Counting Independent Sets up to the Tree Threshold

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What is this work about?

Novel exact tree representation for the marginal probability at a vertex in any binary spin system.

➡ The regular tree is the worst-case graph for an appropriate notion of spatial decay of correlations (Strong Spatial Mixing).

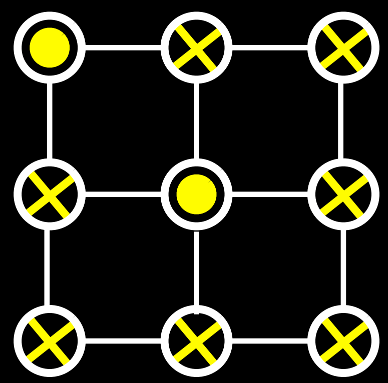
➡ New efficient algorithm for approximating marginals (and hence the partition function) in the regime where the regular tree exhibits SSM.

- Strong application: hard-core model (independent sets).

The Hard-Core Model (Independent Sets)

- Count/sample weighted independent sets of a graph G.
- Weights are determined by an activity parameter λ:

$$w(I) = \lambda^{|I|}$$



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- Unoccupied vertex

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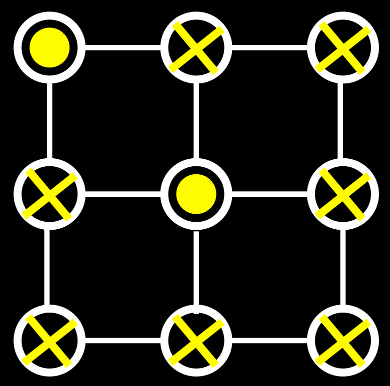

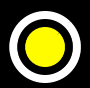 - Occupied vertex

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Computational Problem

- Aim: $(1+\epsilon)$-approximation of the partition function -

$$Z \equiv Z_G^\lambda = \sum_I \lambda^{|I|}$$

Equivalently: approximately sample independent sets where $\Pr(I) = \lambda^{|I|}/Z$.

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- Intuitively, the problem becomes harder as λ grows.
(Sampling with large λ will output a maximum ind. set.)

Known bounds

- NP-hard to approximate Z within a polynomial factor for: max degree Δ and $\lambda \geq c/\Delta$, where c is a (large enough) constant. [Luby-Vigoda]

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  easy: $\lambda \leq \frac{1}{\Delta - 1}$ (Dobrushin's uniqueness condition)

  best: $\lambda \leq \frac{2}{\Delta - 2}$ [Dyer-Greenhill, Vigoda]

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- Finding out exact constants is important -

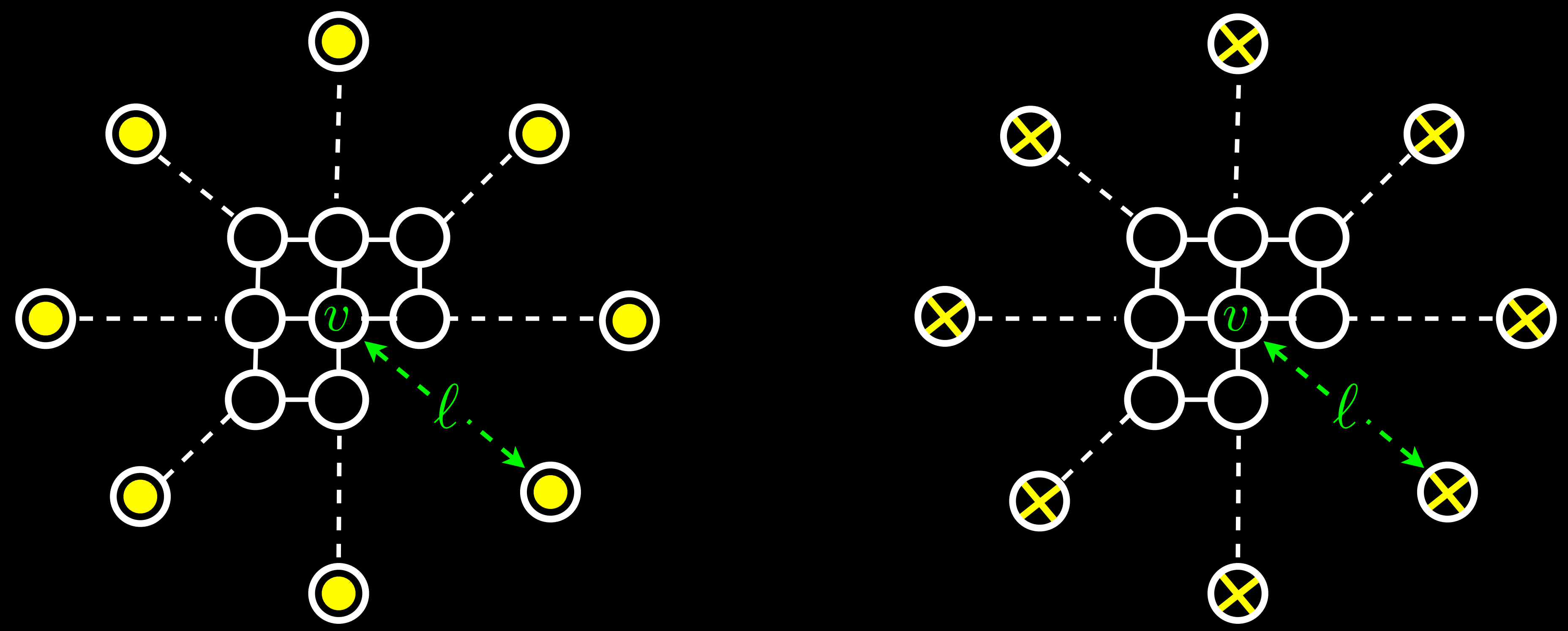
  most interesting graphs are low dimensional lattices.

Combinatorial Problem

For what values of λ is the 'Gibbs' measure unique?

uniqueness of Gibbs measure:

$$|\Pr(v \text{ is occupied} \,|\, \sigma_\ell) - \Pr(v \text{ is occupied} \,|\, \tau_\ell)| \underset{\ell\to\infty}{\longrightarrow} 0$$



Uniqueness for General Graphs

For what values of λ is there a decaying rate $\delta(\ell) \underset{\ell \to \infty}{\longrightarrow} 0$

such that for every graph G of maximum degree Δ

and every $v \in G$,

$$|\Pr(v \text{ is occupied} \,|\, \sigma_\ell) - \Pr(v \text{ is occupied} \,|\, \tau_\ell)| \;\le\; \delta(\ell)$$

?

Known Bounds

- Gibbs measure is unique on all graphs of maximum degree Δ for $\lambda < \frac{2}{\Delta-2}$. [Vigoda]

Same bound as the algorithmic one; uses essentially the same argument. (Part of a general correspondence between computational complexity and decay of correlations in the Gibbs distribution.)

Known Bounds

- Gibbs measure is unique on all graphs of maximum degree Δ for $\lambda < \frac{2}{\Delta-2}$. [Vigoda]

- On the Δ-regular tree, Gibbs measure is unique if and only if $\lambda \le \lambda_c = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \quad \left(\ge \frac{e}{\Delta-2}\right)$.

Algorithmic implications: although it is easy to count independent sets of the tree for arbitrary λ, arguments that imply uniqueness are bound to fail above λ_c .

