IQI 04, Seminar 3

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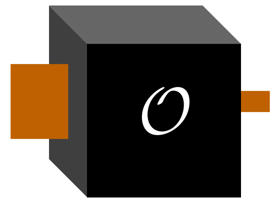
- Oracles
- The Classical Parity Problem.
- Quantum Oracles.
- The Quantum Parity Problem.
- Gate Set Limitations.
- Universality.

E. "Manny" Knill: knill@boulder.nist.gov

Classical Oracles

- A classical oracle $\mathcal{O}$ is a device that takes an input x and outputs an answer $\mathcal{O}(x)$.

x  $\mathcal{O}(x)$

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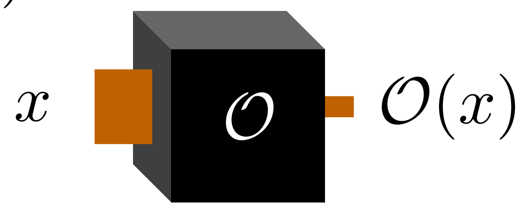

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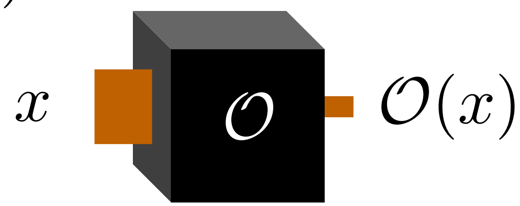

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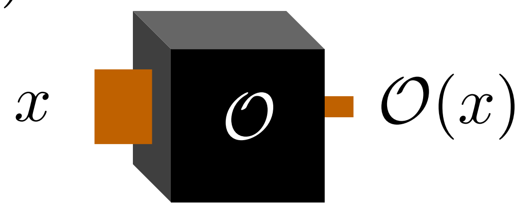

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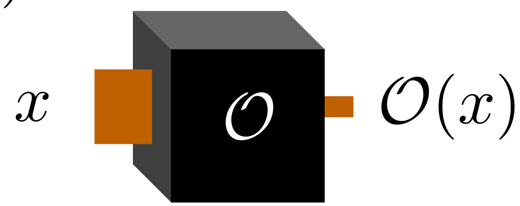

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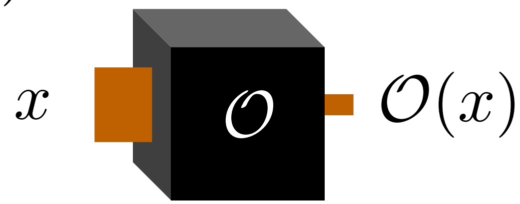

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...Oracles can act as black boxes to be analyzed.

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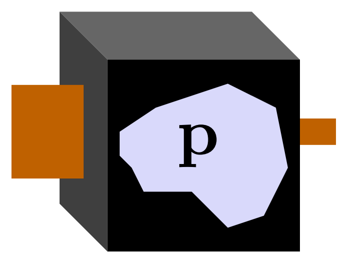
$\mathbf{s}$  $\mathbf{p} \cdot \mathbf{s}$

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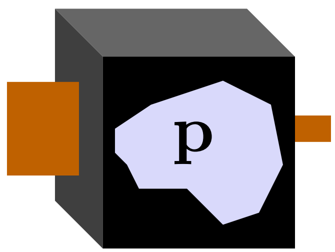
- A parity oracle. $(a,b)^T$ 

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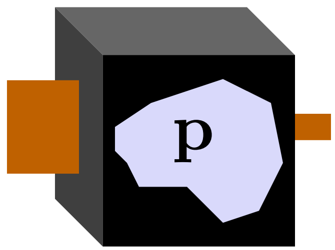

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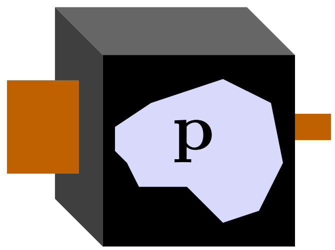

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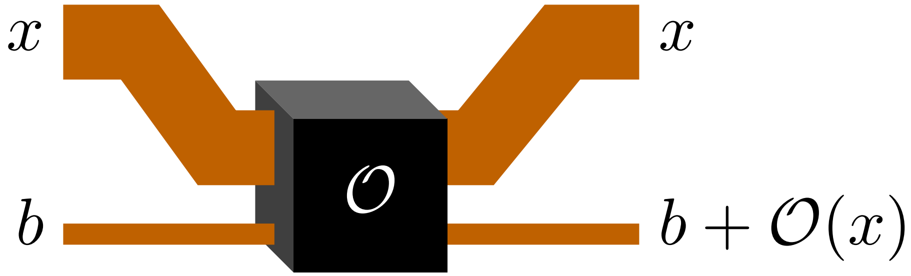

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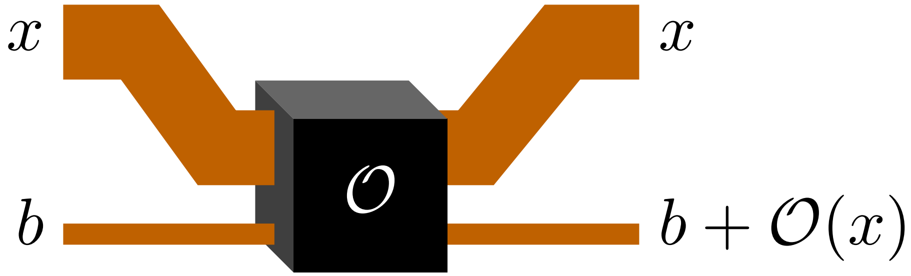
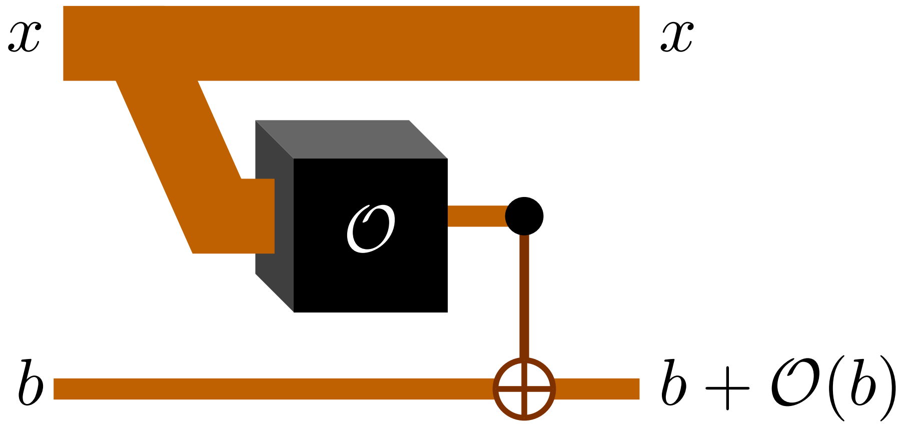


- Simulation, using a standard oracle.



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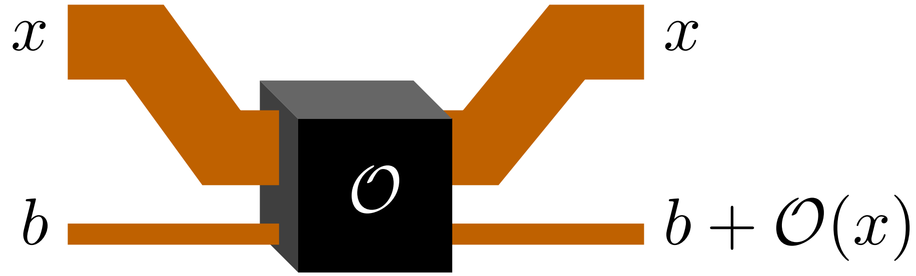

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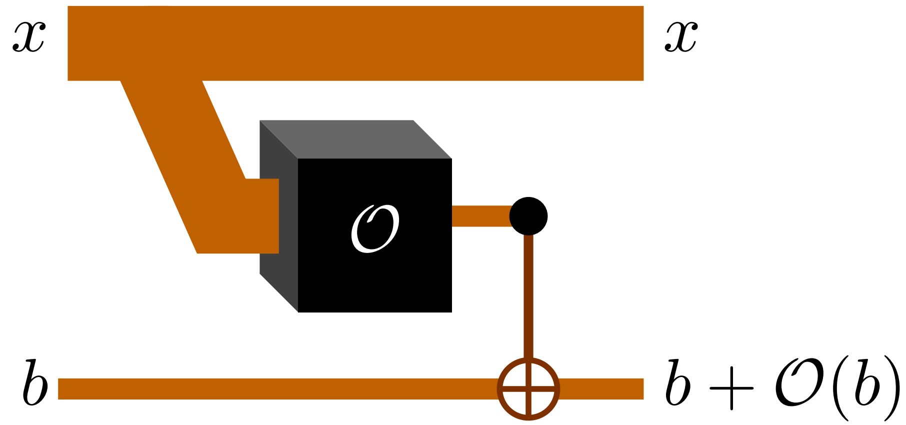

- Is the simulation equivalent to a reversible oracle?

Quantum Oracles

- A Quantum Oracle is the linear extension of a classical reversible oracle.

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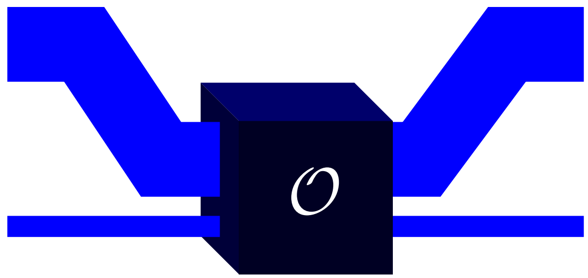

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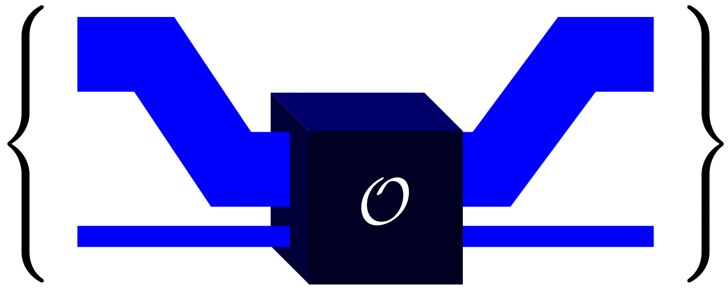

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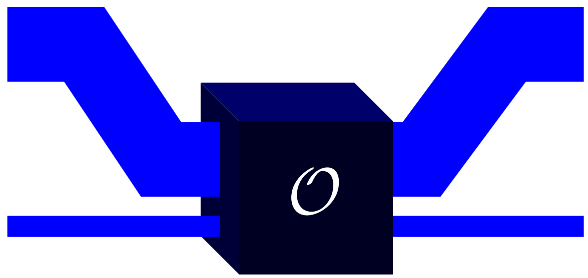

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- Quantum oracles versus classical reversible oracles?
  - Does it help to use a quantum computer to analyze a classical reversible oracle?

