbf{x}-sz\mathbf{m})\langle(y+z)\mathbf{m}\rangle \\
&\qquad\qquad = \hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle(y+z)\mathbf{m}\rangle + O(s).
\end{aligned} \tag{30}$$

From (26) and (30),

$$\begin{aligned}
\boldsymbol{\nu}^s(\mathbf{x}) &= \int_{\mathcal{S}}\int_0^\delta\int_0^\delta (y+z)^2\hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle(y+z)\mathbf{m}\rangle\otimes\mathbf{m}\;dz\;dy\;d\Omega_{\mathbf{m}} + O(s) \\
&= \int_{\mathcal{S}}\int_0^\delta\int_z^\delta p^2\hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle p\mathbf{m}\rangle\otimes\mathbf{m}\;dp\;dz\;d\Omega_{\mathbf{m}} + O(s)
\end{aligned} \tag{31}$$

where the change of variables $p = y + z$ has been used. The upper limit of integration on the integral over p is shown as δ instead of $\delta + z$ because of (3).

Using Lemma 1 in (31) with $g(p) = p^2\hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle p\mathbf{m}\rangle$, it follows that

$$\begin{aligned}
\boldsymbol{\nu}^s(\mathbf{x}) &= \int_{\mathcal{S}}\int_0^\delta p^3\hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle p\mathbf{m}\rangle\otimes\mathbf{m}\;dp\;d\Omega_{\mathbf{m}} + O(s) \\
&= \int_{\mathcal{H}} |\mathbf{p}|\;\hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle\mathbf{p}\rangle\otimes\mathbf{m}\;dV_{\mathbf{p}} + O(s) \\
&= \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle\mathbf{p}\rangle\otimes\mathbf{p}\;dV_{\mathbf{p}} + O(s)
\end{aligned}$$

in which the change of variables $\mathbf{p} = p\mathbf{m}$ was used, hence $dV_{\mathbf{p}} = p^2 d\Omega_{\mathbf{m}}$. Now define the *collapsed peridynamic stress tensor field* $\boldsymbol{\nu}^0$ by

$$\boldsymbol{\nu}^0(\mathbf{x}) = \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{F}(\mathbf{x})\underline{\mathbf{X}},\mathbf{x})\langle\boldsymbol{\xi}\rangle\otimes\boldsymbol{\xi}\;dV_{\boldsymbol{\xi}} \qquad \forall\mathbf{x}\in\mathcal{B}. \tag{32}$$

Geometrically, $\mathbf{F}(\mathbf{x})\underline{\mathbf{X}}$ represents the deformation state $\underline{\mathbf{Y}}$ that would be obtained by observing the deformation at $\mathbf{x}$ "through a microscope" as suggested by Figure 1. The discussion above has established the following proposition.

Proposition 1. Let $\mathcal{B}$ be an open region occupied by the reference configuration of an elastic peridynamic body, and let $\boldsymbol{\Phi}$ be a deformation of $\mathcal{B}$. Let $\hat{W}$ be a strain energy density function for the body with horizon δ, and let $\hat{\underline{\mathbf{T}}}$ be the corresponding constitutive model derived from (13). Suppose that assumptions (i) and (ii) are satisfied. Let a family of constitutive models parameterized by horizon $\delta' = s\delta$ be given by (14) for any $s > 0$. Let $\boldsymbol{\nu}^s$ be the corresponding family of peridynamic stress tensor fields defined by (25). Then

$$\lim_{s\to 0}\boldsymbol{\nu}^s = \boldsymbol{\nu}^0 \qquad \text{on } \mathcal{B}$$

where $\boldsymbol{\nu}^0$ is the tensor field defined by (32).

The condition stated in Proposition 1 that $\mathcal{B}$ is an open set is required so that for sufficiently small s, the neighborhood of radius s centered at any $\mathbf{x} \in \mathcal{B}$ is contained in $\mathcal{B}$. This is needed for statements such as (19) to be true. Proposition 1 still holds if assumption (i) is replaced by the weaker assumption that $\mathbf{\Phi}$ is a continuously differentiable function of $\mathbf{x}$. However, the stronger assumptions will be needed for subsequent results below. The following proposition follows immediately from (32) and assumptions (i) and (ii):

Proposition 2. Under the conditions of Proposition 1, $\boldsymbol{\nu}^0$ is a continuously differentiable function of $\mathbf{x}$.