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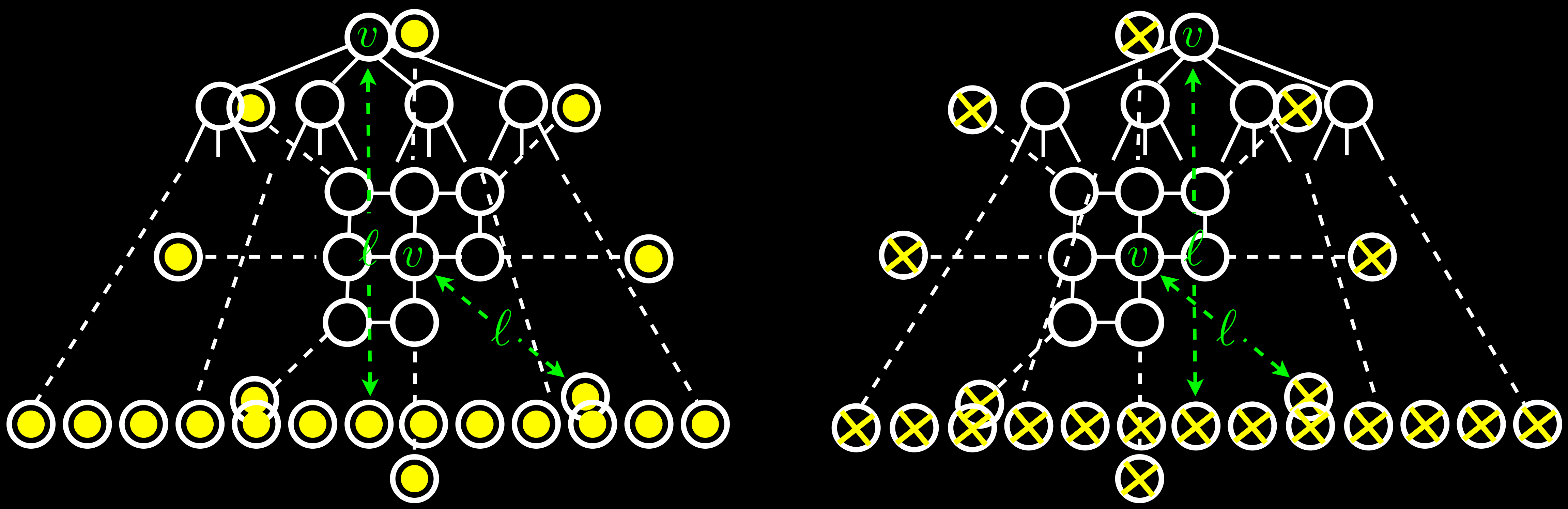
- Conjecture [Sokal]: the tree is the worst case - uniqueness on all graphs for $\lambda \leq \lambda_c$.

Main Result

Theorem: Fix Δ and λ. For a general graph G of maximum degree Δ, consider the influence of placing conditions at any given distance. This influence is maximized by taking G to be the regular tree.



Main Result

Theorem: Fix Δ and λ. For a general graph G of maximum degree Δ, consider the influence of placing conditions at any given distance. This influence is maximized by taking G to be the regular tree.

Corollary: The Gibbs measure is unique for all graphs of maximum degree Δ and $\lambda \leq \lambda_c = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$.

Algorithmic Implications

<u>**New algorithm**</u>: fix Δ and $\lambda < \lambda_c$; deterministic $(1+\epsilon)$-approximation for any graph of max degree Δ in time $\mathrm{poly}(n, 1/\epsilon)$.
(degree of poly depends on Δ and λ.)

<u>**Corollary:**</u> For all graphs of *'sub-exponential growth'* and $\lambda < \lambda_c$ the Glauber dynamics is rapidly mixing.
$(\Rightarrow \mathbf{FPRAS})$

Interesting Specific Cases

- Uniformly weighted independent sets ($\lambda = 1$):
  - New: efficient approximation scheme for $\Delta \leq 5$.
  - Previous bound is $\Delta \leq 4$.
  - Believed to be hard for $\Delta \geq 6$.
  - First <u>deterministic</u> approx scheme for #P-complete problem.

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  - First deterministic approx scheme for #P-complete problem.

- The sqaure lattice $\mathbb{Z}^2$:
  - Believed to have a critical activity at ~ 3.79.
  - Previously best known lower bound: 1.25 $\{1.45\}$ (site-perc.)
  - New bound: 1.6875.

Proof of Main Theorem

Theorem: Fix Δ and λ. For a general graph G of maximum degree Δ, consider the influence of placing conditions at any given distance. This influence is maximized by taking G to be the regular tree.

Part 1: prove the theorem when G is a general (irregular) tree.

In other words: on the regular tree SSM holds all the way up to the uniqueness threshold.

Tree Representation for General Graphs

Theorem: For every graph G and vertex $v \in G$ there exists a tree $T(G,v)$ of the same maximum degree such that

$$\Pr_G(v \text{ is occupied}) = \Pr_{T(G,v)}(\text{root is occupied}).$$

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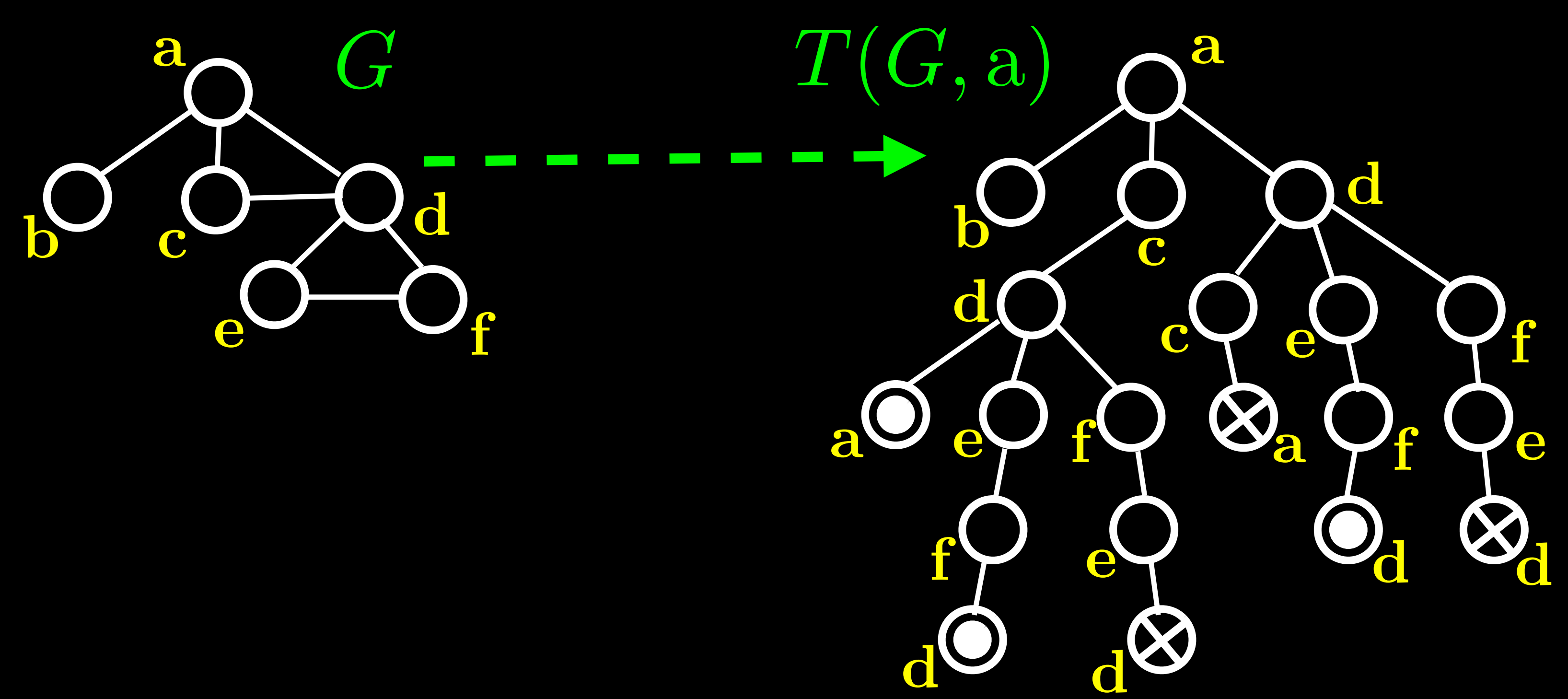
$$\Pr_G(v \text{ is occupied} \mid \sigma_\ell) \;=\; \Pr_{T(G,v)}(\text{root is occupied} \mid \widehat{\sigma_\ell}).$$

Furthermore, the correspondence (with the same tree) continues to hold when placing a condition on G (and a corresponding condition on $T(G,v)$).

Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at v:

- order the neighbors of each vertex;
- construct the tree of paths originating at v;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



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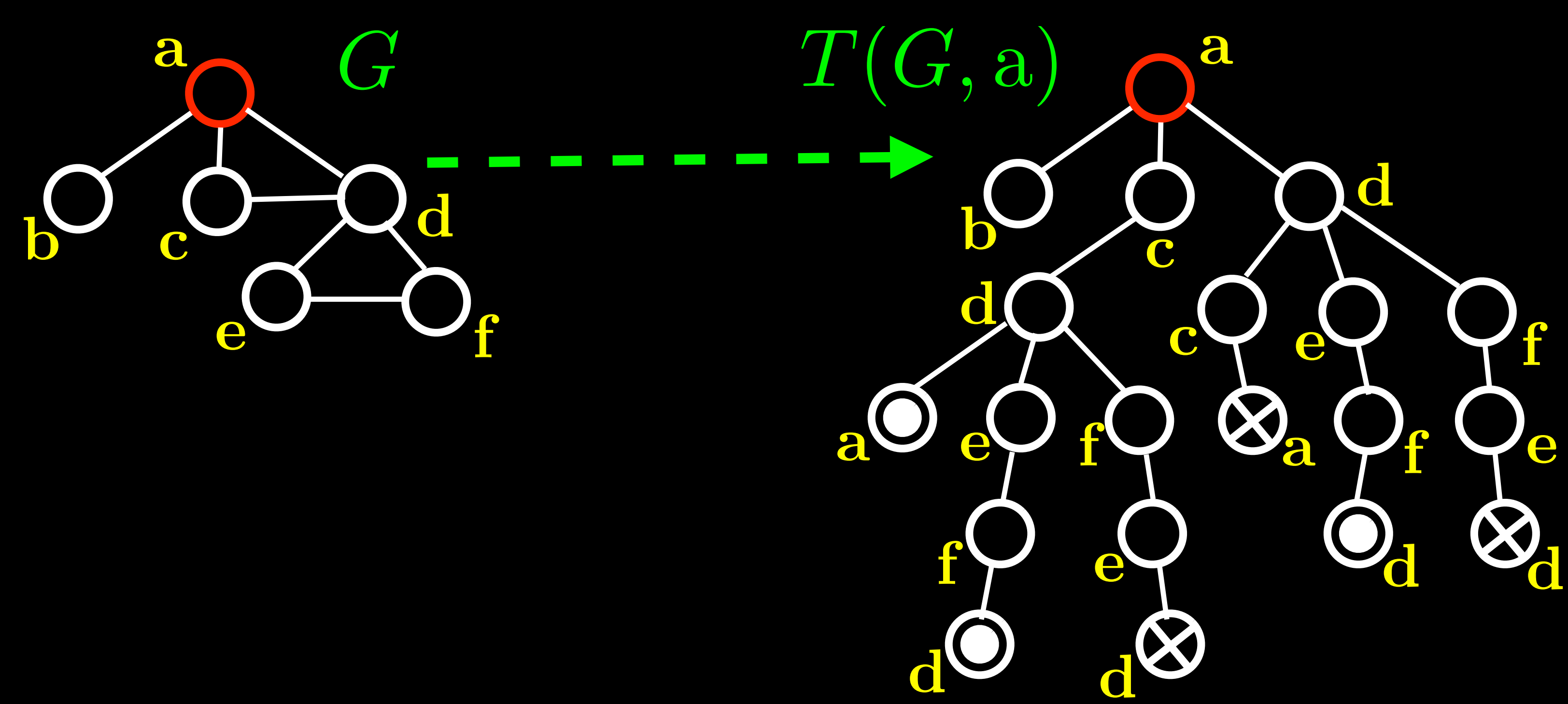

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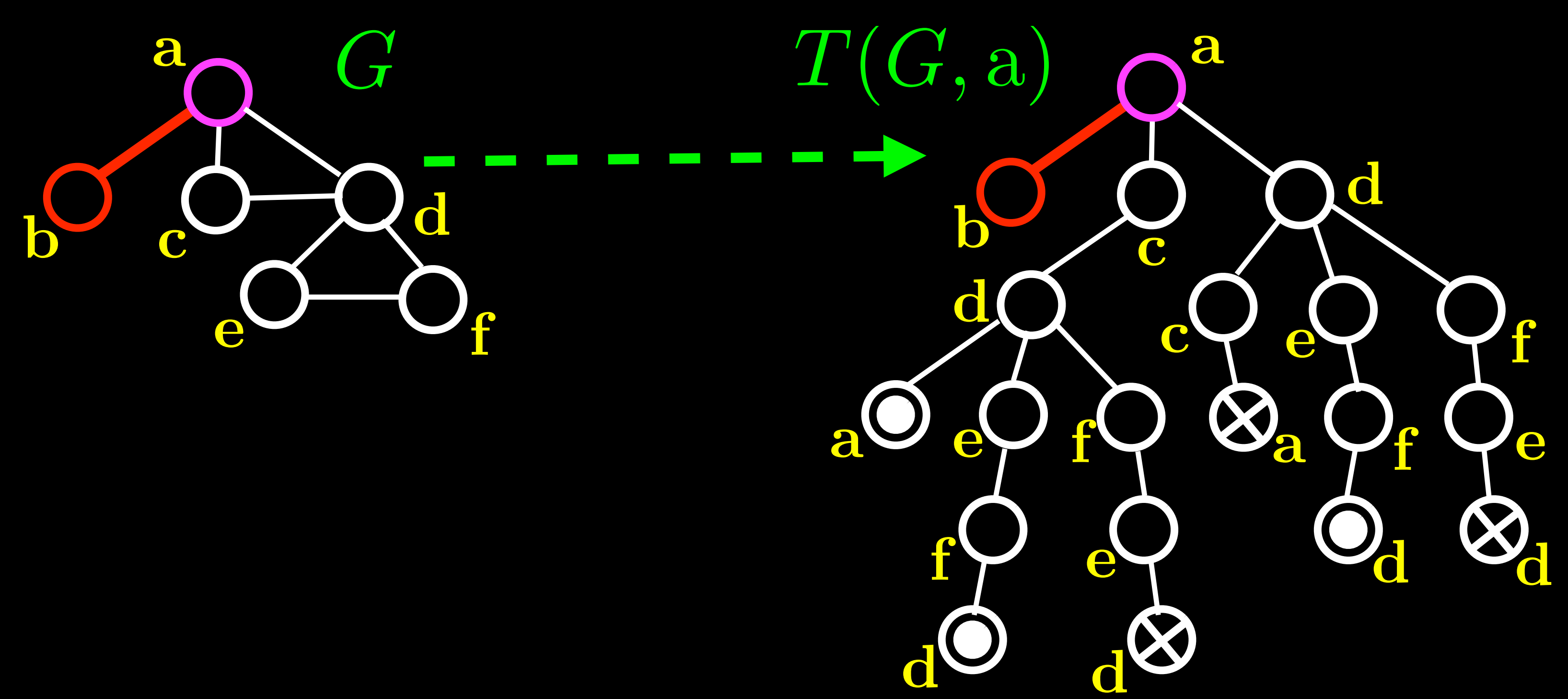