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  Problem: Determine the parity vector with one query.

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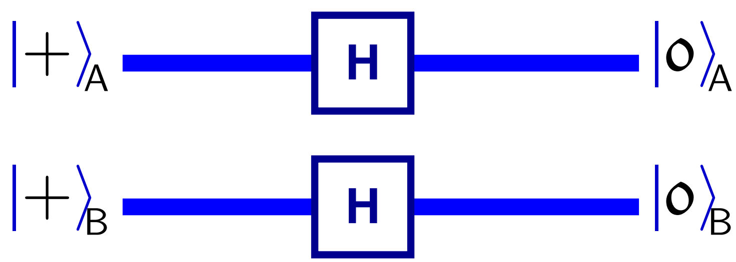

Product state convention:
Multiply states associated with different qubit lines.

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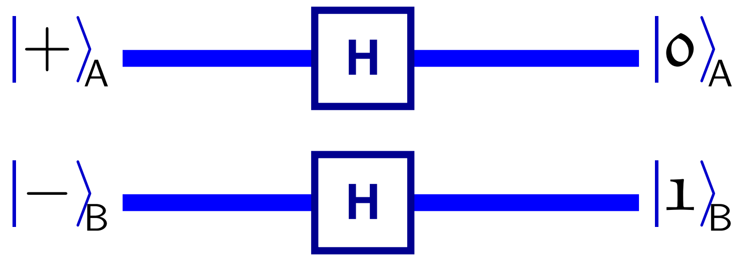

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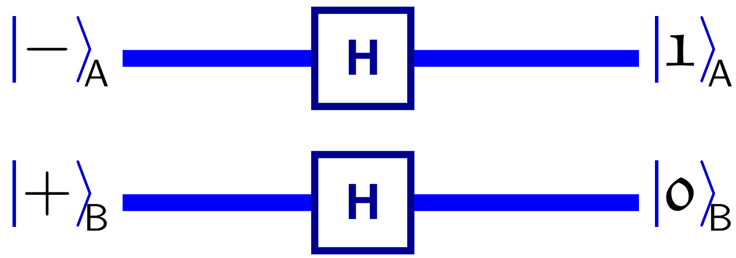

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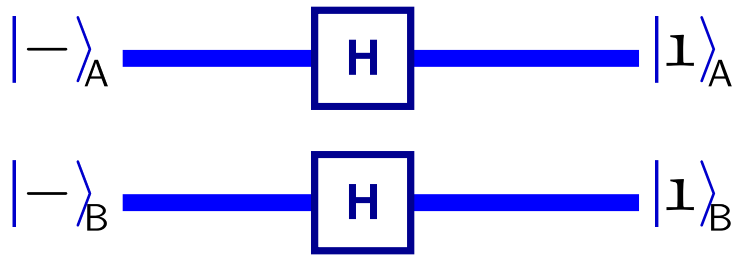
     - Ans.: States with odd parity w.r.t. the $|-\rangle$-qubits.
     - Are these states distinguishable?



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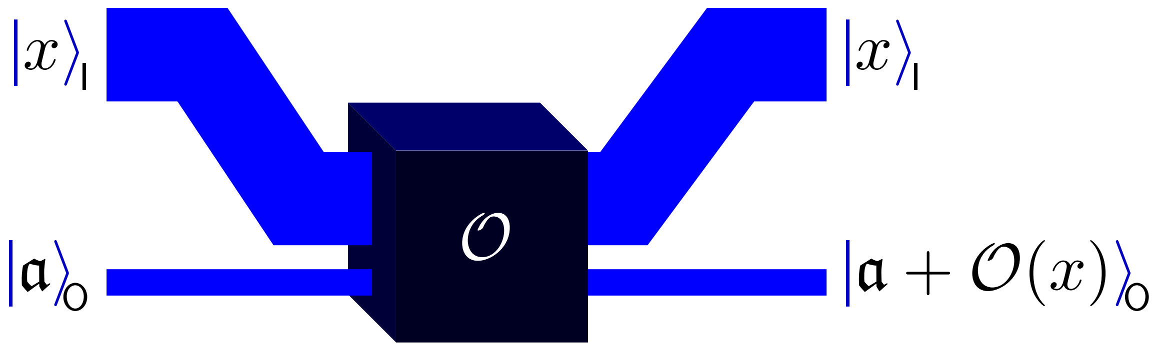

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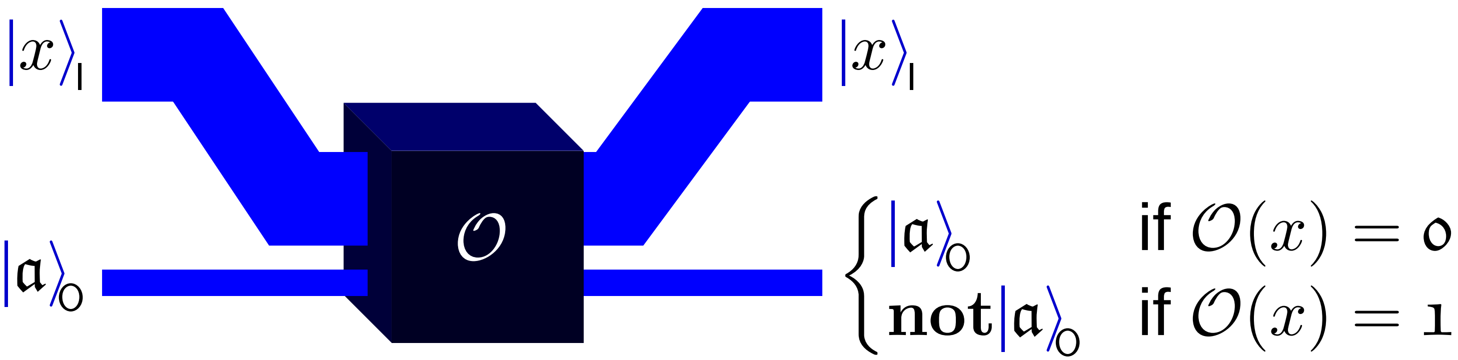

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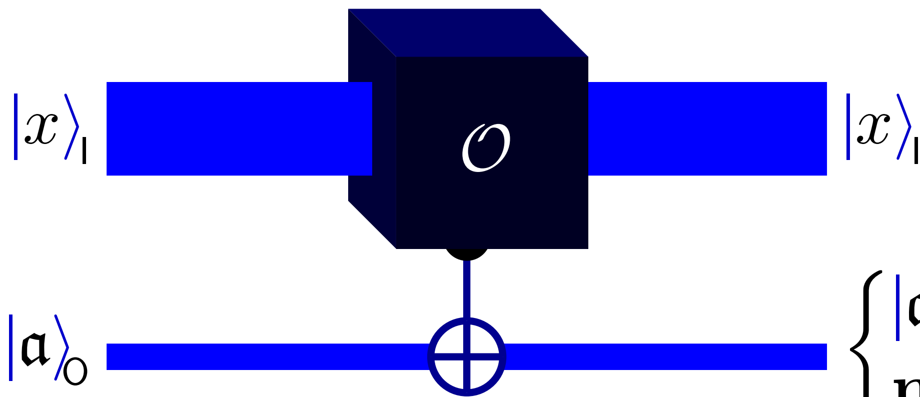

$$\begin{cases} |\mathfrak{a}\rangle_{\mathsf{0}} & \text{if } \mathcal{O}(x) = 0 \\ \mathbf{not}|\mathfrak{a}\rangle_{\mathsf{0}} & \text{if } \mathcal{O}(x) = 1 \end{cases}$$

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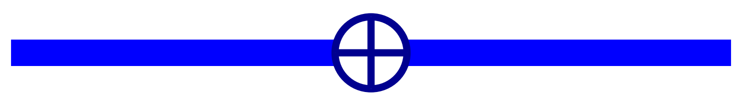

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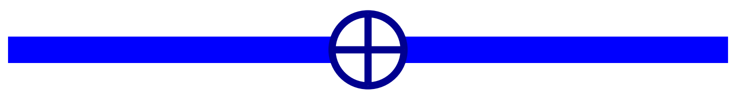

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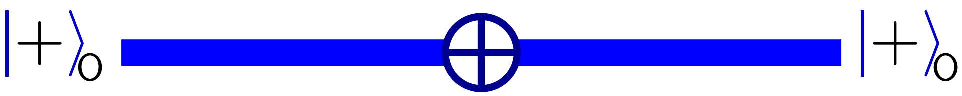

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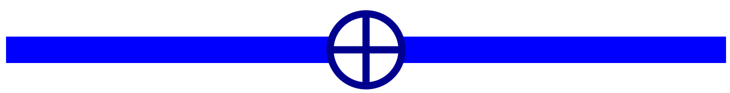

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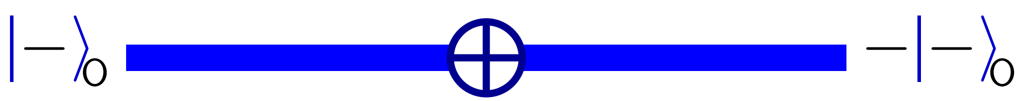

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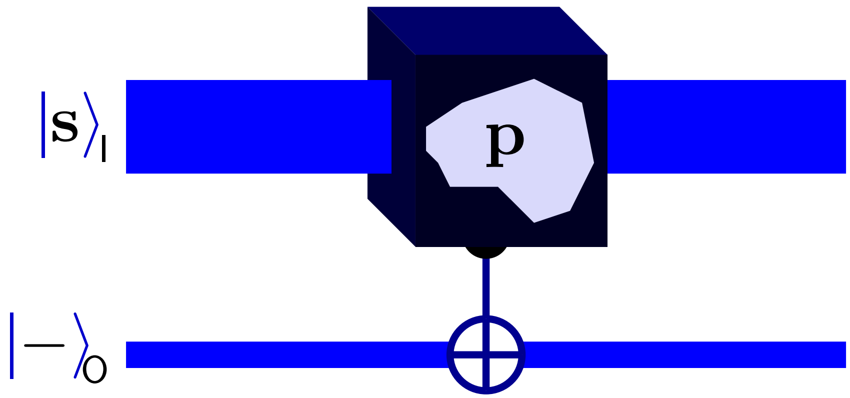