6 Convergence of the divergence of the peridynamic stress field

Propositions 1 and 2 do not by themselves establish that the integral in the equation of motion (1) converges to $\nabla \cdot \boldsymbol{\nu}^0$ as $s \to 0$. However, this convergence will now be shown directly. Let a deformation $\mathbf{\Phi}$ on $\mathcal{B}$ be given. Define the following functional of $\mathbf{u}$, parameterized by s:

$$\mathbf{L}^s_{\mathbf{u}}(\mathbf{x}) = \int_{\mathcal{H}^s} \{\underline{\mathbf{T}}^s[\mathbf{x}]\langle\boldsymbol{\zeta}\rangle - \underline{\mathbf{T}}^s[\mathbf{x}+\boldsymbol{\zeta}]\langle -\boldsymbol{\zeta}\rangle\} \; dV_{\boldsymbol{\zeta}} \quad \forall \mathbf{x} \in \mathcal{B}. \tag{33}$$

(Time labels will be omitted to simplify the notation, although it is understood that $\mathbf{u}$ can depend on time.) From (17) and (33), and setting $\boldsymbol{\zeta} = s\boldsymbol{\xi}$,

$$\begin{aligned} \mathbf{L}^s_{\mathbf{u}}(\mathbf{x}) &= \int_{\mathcal{H}} \left\{ s^{-4}\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle - s^{-4}\underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}]\langle -\boldsymbol{\xi}\rangle \right\} \; s^3 dV_{\boldsymbol{\xi}} \\ &= s^{-1} \int_{\mathcal{H}} \{\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle - \underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}]\langle -\boldsymbol{\xi}\rangle\} \; dV_{\boldsymbol{\xi}} \end{aligned} \tag{34}$$

where

$$\underline{\mathbf{T}}[\mathbf{x}] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}]), \mathbf{x}), \qquad \underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}] = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}+s\boldsymbol{\xi}]), \mathbf{x}).$$

Setting

$$\Delta\underline{\mathbf{Y}} = \underline{\mathbf{Y}}[\mathbf{x}+s\boldsymbol{\xi}] - \underline{\mathbf{Y}}[\mathbf{x}], \qquad \Delta\underline{\mathbf{E}} = \underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}+s\boldsymbol{\xi}]) - \underline{\mathbf{E}}(\underline{\mathbf{Y}}[\mathbf{x}]),$$

it follows from (15) that

$$\Delta\underline{\mathbf{E}} = \underline{\mathbf{E}}(\Delta\underline{\mathbf{Y}}). \tag{35}$$

Applying (15) again to write out $\Delta\underline{\mathbf{E}}$ explicitly, and using (22) to approximate the result for small s, for any $\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathcal{H}$,

$$\begin{aligned} \Delta\underline{\mathbf{E}}\langle\boldsymbol{\xi}'\rangle &= s^{-1}\Delta\underline{\mathbf{Y}}\langle s\boldsymbol{\xi}'\rangle \\ &= s^{-1}\left((\nabla_{\mathbf{x}}\mathbf{F})(s\boldsymbol{\xi}')(s\boldsymbol{\xi}) + O(s^3) \right) \\ &= s(\nabla_{\mathbf{x}}\mathbf{F})\boldsymbol{\xi}'\boldsymbol{\xi} + O(s^2) \end{aligned}$$

or

$$\Delta\underline{\mathbf{E}} = s(\nabla_{\mathbf{x}}\mathbf{F}\underline{\mathbf{X}})\boldsymbol{\xi} + O(s^2). \tag{36}$$

(Recall from (29) that $\mathbf{F}\underline{\mathbf{X}}\langle\boldsymbol{\xi}'\rangle = \mathbf{F}\boldsymbol{\xi}'$. The identity vector state $\underline{\mathbf{X}}$ does not depend on $\mathbf{x}$.) To evaluate the second term in the integrand in (34), use assumptions (i) and (ii), the first two terms of a Taylor series with remainder, and (36) to obtain

$$\begin{aligned}\underline{\mathbf{T}}[\mathbf{x}+s\boldsymbol{\xi}] &= \underline{\mathbf{T}}[\mathbf{x}] + \nabla\hat{\underline{\mathbf{T}}} \bullet \Delta\underline{\mathbf{E}} + (\nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}})s\boldsymbol{\xi} + O(s^2) \\ &= \underline{\mathbf{T}}[\mathbf{x}] + s(\nabla\hat{\underline{\mathbf{T}}} \bullet (\nabla_{\mathbf{x}}\mathbf{F}\underline{\mathbf{X}}) + \nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}})\boldsymbol{\xi} + O(s^2). \end{aligned} \tag{37}$$