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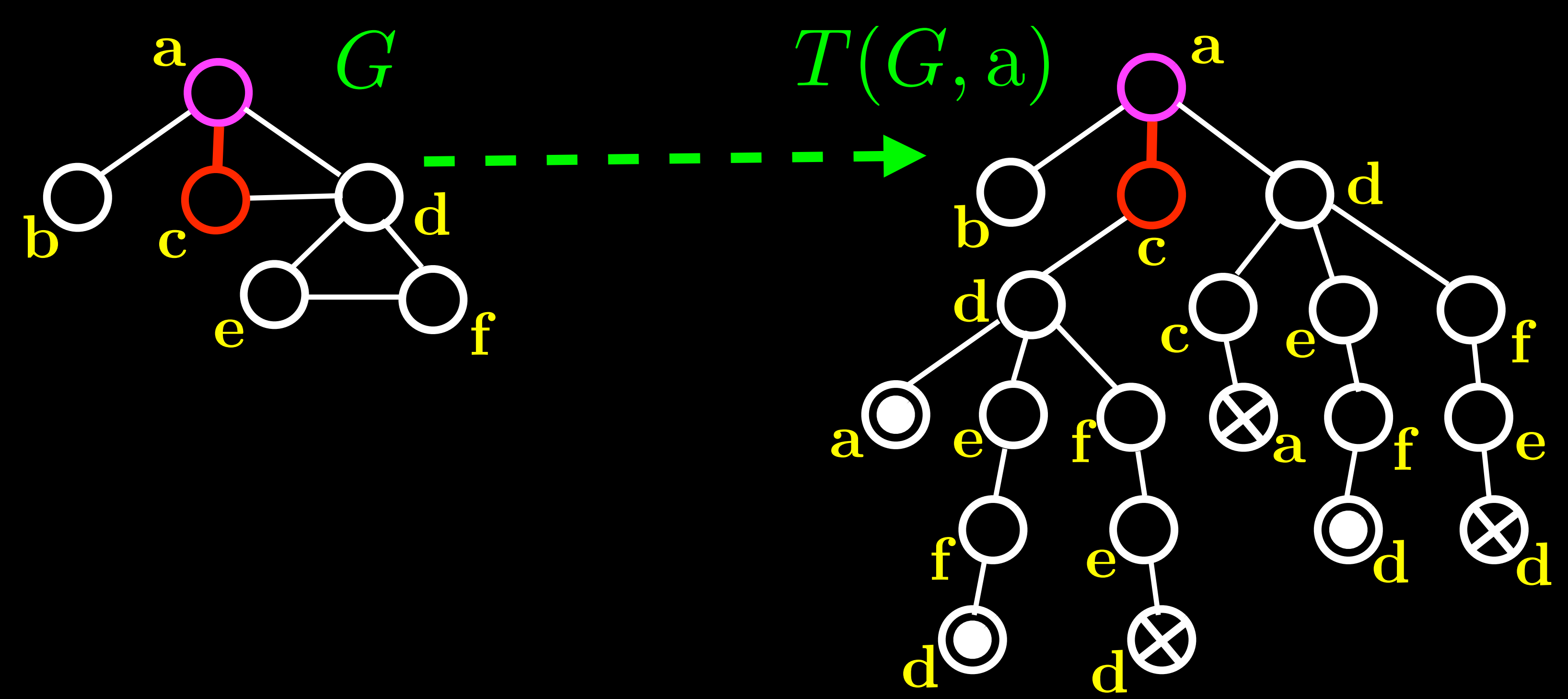

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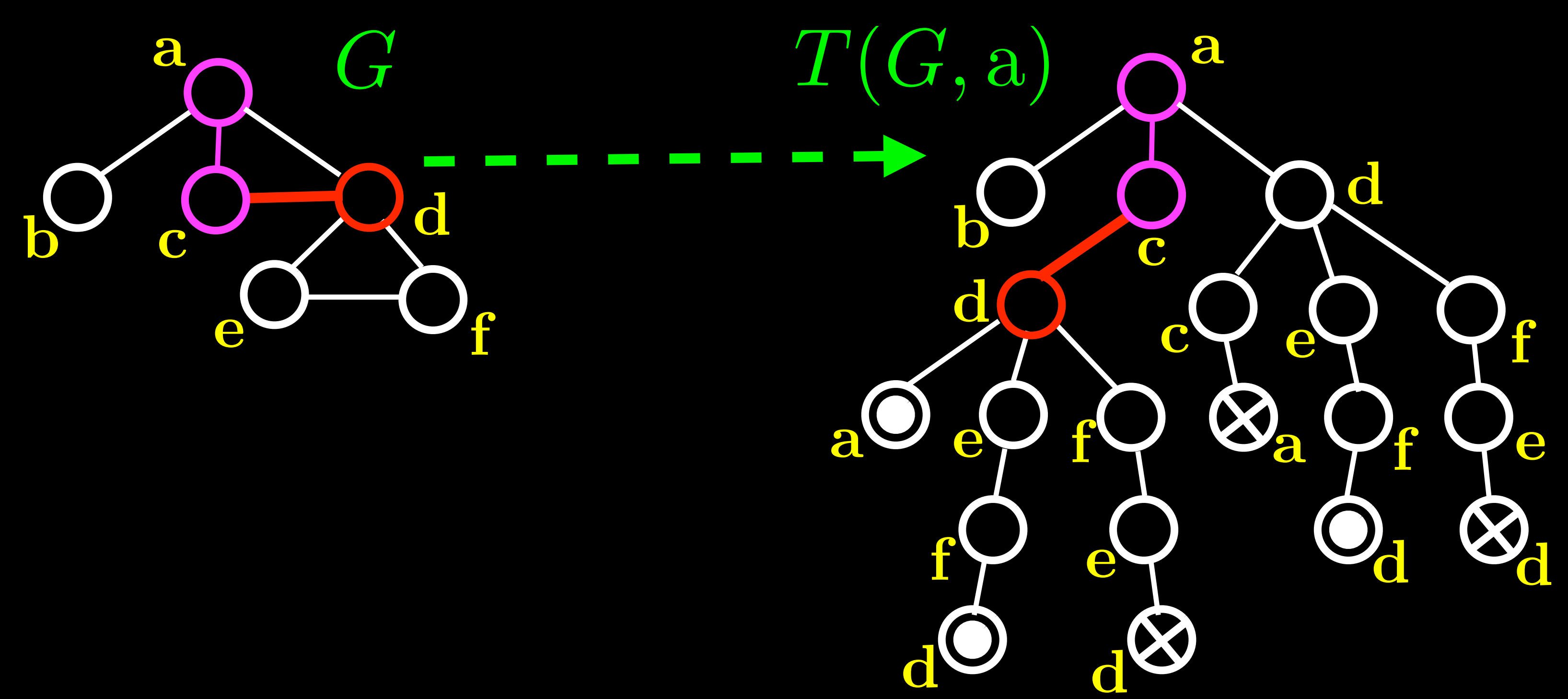

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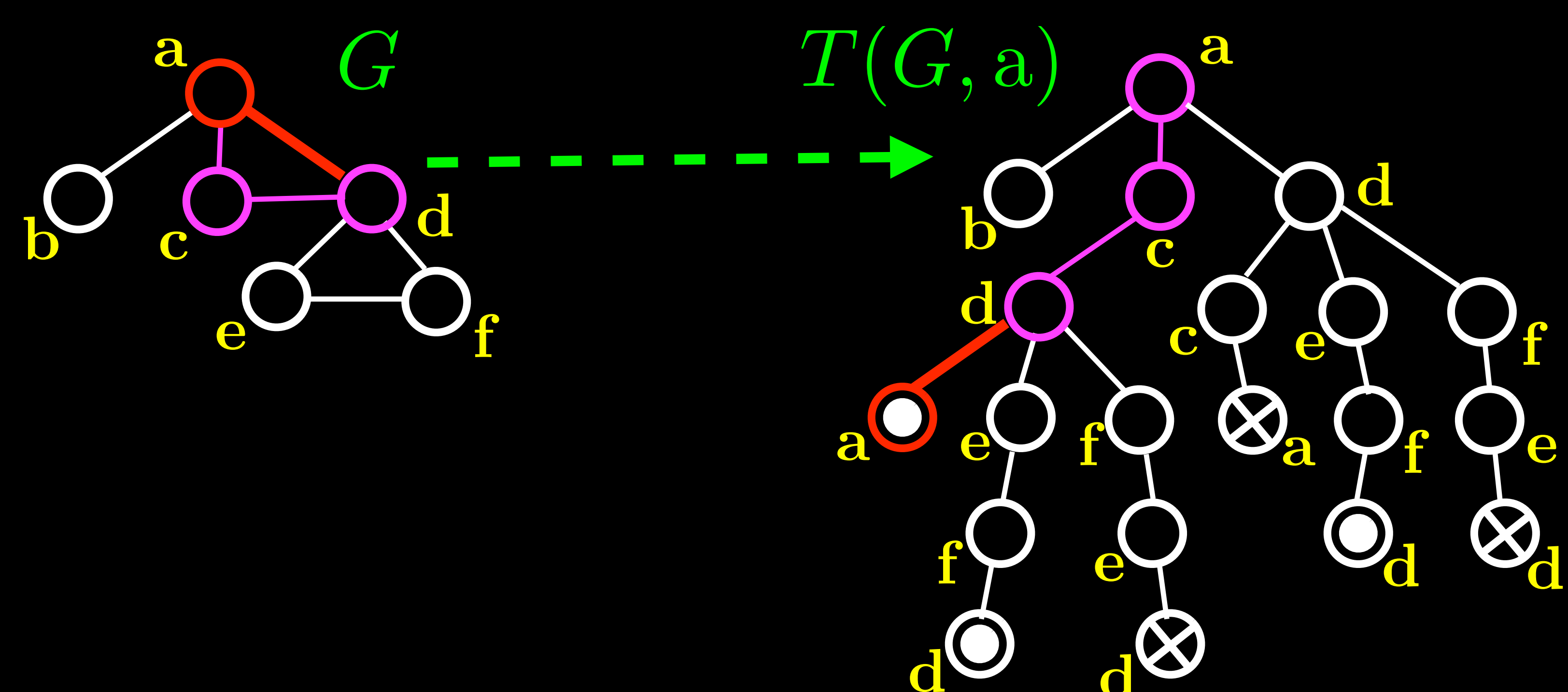