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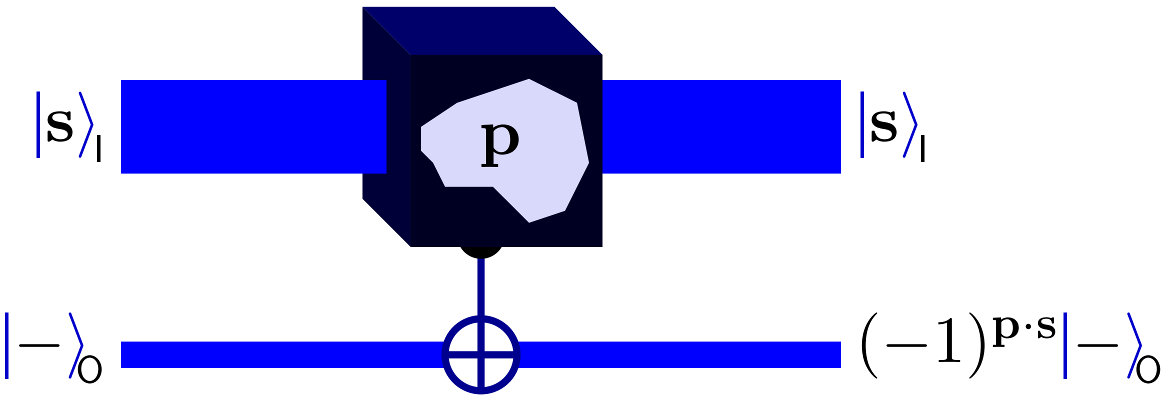

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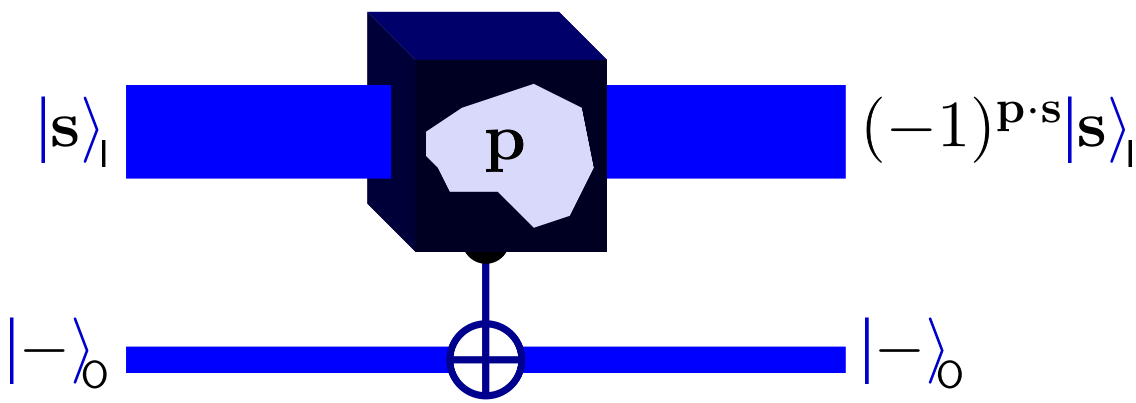

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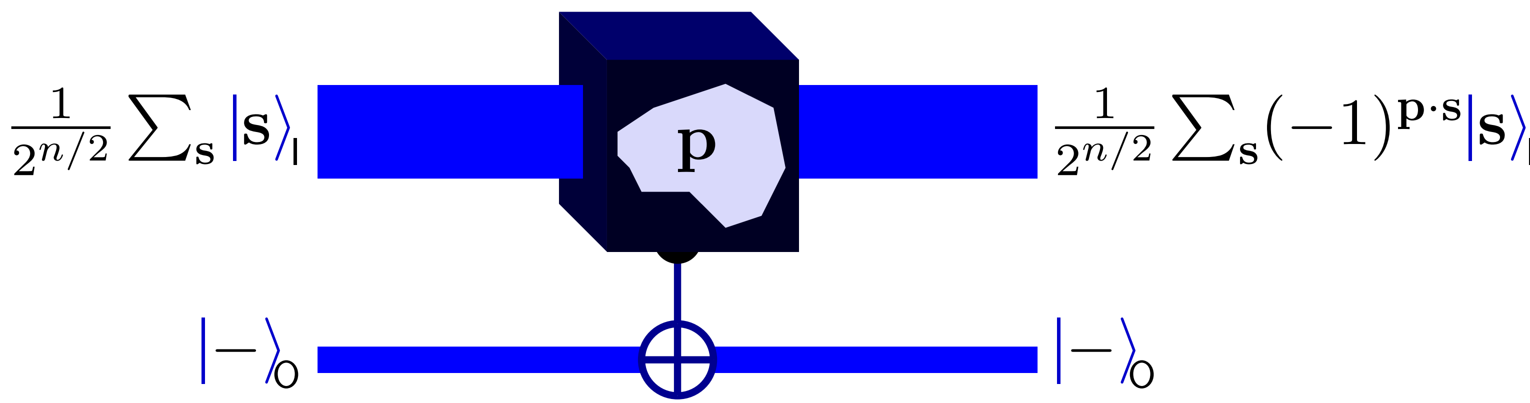

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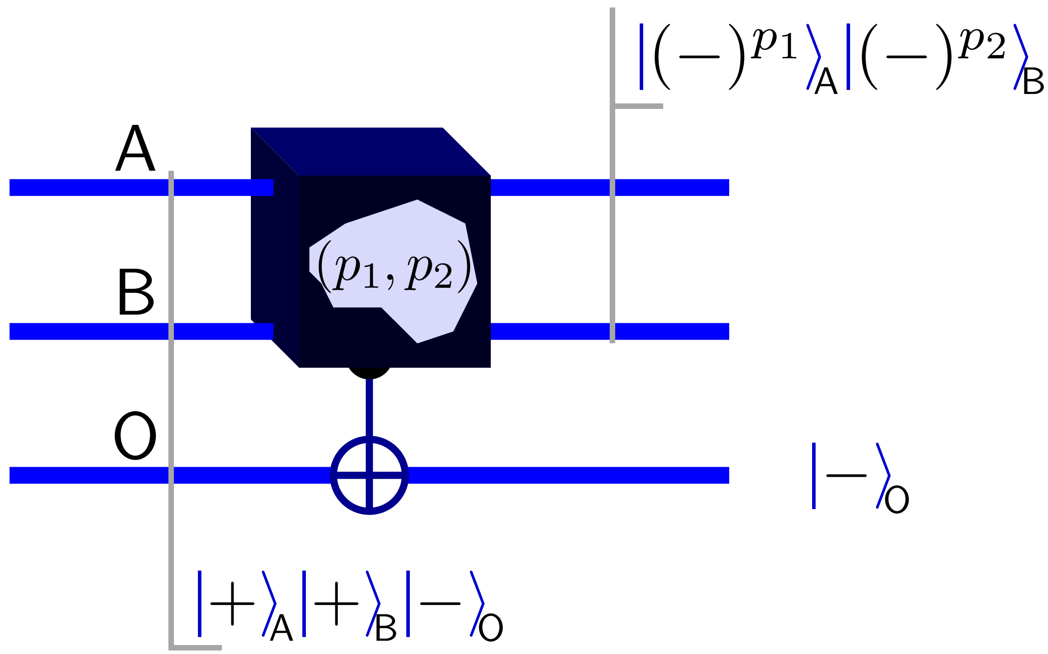

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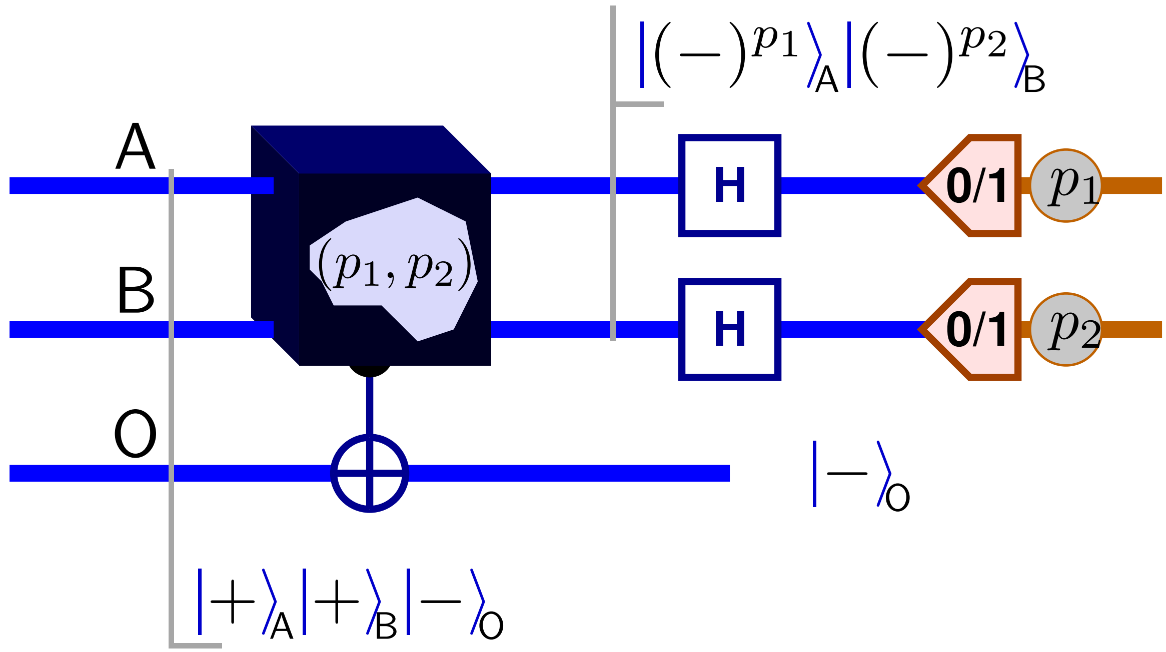

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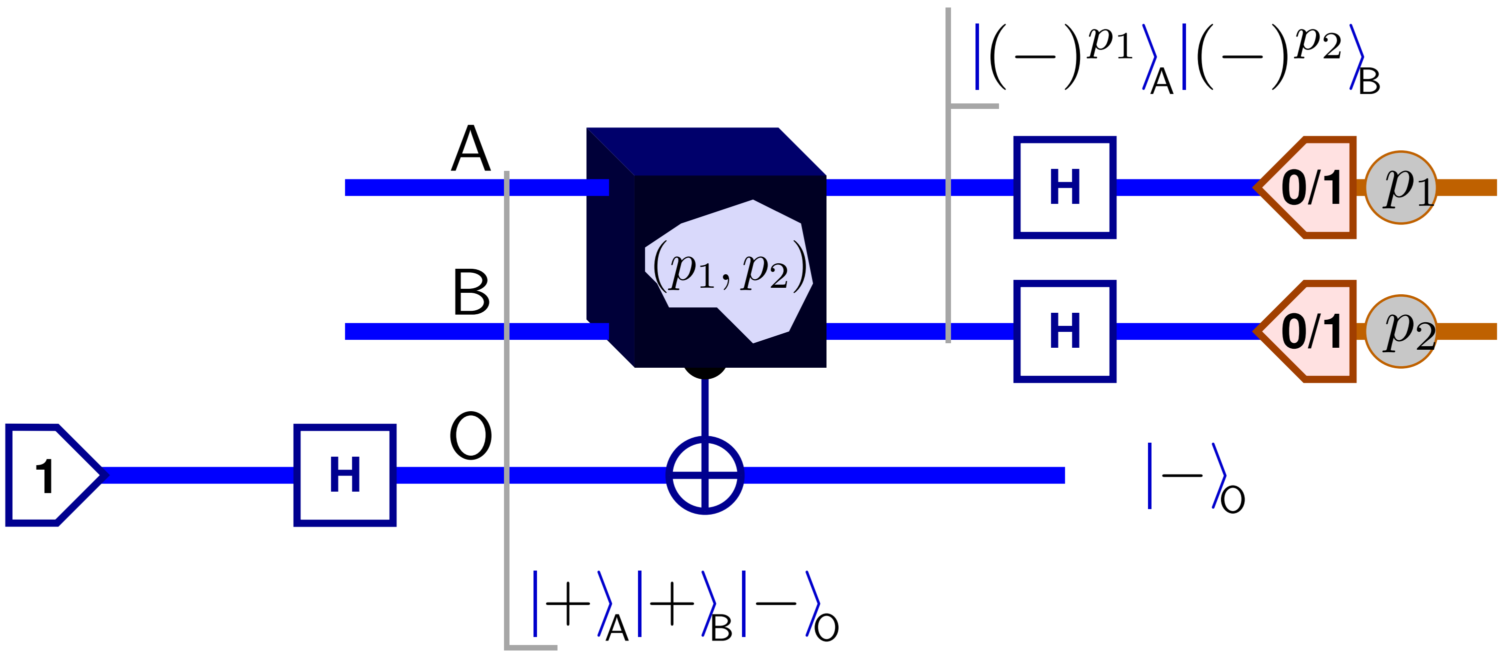