The term $\nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}}$ refers to the explicit dependence of $\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})$ on $\mathbf{x}$ due to nonhomogeneity. Using (37) in (34) with the change of dummy variable $\boldsymbol{\xi} \to -\boldsymbol{\xi}$, applying the chain rule, and noting that the zero-order terms $\underline{\mathbf{T}}[\mathbf{x}]\langle\boldsymbol{\xi}\rangle$ and $\underline{\mathbf{T}}[\mathbf{x}]\langle -\boldsymbol{\xi}\rangle$ cancel each other when the integration is carried out,

$$\begin{aligned}\mathbf{L}^s_{\mathbf{u}}(\mathbf{x}) &= \int_{\mathcal{H}} (\nabla\hat{\underline{\mathbf{T}}} \bullet (\nabla_{\mathbf{x}}\mathbf{F}\underline{\mathbf{X}}) + \nabla_{\mathbf{x}}\hat{\underline{\mathbf{T}}})\langle\boldsymbol{\xi}\rangle\boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} + O(s) \\ &= \nabla \cdot \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} + O(s) \\ &= \nabla \cdot \boldsymbol{\nu}^0(\mathbf{x}) + O(s) \end{aligned} \tag{38}$$

where $\nabla\cdot$ denotes the divergence operator and the last step comes from (32). This proves the following result.

Proposition 3. Under the conditions of Proposition 1,

$$\lim_{s\to 0} \mathbf{L}^s_{\mathbf{u}} = \nabla \cdot \boldsymbol{\nu}^0 \qquad \text{on } \mathcal{B}$$

where $\mathbf{L}^s_{\mathbf{u}}$ is defined by (33).

7 Constitutive model for the collapsed peridynamic stress tensor

Since (32) provides an expression for the collapsed peridynamic stress tensor at $\mathbf{x}$ that depends only on $\mathbf{F}(\mathbf{x})$, we can now define a constitutive model for this $\boldsymbol{\nu}^0$ as follows:

$$\hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x}) = \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \qquad \forall \mathbf{F} \in \mathcal{L},\ \forall \mathbf{x} \in \mathcal{B} \tag{39}$$

where $\mathcal{L}$ is the set of all second order tensors. Recall that $\mathbf{F}\underline{\mathbf{X}}$ is a vector state; see (4) and (29) regarding notation.

Equation (39) is a *local* constitutive model in the sense that it depends on the deformation only through the deformation gradient tensor. (It can also depend on $\mathbf{x}$ explicitly to reflect nonhomogeneity of the body.) As shown by Proposition 3 and (1), the $\boldsymbol{\nu}^0$ field provided by this constitutive model describes the internal forces to which the peridynamic model converges (subject to assumptions (i) and (ii)) in the limit of small horizon. In the remainder of this section we consider the properties of $\hat{\boldsymbol{\nu}}^0$, in the sense of the classical theory, with regard to angular momentum balance, isotropy, and objectivity. The function $\hat{\boldsymbol{\nu}}^0$ will be referred to below as a "stress-strain relation" to distinguish it from a peridynamic constitutive model and to reflect its dependence on the strain-like quantity $\mathbf{F}$.

7.1 Angular momentum balance

To complete the identification of $\hat{\boldsymbol{\nu}}^0$ defined by (39) with the Piola-Kirchhoff stress in the classical theory, we now investigate the properties of the corresponding Cauchy stress defined by

$$\hat{\boldsymbol{\tau}}^0 = \frac{1}{J}\hat{\boldsymbol{\nu}}^0\mathbf{F}^T, \qquad J = \det \mathbf{F} \qquad \forall \mathbf{F} \in \mathcal{L} \tag{40}$$

where it is assumed that $J \neq 0$ [5]. As described in [4], a sufficient condition for global balance of angular momentum to hold in a peridynamic body is that the constitutive model obey

$$\int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \times \underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}} = \mathbf{0} \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}, \; \forall \mathbf{x} \in \mathcal{B} \tag{41}$$

or using components,

$$\varepsilon_{ijk} \int_{\mathcal{H}} \hat{\underline{T}}_j(\underline{\mathbf{Y}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \underline{Y}_k\langle\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}} = 0 \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}, \; \forall \mathbf{x} \in \mathcal{B} \tag{42}$$

where ε_{ijk} is the alternator symbol. Geometrically, the condition (41) means that force states individually satisfy balance of angular momentum; *i.e.,* the forces on $\mathbf{x}$ due to $\underline{\mathbf{T}}[\mathbf{x}]$ exert no net moment.