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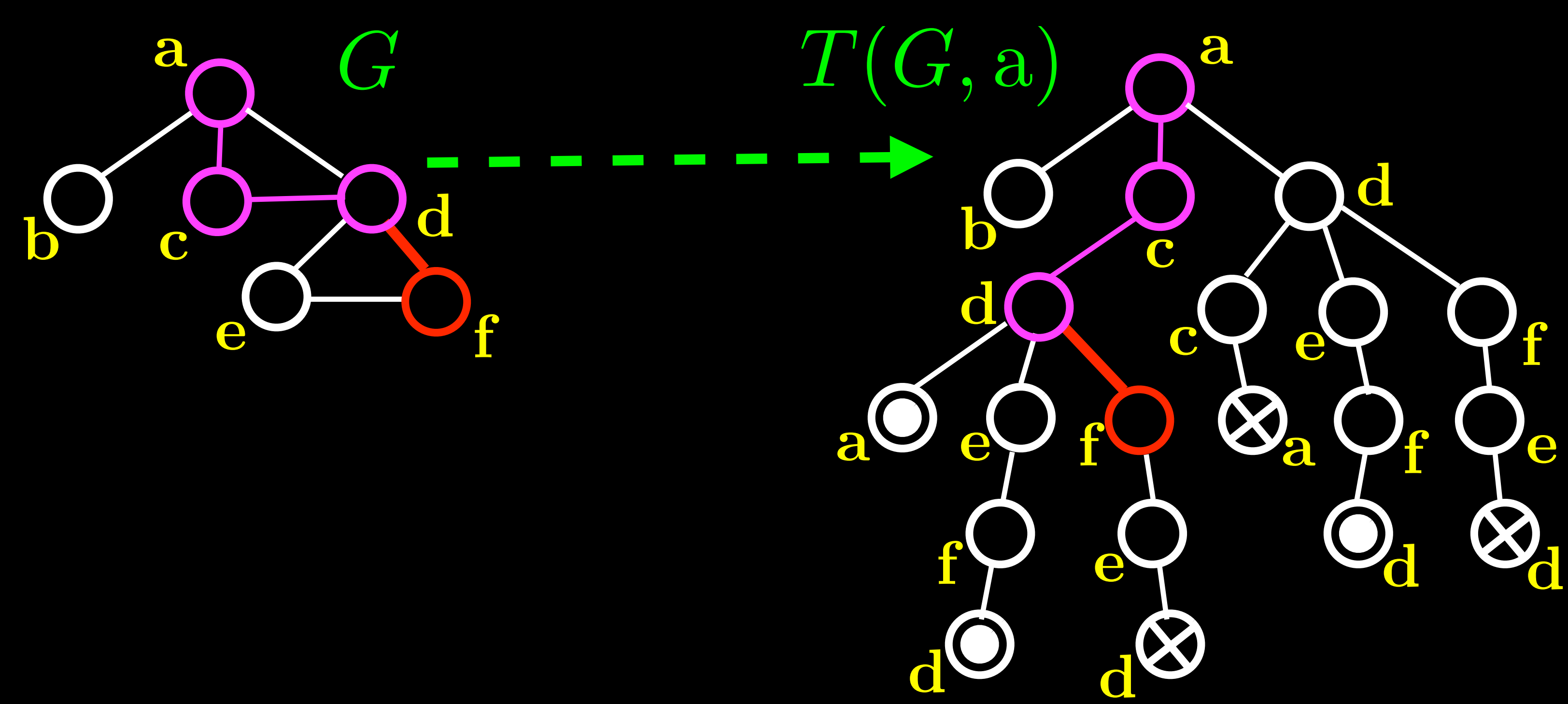

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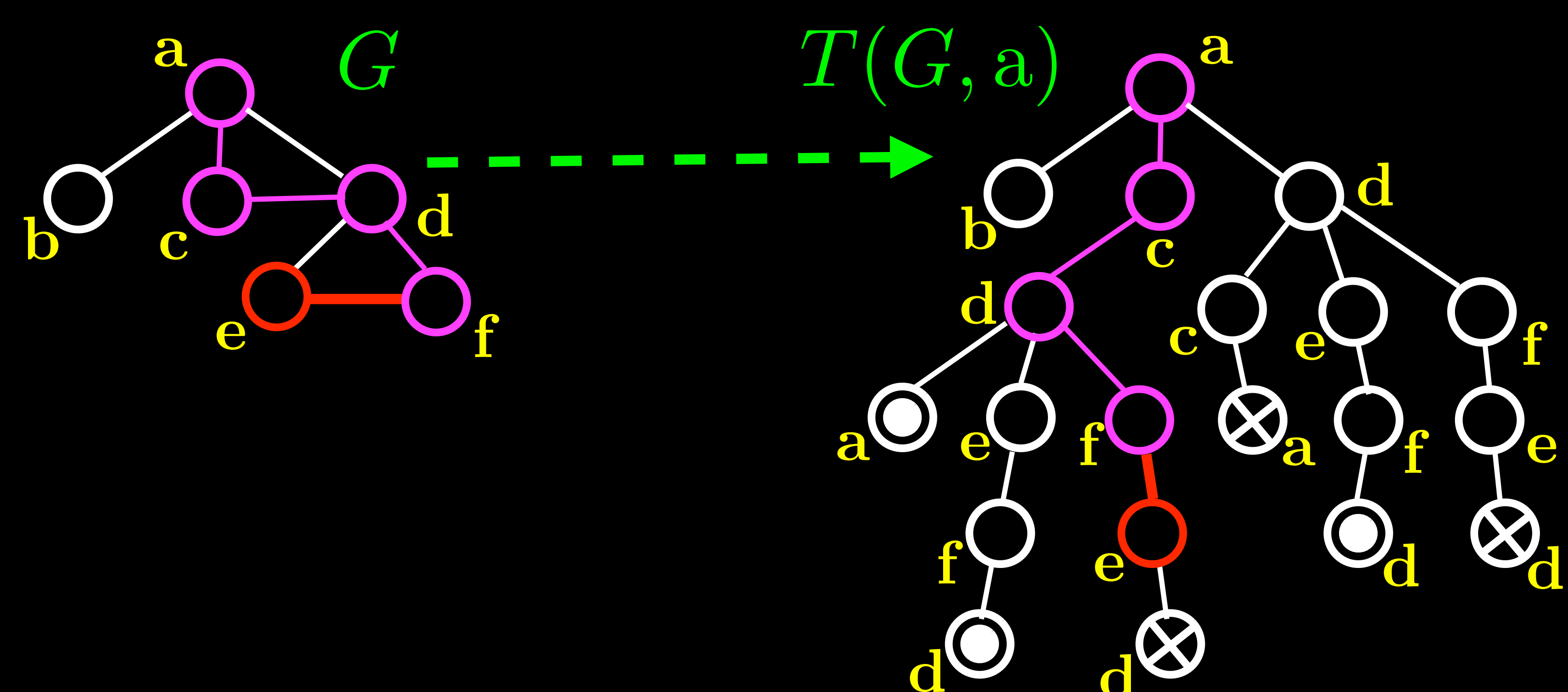