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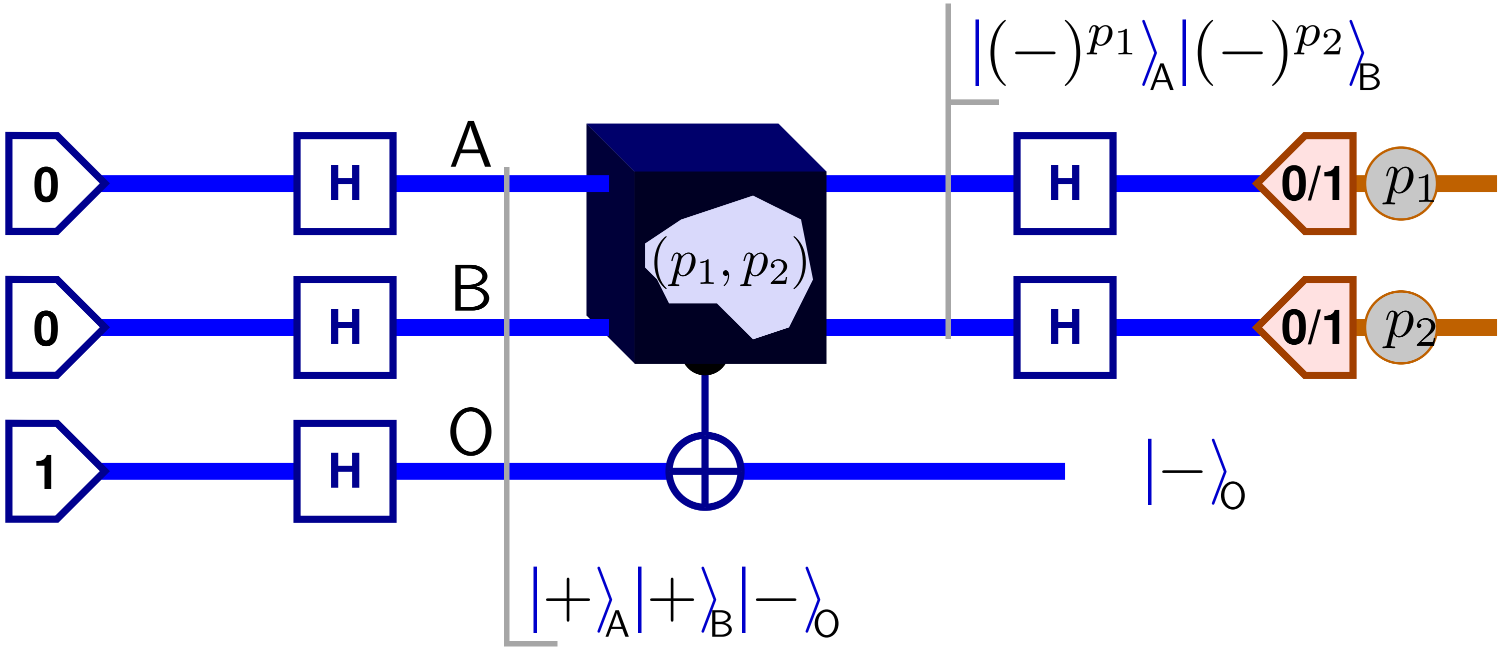

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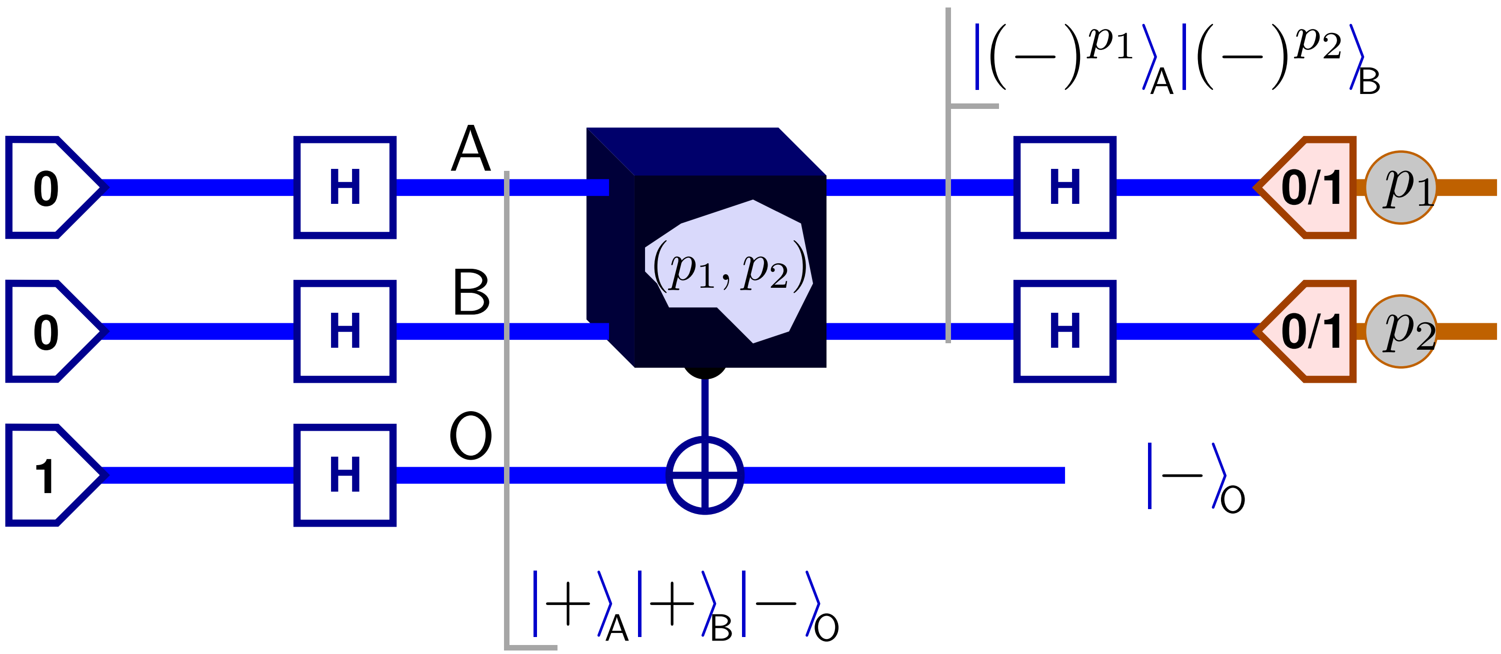

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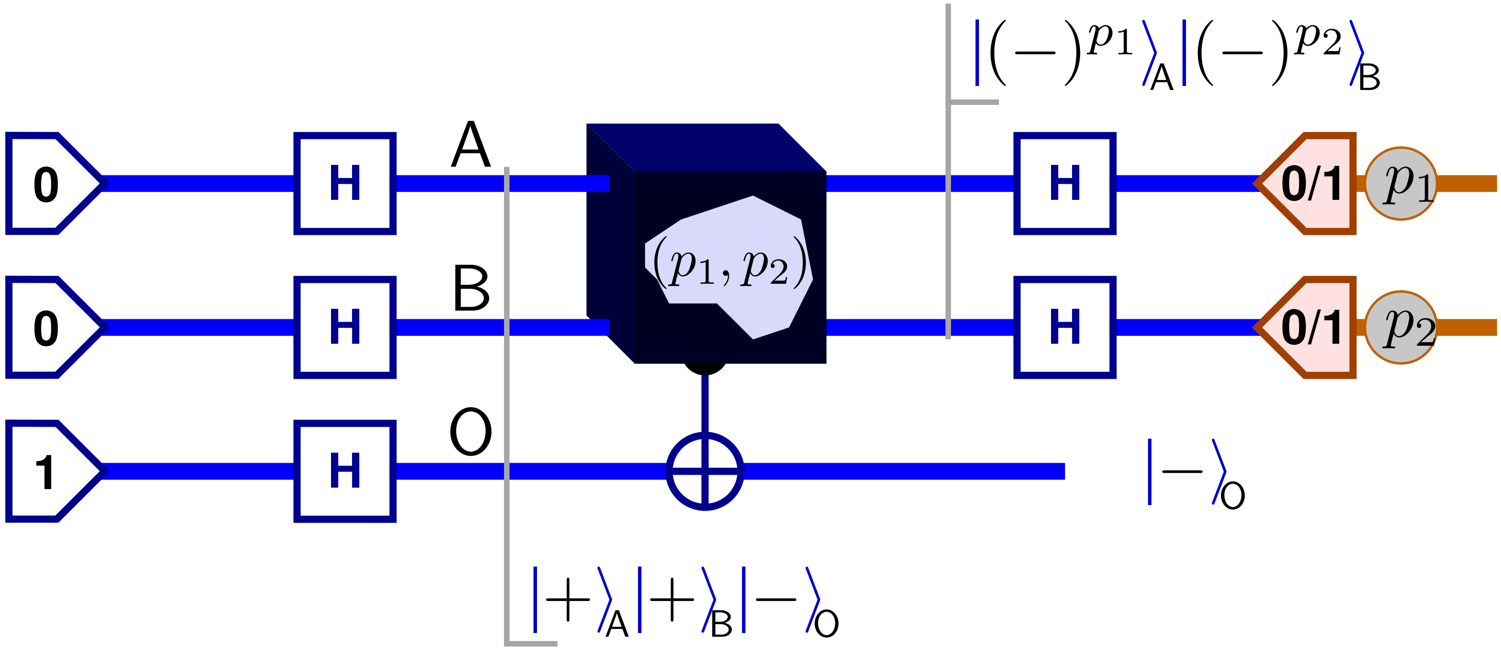

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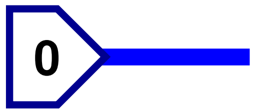
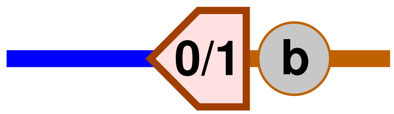
. . . note use of “quantum parallelism”.