Proposition 4. Under the conditions of Proposition 1, suppose $\hat{\underline{\mathbf{T}}}$ satisfies (41). For any $\mathbf{x} \in \mathcal{B}$, let $\hat{\boldsymbol{\nu}}^0$ be given by (39), and let $\hat{\boldsymbol{\tau}}^0$ be given by (40). Then $\hat{\boldsymbol{\tau}}^0$ is symmetric on $\mathcal{B}$.

Proof. Setting $\underline{\mathbf{Y}} = \mathbf{F}\underline{\mathbf{X}}$ in (42) and using (39) leads to

$$\begin{aligned} 0 &= \varepsilon_{ijk} \int_{\mathcal{H}} \hat{\underline{T}}_j(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle (\mathbf{F}\underline{\mathbf{X}})_{km}\langle\boldsymbol{\xi}\rangle \; dV_{\boldsymbol{\xi}} \\ &= \varepsilon_{ijk} \int_{\mathcal{H}} \hat{\underline{T}}_j(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle F_{km}\xi_m \; dV_{\boldsymbol{\xi}} \\ &= \varepsilon_{ijk} \left(\int_{\mathcal{H}} \hat{\underline{T}}_j(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle\boldsymbol{\xi}\rangle \xi_m \; dV_{\boldsymbol{\xi}} \right) F_{km} \\ &= \varepsilon_{ijk}\hat{\nu}^0_{jm}F_{km} \\ &= J\varepsilon_{ijk}\hat{\tau}^0_{jk} \end{aligned}$$

so $\hat{\tau}^0_{jk} = \hat{\tau}^0_{kj}$. $\square$

7.2 Isotropy

If $\mathbf{Q}$ is any orthogonal tensor, then the corresponding orthogonal state $\underline{\mathbf{Q}}$ is defined by

$$\underline{\mathbf{Q}}\langle\boldsymbol{\xi}\rangle = \mathbf{Q}\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi}.$$

As discussed in [4], the definition of isotropy in a peridynamic body is

$$\hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}} \circ \underline{\mathbf{Q}}, \mathbf{x}) = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}) \circ \underline{\mathbf{Q}} \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V}, \; \forall \mathbf{x} \in \mathcal{B} \tag{43}$$

or

$$\underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}} \circ \underline{\mathbf{Q}}, \mathbf{x})\langle \boldsymbol{\xi} \rangle = \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})\langle \mathbf{Q}\boldsymbol{\xi} \rangle \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V},\ \forall \mathbf{x} \in \mathcal{B},\ \forall \boldsymbol{\xi} \in \mathcal{H}$$

for all orthogonal states $\underline{\mathbf{Q}}$.

Proposition 5. Under the conditions of Proposition 1, suppose $\underline{\hat{\mathbf{T}}}$ satisfies (43). For any $\mathbf{x} \in \mathcal{B}$, let $\hat{\boldsymbol{\nu}}^0$ be given by (39). Then

$$\hat{\boldsymbol{\nu}}^0(\mathbf{FQ}, \mathbf{x}) = \hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x})\mathbf{Q} \tag{44}$$

for all orthogonal tensors $\mathbf{Q}$ and all $\mathbf{F}$.

Proof. For any orthogonal tensor $\mathbf{Q}$ and any $\mathbf{F}$, using (39) and (43) and the change of variables $\boldsymbol{\xi}' = \mathbf{Q}\boldsymbol{\xi}$,

$$\begin{aligned}
\hat{\boldsymbol{\nu}}^0(\mathbf{FQ}, \mathbf{x}) &= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{FQ}\underline{\mathbf{X}}, \mathbf{x})\langle \boldsymbol{\xi} \rangle \otimes \boldsymbol{\xi}\ dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}} \circ \underline{\mathbf{Q}}, \mathbf{x})\langle \boldsymbol{\xi} \rangle \otimes \boldsymbol{\xi}\ dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle \mathbf{Q}\boldsymbol{\xi} \rangle \otimes \boldsymbol{\xi}\ dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle \boldsymbol{\xi}' \rangle \otimes (\mathbf{Q}^T\boldsymbol{\xi}')\ dV_{\boldsymbol{\xi}'} \\
&= \left(\int_{\mathcal{H}} \underline{\hat{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}}, \mathbf{x})\langle \boldsymbol{\xi}' \rangle \otimes \boldsymbol{\xi}'\ dV_{\boldsymbol{\xi}'} \right) \mathbf{Q} \\
&= \hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x})\mathbf{Q}. \qquad \square
\end{aligned}$$