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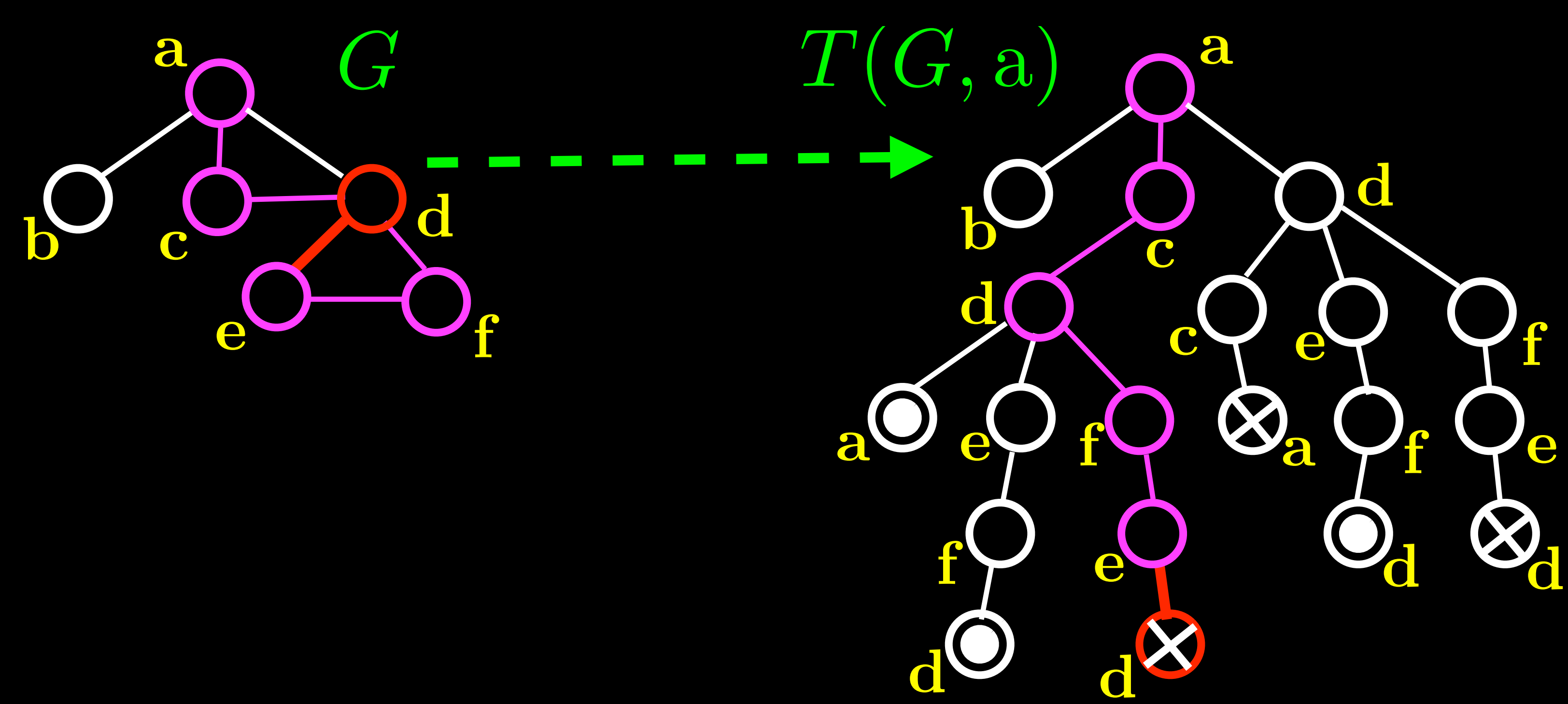
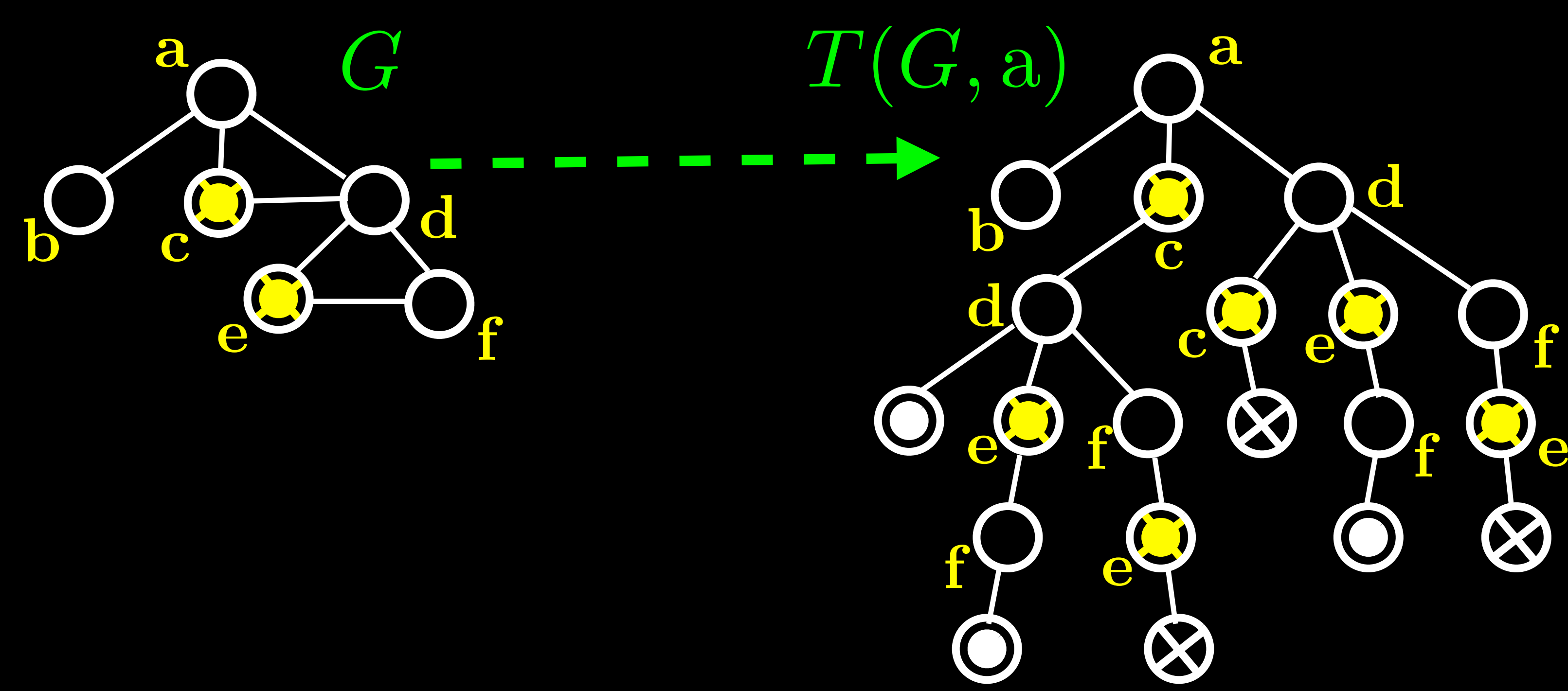


Construction of $T(G, v)$

Condition on G $\longrightarrow$ Condition on $T(G, v)$



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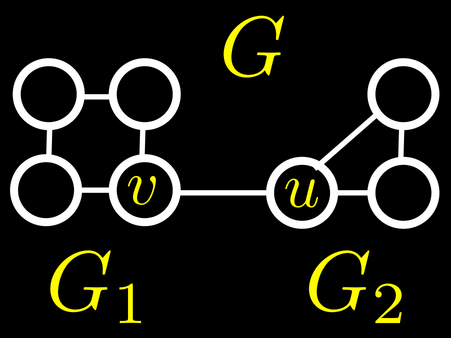

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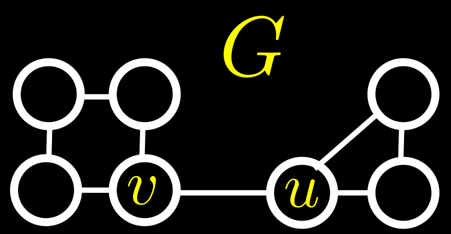

- Standard recursive procedure for trees:

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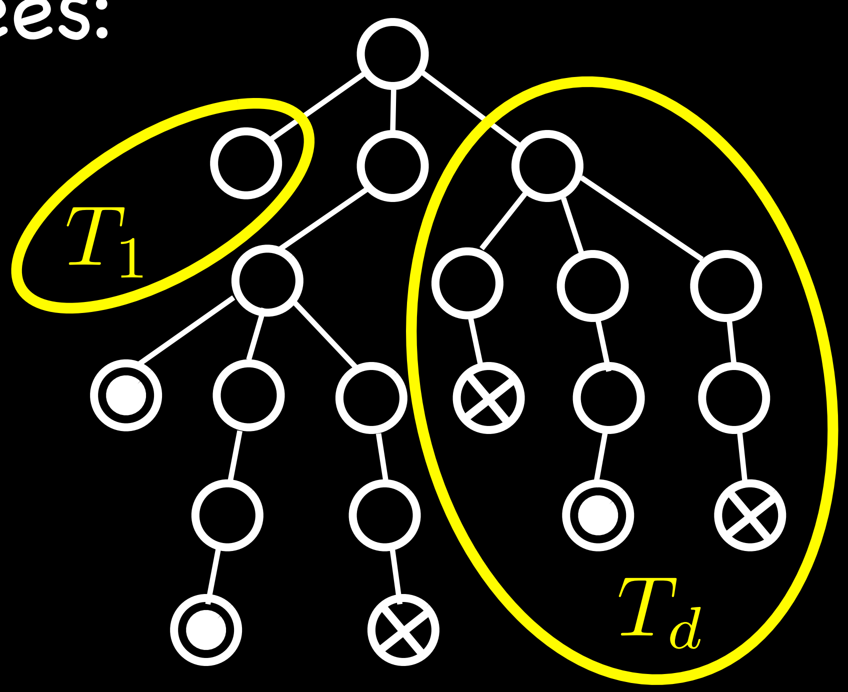

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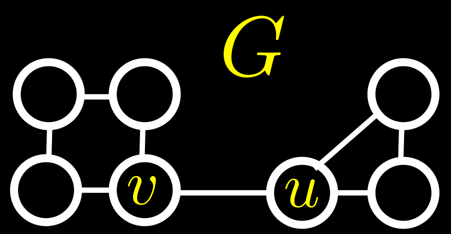

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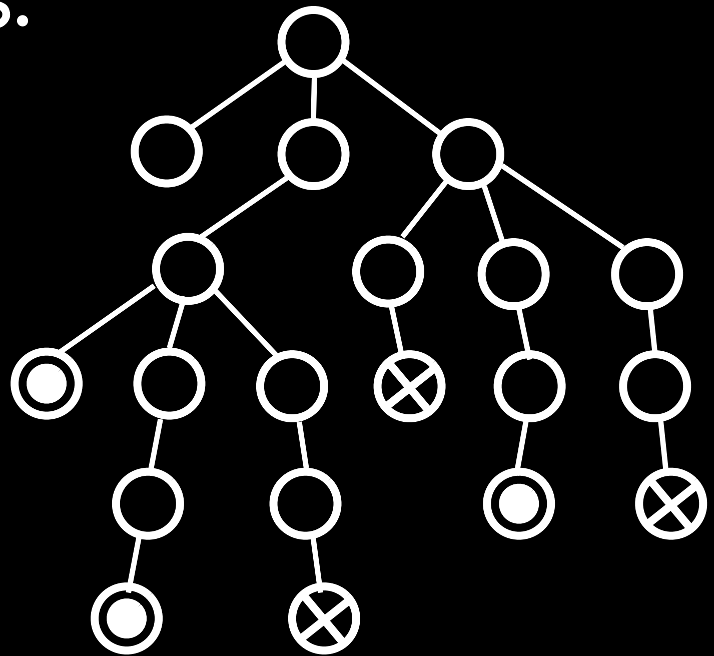

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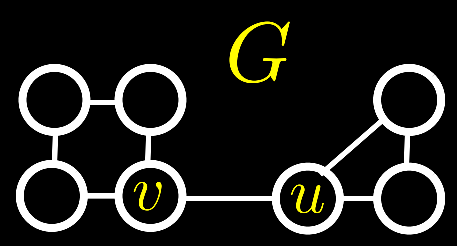

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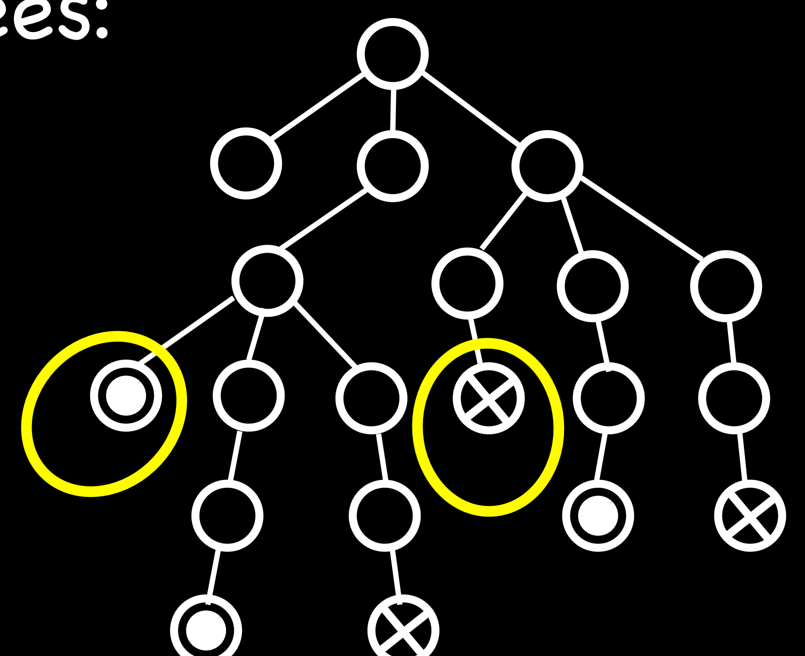

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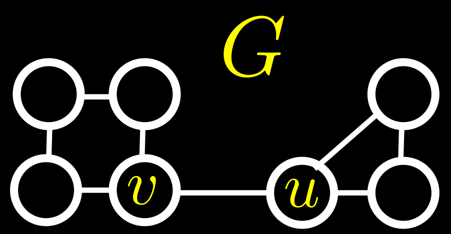

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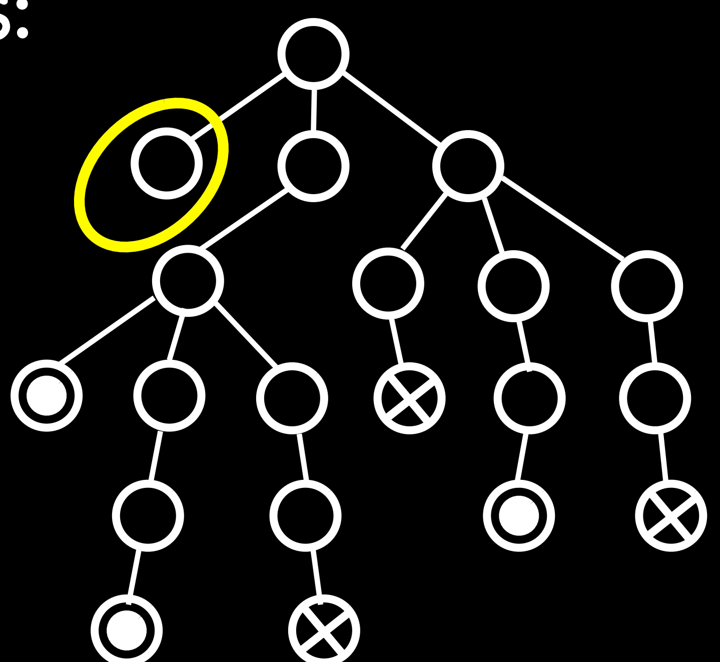
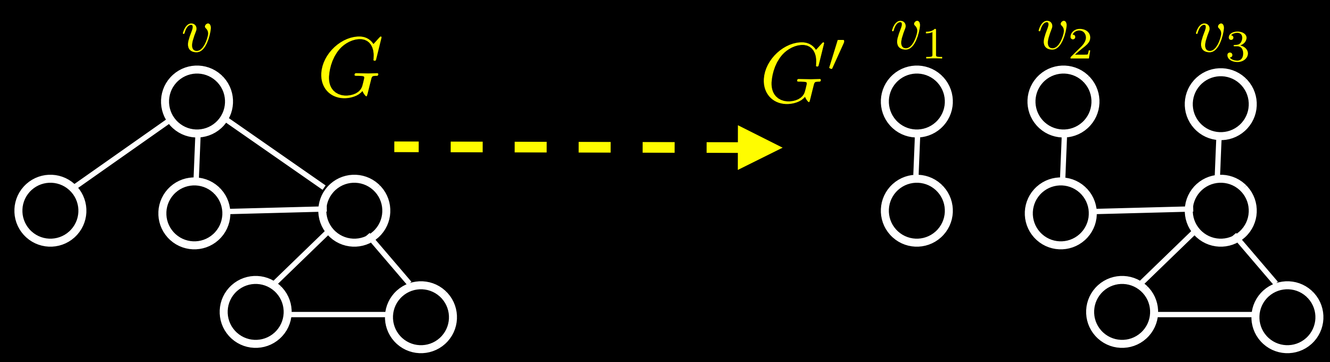


Stopping rules -

- fixed vertices: $R = \infty$ or 0;
- (unfixed) leaves: $R = \lambda$.