Summary of Gates Introduced So Far

Gate picture	Symbol	Matrix form
	$\mathbf{prep}(0)$	
	$\mathbf{meas}(Z\mapsto b)$	

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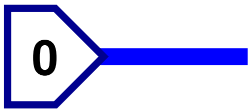
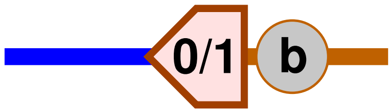
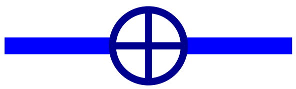
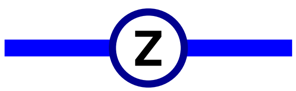
	$\mathbf{prep}(\mathtt{0})$	
	$\mathbf{meas}(Z\mapsto b)$	
	$\mathbf{not}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	$\mathbf{sgn}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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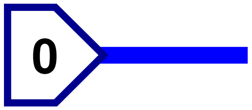
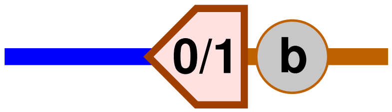
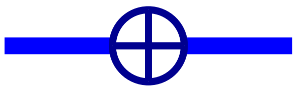
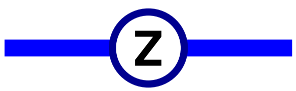
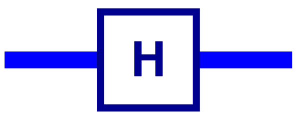
	$\mathbf{prep}(0)$	
	$\mathbf{meas}(Z\mapsto b)$	
	$\mathbf{not}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	$\mathbf{sgn}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
	$\mathbf{had}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

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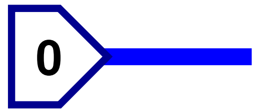
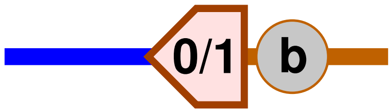
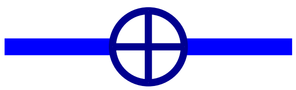
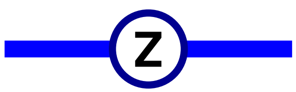
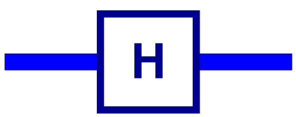
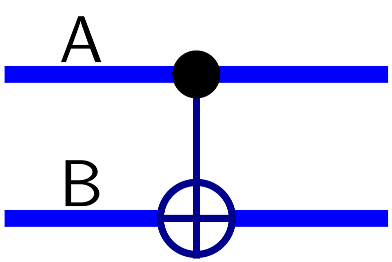
	$\mathbf{prep}(0)$	
	$\mathbf{meas}(Z \mapsto b)$	
	$\mathbf{not}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	$\mathbf{sgn}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
	$\mathbf{had}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
	$\mathbf{cnot}^{(\mathsf{AB})}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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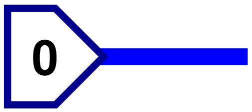
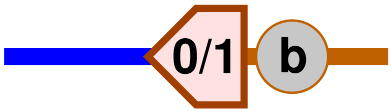
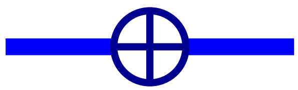
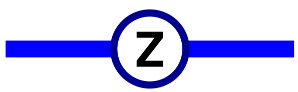
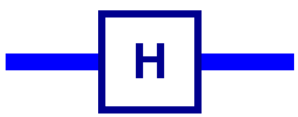
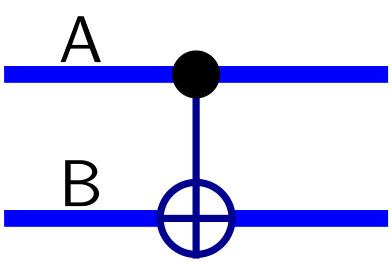
	**prep**(0)	
	**meas**$(Z{\mapsto}b)$	
	**not**	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	**sgn**	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
	**had**	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
	**cnot**$^{(\mathrm{AB})}$	$\begin{array}{c} \\ \lvert 00\rangle_{\mathrm{AB}} \\ \lvert 01\rangle_{\mathrm{AB}} \\ \lvert 10\rangle_{\mathrm{AB}} \\ \lvert 11\rangle_{\mathrm{AB}} \end{array} \begin{array}{c} \begin{array}{cccc} \lvert 00\rangle_{\mathrm{AB}} & \lvert 01\rangle_{\mathrm{AB}} & \lvert 10\rangle_{\mathrm{AB}} & \lvert 11\rangle_{\mathrm{AB}} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$

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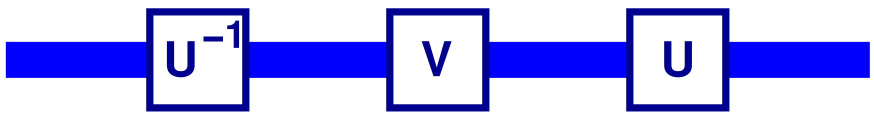

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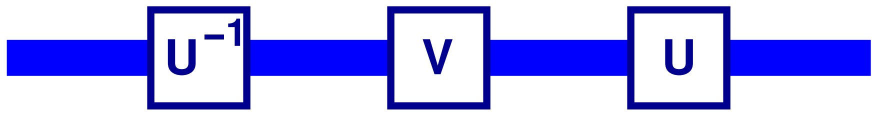

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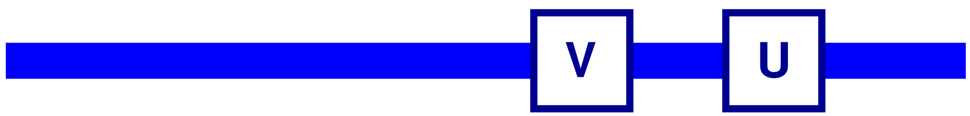

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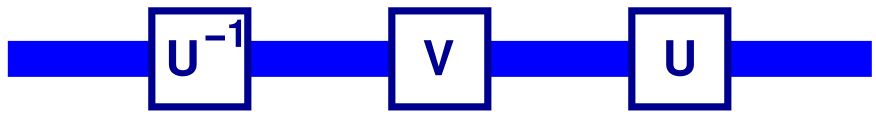

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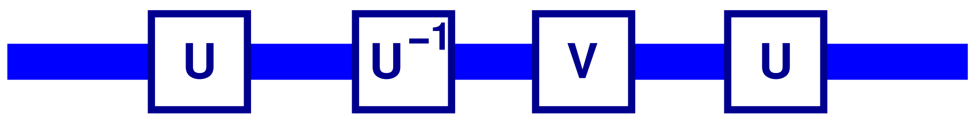

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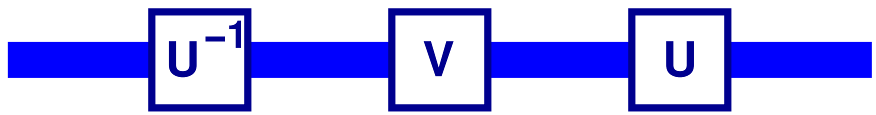

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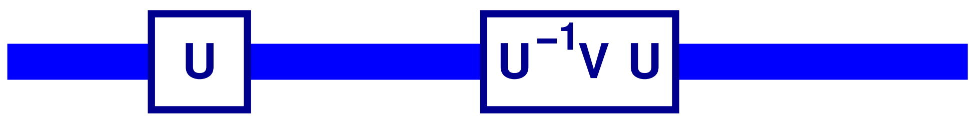

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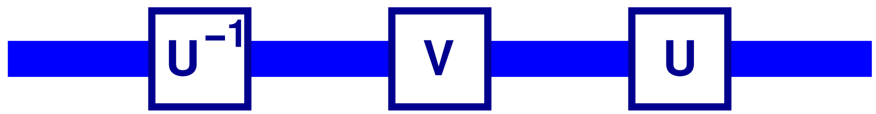

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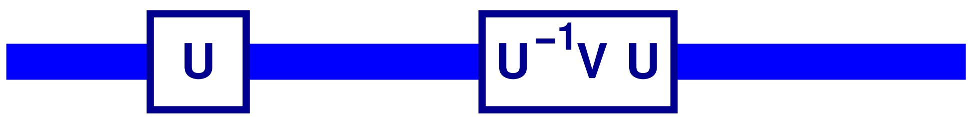

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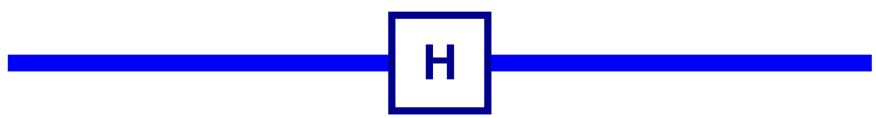

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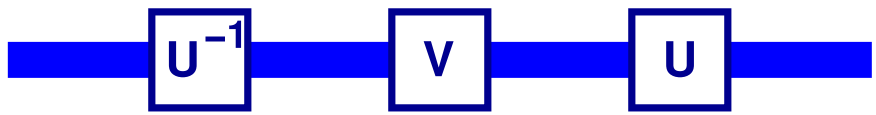

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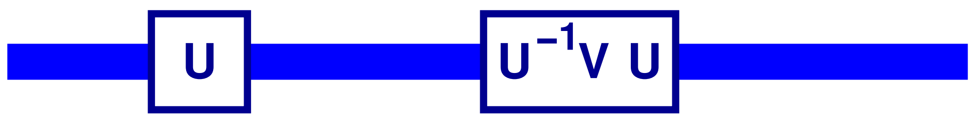

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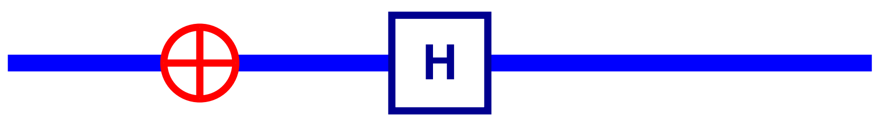

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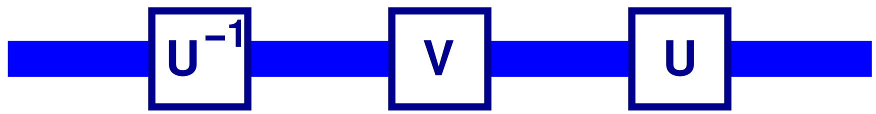