Equation (44) is the condition for isotropy of a (local) material model in the classical theory in terms of the Piola-Kirchhoff stress [5]. So, the conclusion is that if the peridynamic material model is isotropic, then the corresponding $\hat{\boldsymbol{\nu}}^0$ is also isotropic in the sense of the classical theory.

7.3 Objectivity

As discussed in [4], the definition of an objective peridynamic body is

$$\underline{\hat{\mathbf{T}}}(\underline{\mathbf{Q}} \circ \underline{\mathbf{Y}}, \mathbf{x}) = \underline{\mathbf{Q}} \circ \underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x}) \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V},\ \forall \mathbf{x} \in \mathcal{B} \tag{45}$$

or

$$\underline{\hat{\mathbf{T}}}(\underline{\mathbf{Q}} \circ \underline{\mathbf{Y}}, \mathbf{x})\langle \boldsymbol{\xi} \rangle = \mathbf{Q}\underline{\hat{\mathbf{T}}}(\underline{\mathbf{Y}}, \mathbf{x})\langle \boldsymbol{\xi} \rangle \qquad \forall \underline{\mathbf{Y}} \in \mathcal{V},\ \forall \mathbf{x} \in \mathcal{B},\ \forall \boldsymbol{\xi} \in \mathcal{H}$$

for all orthogonal states $\underline{\mathbf{Q}}$.

Proposition 6. Under the conditions of Proposition 1, suppose $\underline{\hat{\mathbf{T}}}$ satisfies (45). For any $\mathbf{x} \in \mathcal{B}$, let $\hat{\boldsymbol{\nu}}^0$ be given by (39). Then

$$\hat{\boldsymbol{\nu}}^0(\mathbf{QF}, \mathbf{x}) = \mathbf{Q}\hat{\boldsymbol{\nu}}^0(\mathbf{F}, \mathbf{x}) \tag{46}$$

for all orthogonal tensors $\mathbf{Q}$ and all $\mathbf{F}$.

Proof. For any orthogonal tensor $\mathbf{Q}$ and any $\mathbf{F}$, from (39) and (45),

$$\begin{aligned}\hat{\boldsymbol{\nu}}^0(\mathbf{QF},\mathbf{x}) &= \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{QF}\underline{\mathbf{X}},\mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \\ &= \int_{\mathcal{H}} \hat{\underline{\mathbf{T}}}(\mathbf{Q}\circ\mathbf{F}\underline{\mathbf{X}},\mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \\ &= \int_{\mathcal{H}} \mathbf{Q}\hat{\underline{\mathbf{T}}}(\mathbf{F}\underline{\mathbf{X}},\mathbf{x})\langle\boldsymbol{\xi}\rangle \otimes \boldsymbol{\xi}\, dV_{\boldsymbol{\xi}} \\ &= \mathbf{Q}\hat{\boldsymbol{\nu}}^0(\mathbf{F},\mathbf{x}). \qquad \square\end{aligned}$$

(46) is the condition for a classical constitutive model to be objective expressed in terms of the Piola-Kirchhoff stress [5]. From this result, the conclusion is that if a peridynamic constitutive model is objective, then the corresponding $\hat{\boldsymbol{\nu}}^0$ is also objective in the sense of the classical theory.

7.4 Hyperelasticity

In this section it will be shown that $\hat{\boldsymbol{\nu}}^0$ is derivable from a scalar valued strain energy density function via the usual tensor gradient within the classical theory. To do this, define the *collapsed strain energy density function* $\hat{W}^0 : \mathcal{L}\times\mathcal{B}\to\mathbb{R}$ by

$$\hat{W}^0(\mathbf{F},\mathbf{x}) = \hat{W}(\mathbf{F}\underline{\mathbf{X}},\mathbf{x}) \qquad \forall\mathbf{F}\in\mathcal{L},\ \forall\mathbf{x}\in\mathcal{B}. \tag{47}$$

where $\hat{W}$ is the peridynamic strain energy density function in (13). Denote the tensor gradient by $\partial/\partial\mathbf{F}$, thus

$$\left(\frac{\partial\hat{W}^0}{\partial\mathbf{F}}\right)_{ij} = \frac{\partial\hat{W}^0}{\partial F_{ij}}. \tag{48}$$