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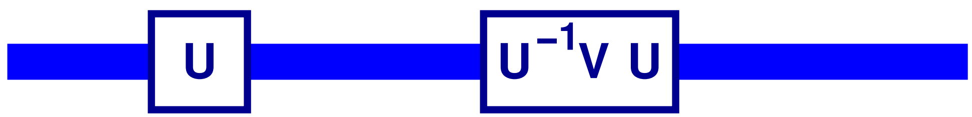

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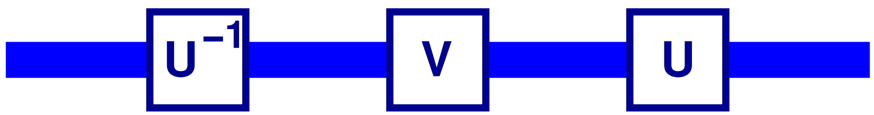

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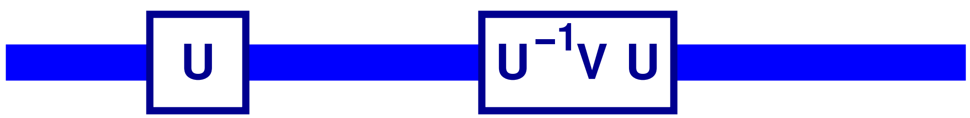

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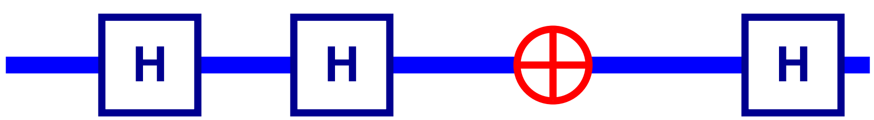

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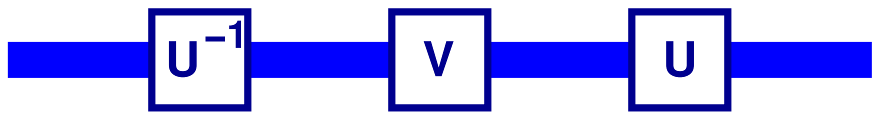

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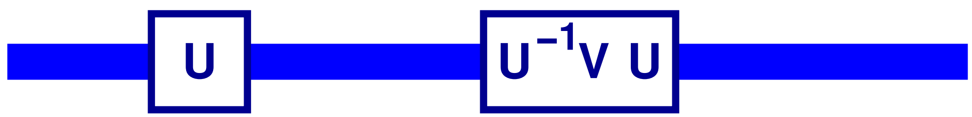

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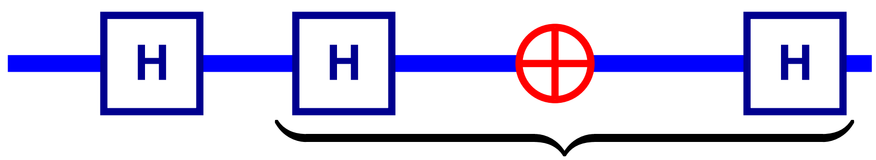

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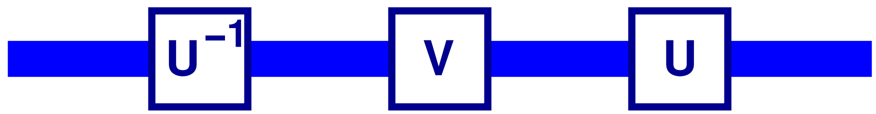

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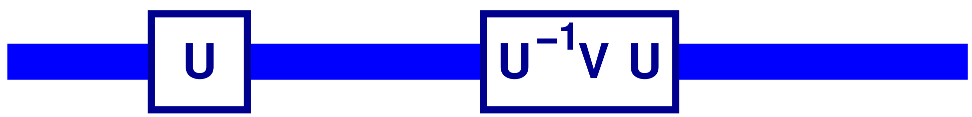

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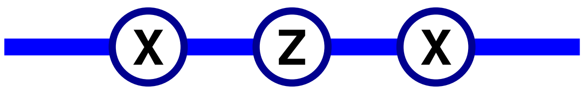

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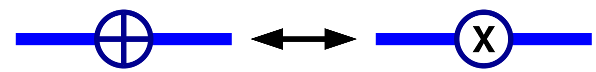

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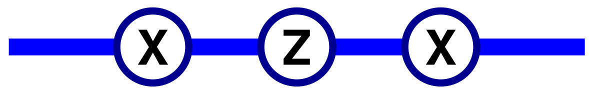

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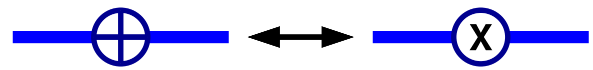

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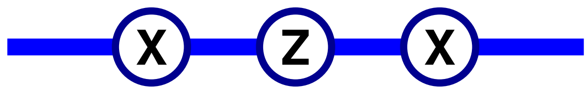

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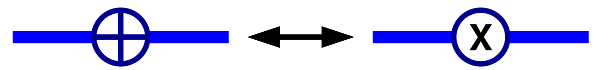

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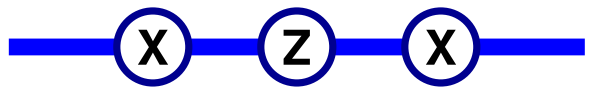

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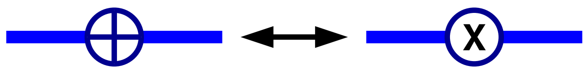

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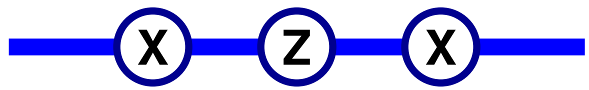

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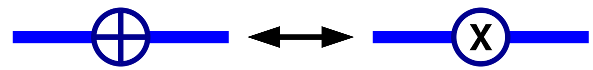

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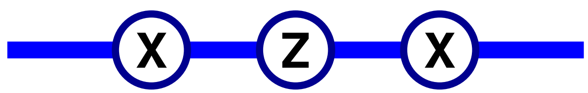

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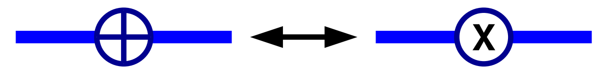

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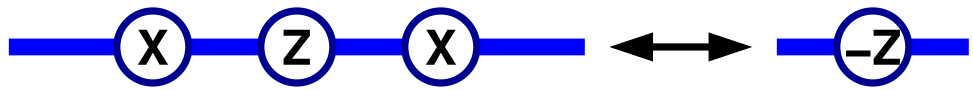

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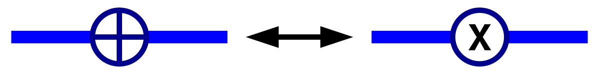

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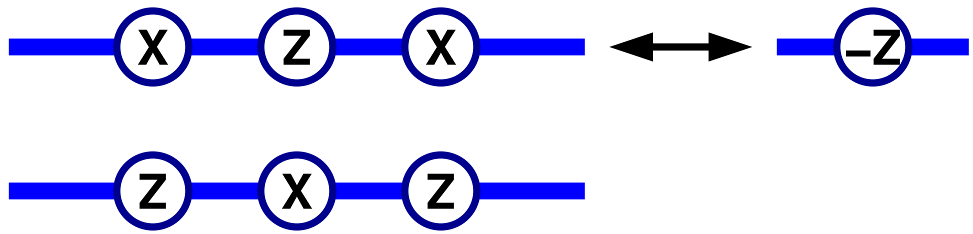

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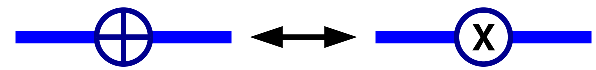

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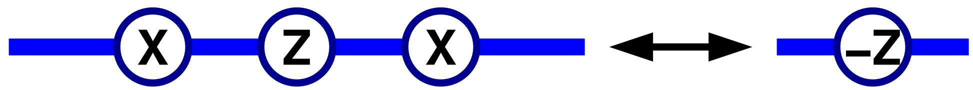

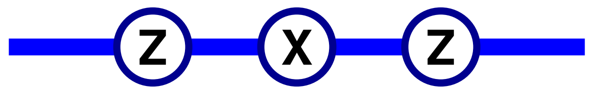

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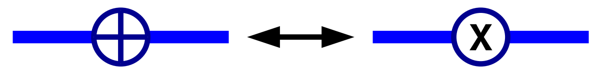

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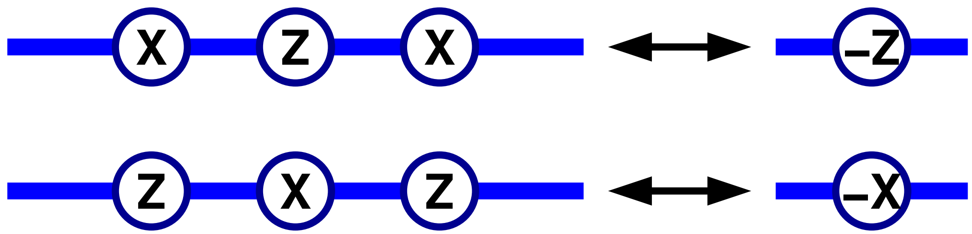

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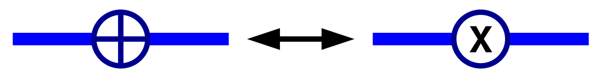

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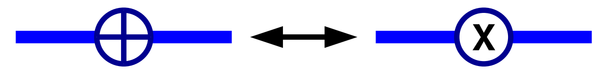

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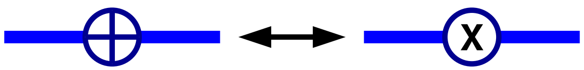

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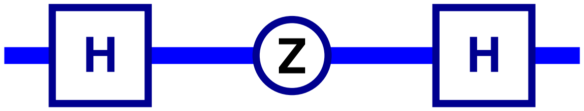

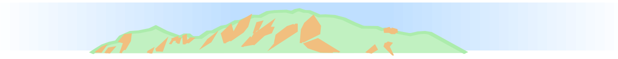

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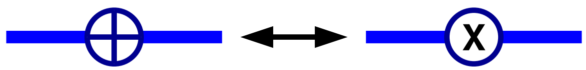

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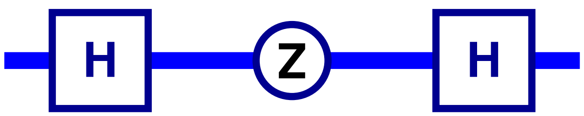

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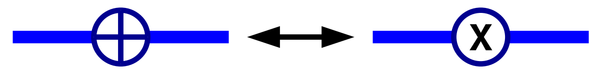

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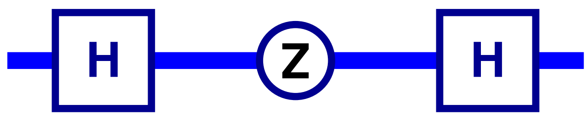

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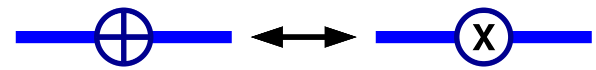

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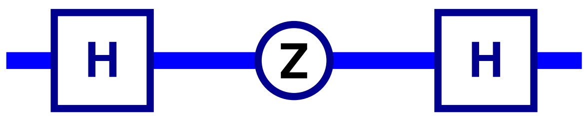

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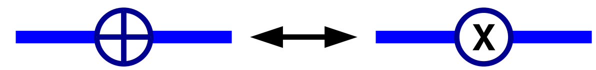

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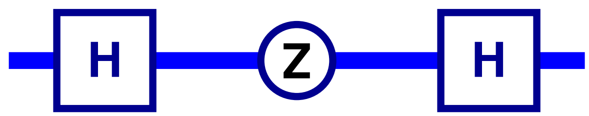

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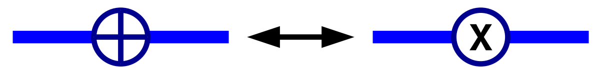

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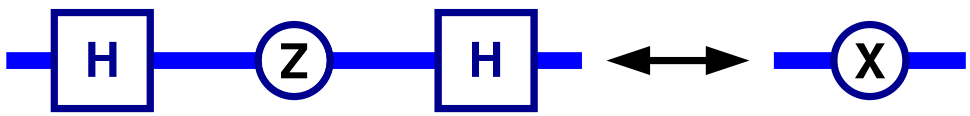

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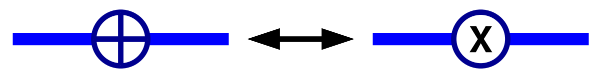

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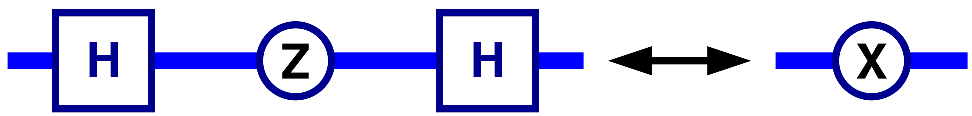

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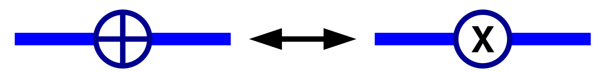

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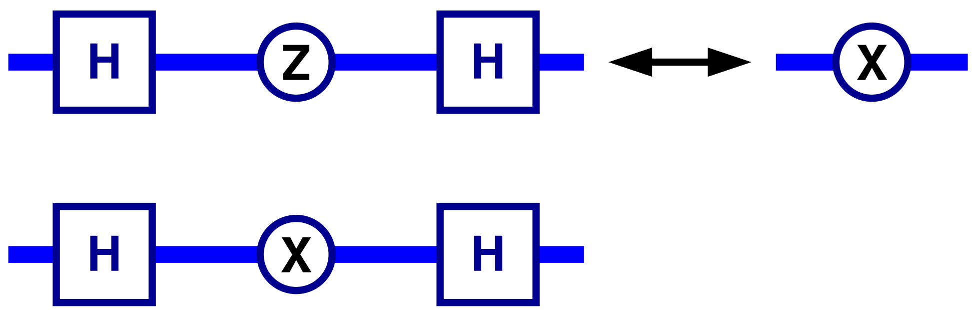

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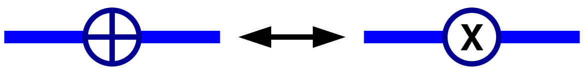

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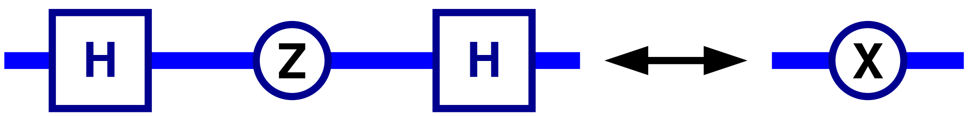

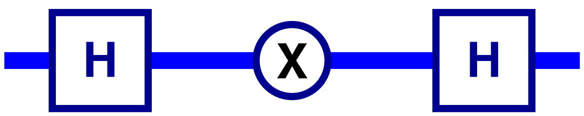

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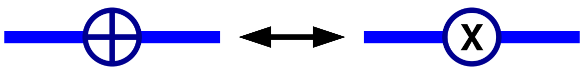

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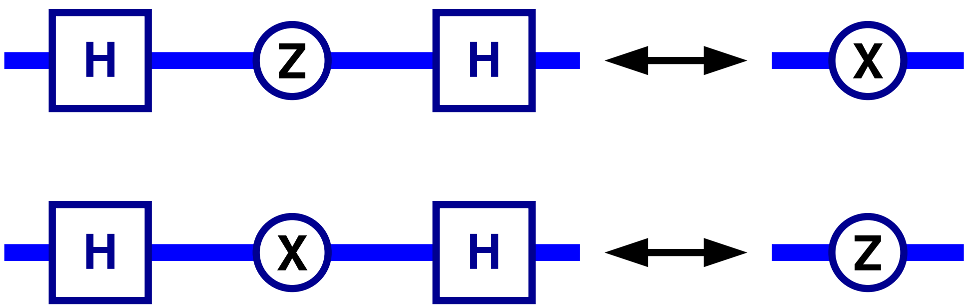

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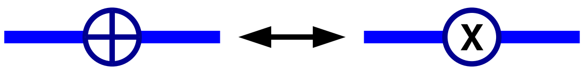

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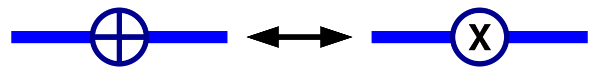

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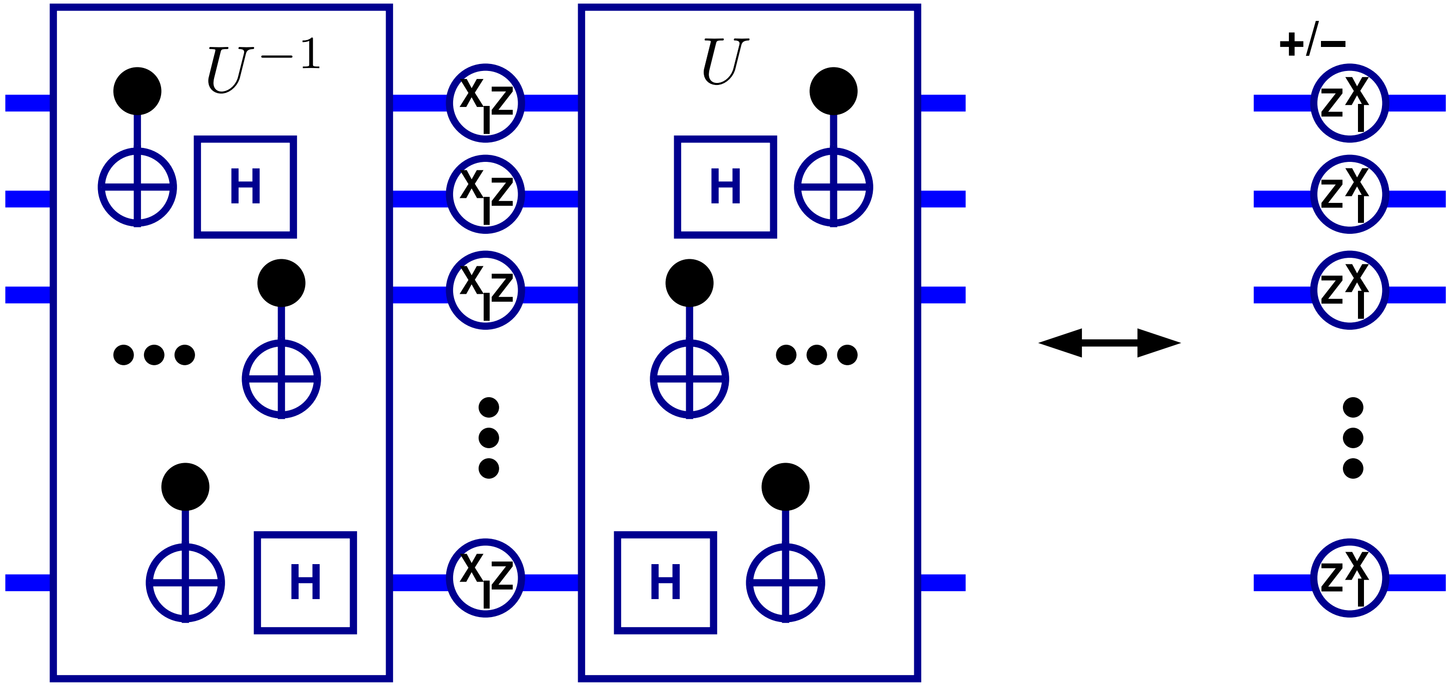

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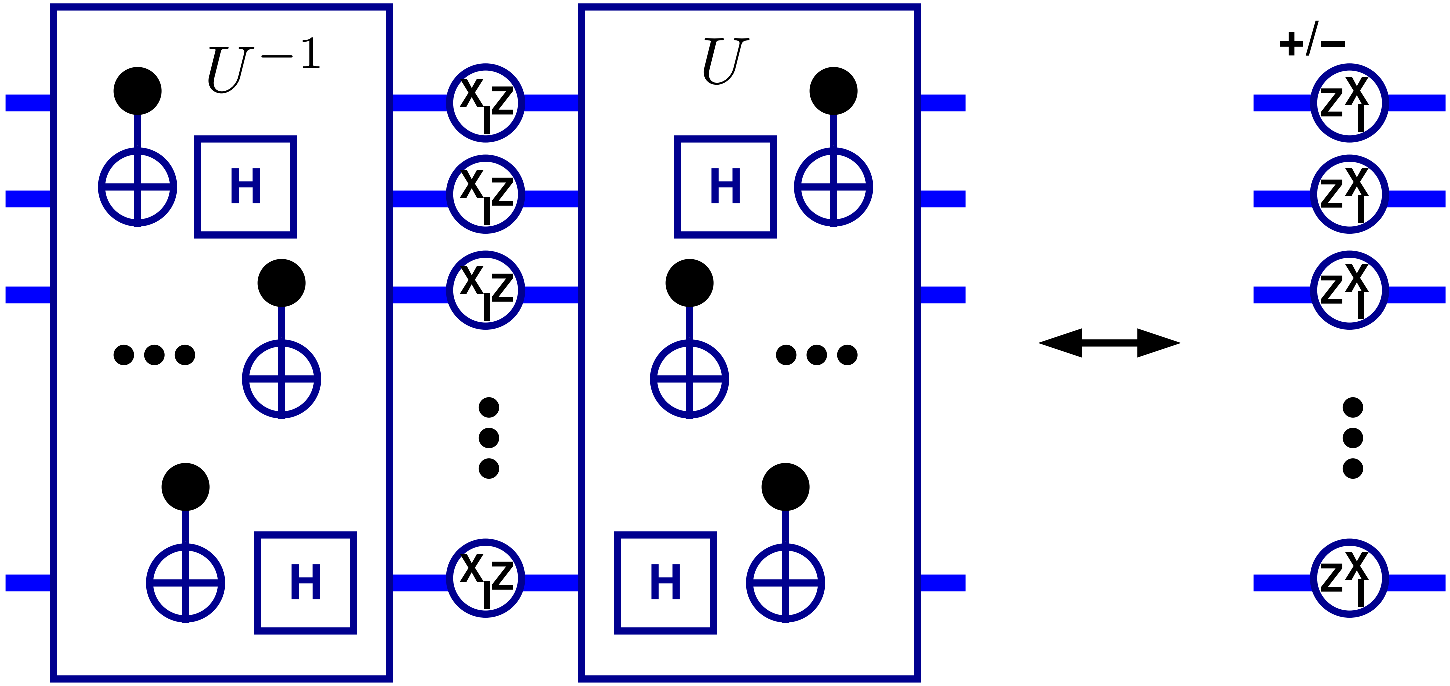

  - What is the power of this gate set?