Proposition 7. Let a peridynamic elastic strain energy density function $\hat{W} : \mathcal{V}\times\mathcal{B}\to\mathbb{R}$ for a nonhomogeneous body $\mathcal{B}$ be given, and let $\underline{\mathbf{T}} = \nabla\hat{W}$, where ∇ denotes the Frechet derivative. Let $\hat{\boldsymbol{\nu}}^0$ be given by (39). Let $\hat{W}^0$ be defined by (47). Then

$$\hat{\boldsymbol{\nu}}^0(\mathbf{F},\mathbf{x}) = \frac{\partial\hat{W}^0}{\partial\mathbf{F}}(\mathbf{F},\mathbf{x}) \qquad \forall\mathbf{F}\in\mathcal{L},\ \forall\mathbf{x}\in\mathcal{B}. \tag{49}$$

Proof. Since

$$(\mathbf{F}\underline{\mathbf{X}})_k\langle\boldsymbol{\xi}\rangle = F_{km}\xi_m,$$

it follows that

$$\left(\frac{\partial(\mathbf{F}\underline{\mathbf{X}})}{\partial\mathbf{F}}\langle\boldsymbol{\xi}\rangle\right)_{kij} = \frac{\partial}{\partial F_{ij}}(F_{km}\xi_m) = \delta_{ik}\delta_{jm}\xi_m = \delta_{ik}\xi_j. \tag{50}$$

By definition, $\hat{W}$ is differentiable. From (47) and the chain rule,

$$\frac{\partial \hat{W}^0}{\partial \mathbf{F}} = \nabla \hat{W} \bullet \frac{\partial (\mathbf{F}\underline{\mathbf{X}})}{\partial \mathbf{F}}.$$

Expressing this in component form, expanding out the dot product, and using (50),

$$\begin{aligned}
\left(\frac{\partial \hat{W}^0}{\partial \mathbf{F}}\right)_{ij} &= \int_{\mathcal{H}} \underline{T}_k\langle\boldsymbol{\xi}\rangle \left(\frac{\partial (\mathbf{F}\underline{\mathbf{X}})}{\partial \mathbf{F}}\langle\boldsymbol{\xi}\rangle\right)_{kij} dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{T}_k\langle\boldsymbol{\xi}\rangle \delta_{ik}\xi_j \, dV_{\boldsymbol{\xi}} \\
&= \int_{\mathcal{H}} \underline{T}_i\langle\boldsymbol{\xi}\rangle \xi_j \, dV_{\boldsymbol{\xi}} \\
&= \hat{\nu}^0_{ij}
\end{aligned}$$

where the final step comes from (39). □

One implication of Proposition 7 is that the collapsed peridynamic stress tensor $\hat{\boldsymbol{\nu}}^0$ is complementary to $\mathbf{F}$, which is consistent with the properties of Piola-Kirchhoff stress tensors in the classical theory for hyperelastic materials.

8 Discussion

The above development has shown that under the assumptions (i) and (ii), the elastic peridynamic model converges to the classical model in the limit of small horizon. Starting with any peridynamic strain energy function $\hat{W}$, and defining a family of peridynamic materials by (14) for variable horizon while holding the bulk properties fixed, the limiting stress tensor is provided by (39). This is a local stress-strain relation. The stress tensor is obtainable from the tensor gradient of the strain energy density function defined by (47). The resulting stress field $\boldsymbol{\nu}^0$ satisfies the classical equation of motion,

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \nabla \cdot \boldsymbol{\nu}^0(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t).$$

As shown in Section 7, the stress-strain relation (39) satisfies the conditions on the Piola-Kirchhoff stress in the classical theory for angular momentum balance, isotropy, and objectivity, provided the underlying peridynamic model meets these conditions.

If the assumptions (i) and (ii) fail to be satisfied, *i.e.,* if either the deformation fails to be twice continuously differentiable, or if the peridynamic constitutive model fails to be continuously differentiable, then the conclusions regarding convergence to a classical model in the limit of small horizon fail to hold. In this case, the peridynamic equations continue to be applicable at any finite horizon, but convergence properties in the limit of zero horizon are undetermined.

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