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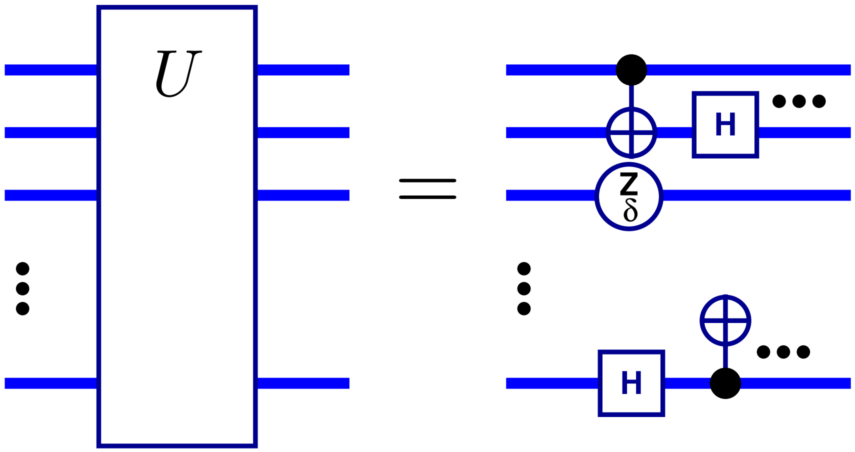

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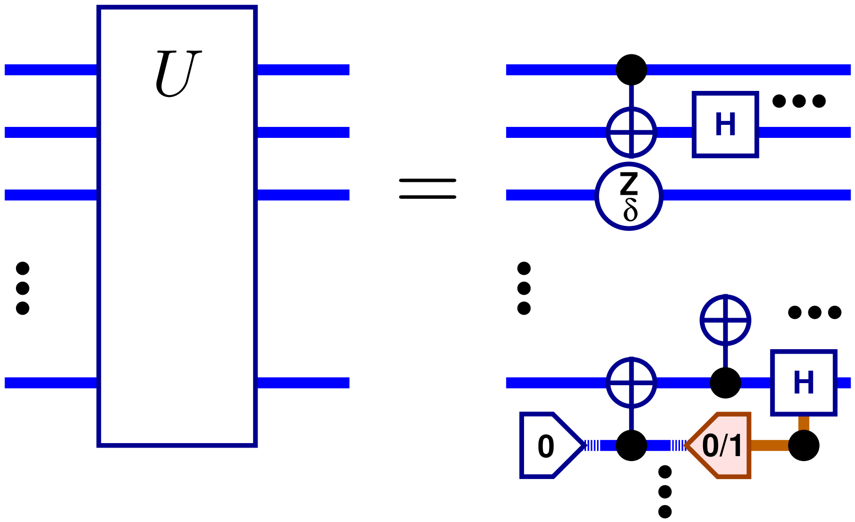

- Other notions of universality:
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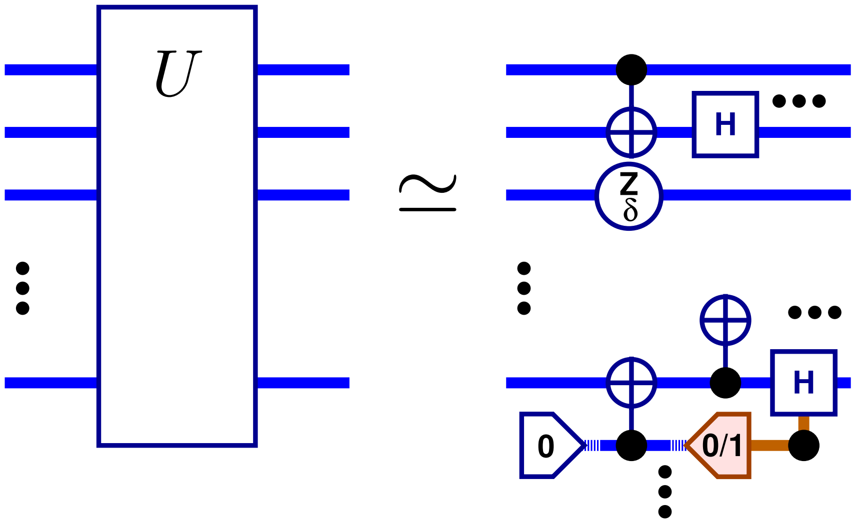
- A set of gates is *universal* if every unitary n-qubit operator can be implemented with a network.



- Other notions of universality:
  - Allow use of ancillas and measurements.
  - Allow approximation to within arbitrarily small error.

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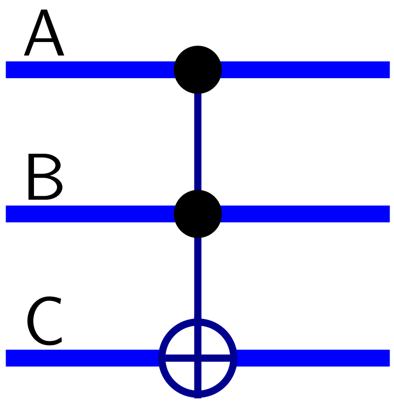

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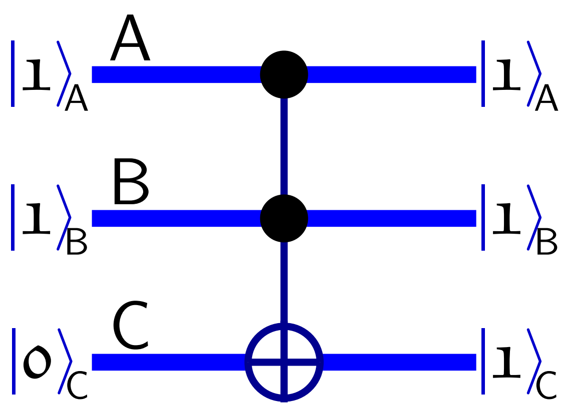

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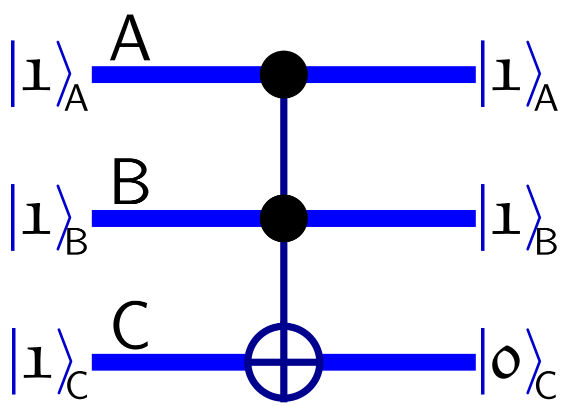

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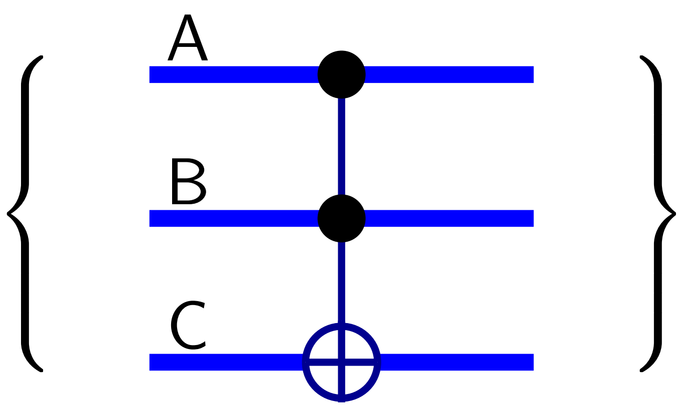

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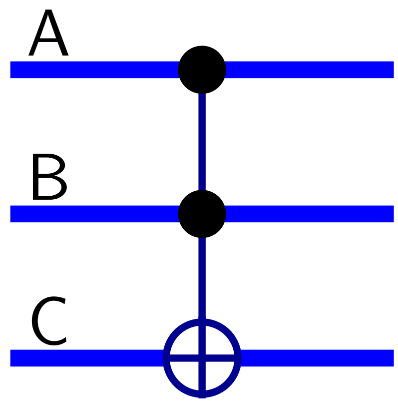

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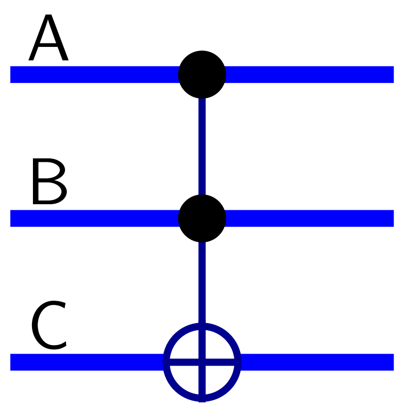

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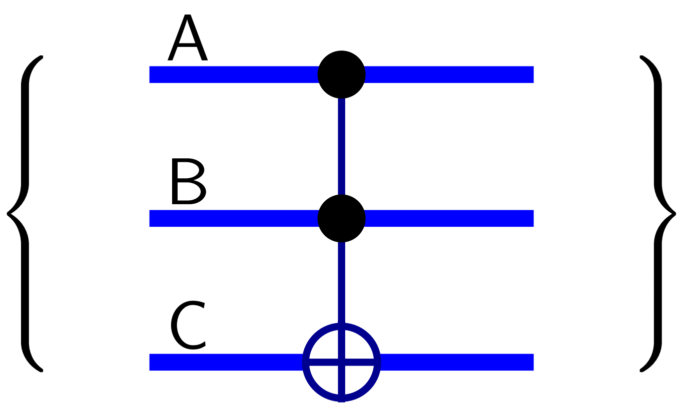

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- . . . but do investigate other gate sets.

Contents

References

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