Calculating $R_{G,v}$

- Split v into $\deg(v)$ degree-one vertices:



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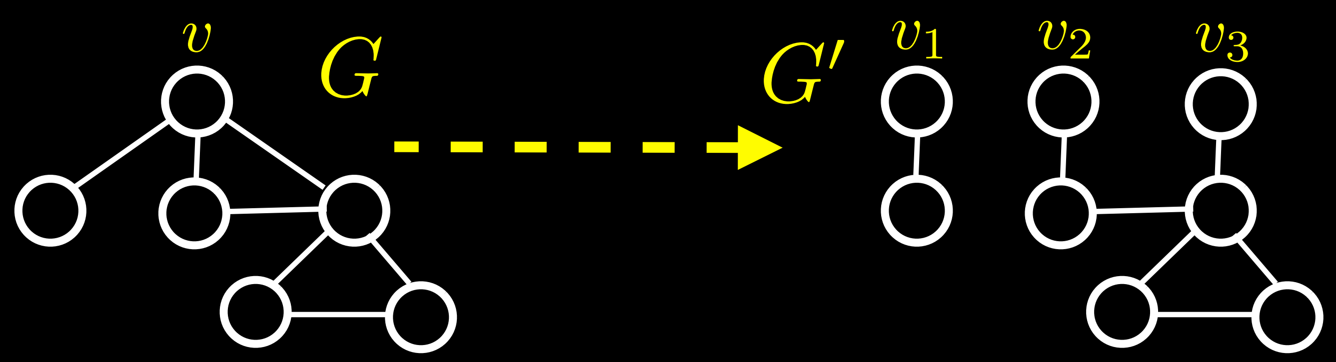
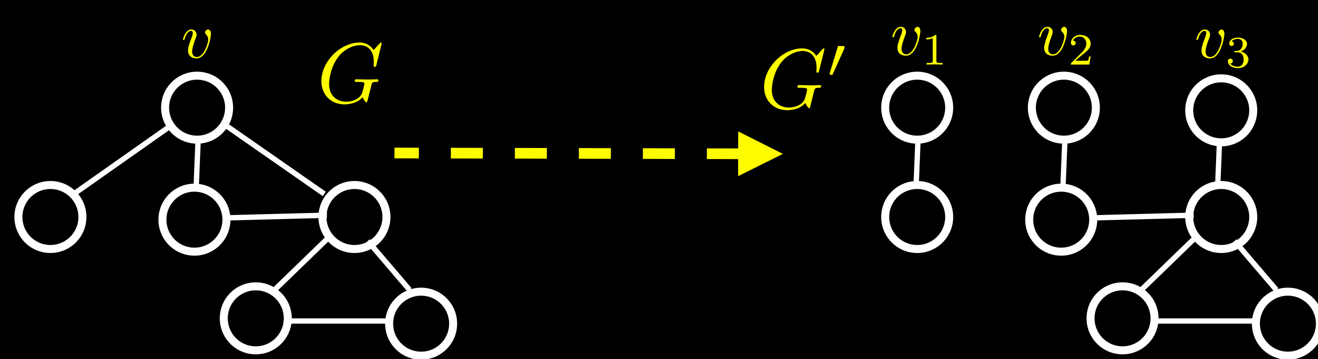


associate the activity $\lambda^{1/d}$ with each v_i.

- Observation:

$$R_{G,v} = \frac{\Pr_G(v \text{ is occupied})}{\Pr_G(v \text{ is unoccupied})} = \frac{\Pr_{G'}(\text{all } v_i \text{ are occupied})}{\Pr_{G'}(\text{all } v_i \text{ are unoccupied})}.$$

Telescopic Product



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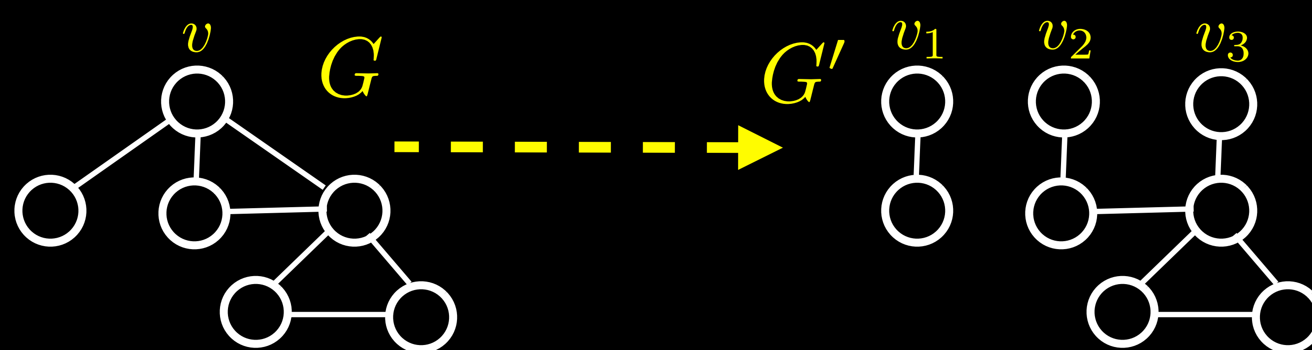
v_i

Conditional Probabilities



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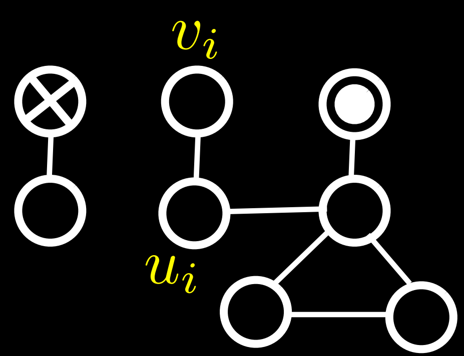
v_i

$$= \prod_{i=1}^{d} R_{G',v_i}^{\tau_i}.$$

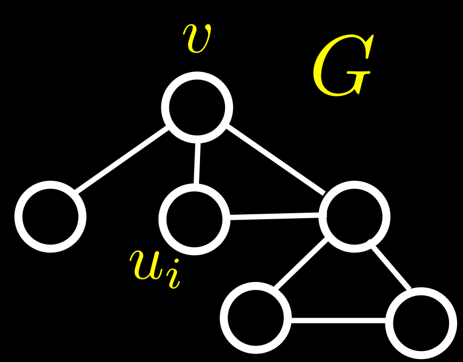
τ_i

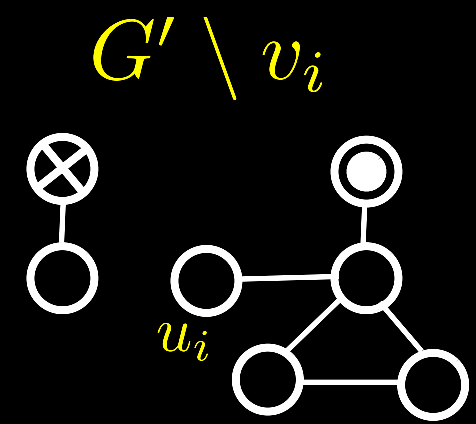
It's all about the Neighbors



$$R^{\tau_i}_{G',v_i} = \frac{\lambda^{1/d}}{1 + R^{\tau_i}_{(G'\setminus v_i),u_i}}$$

Recursive Procedure for Calculating $R_{G,v}$





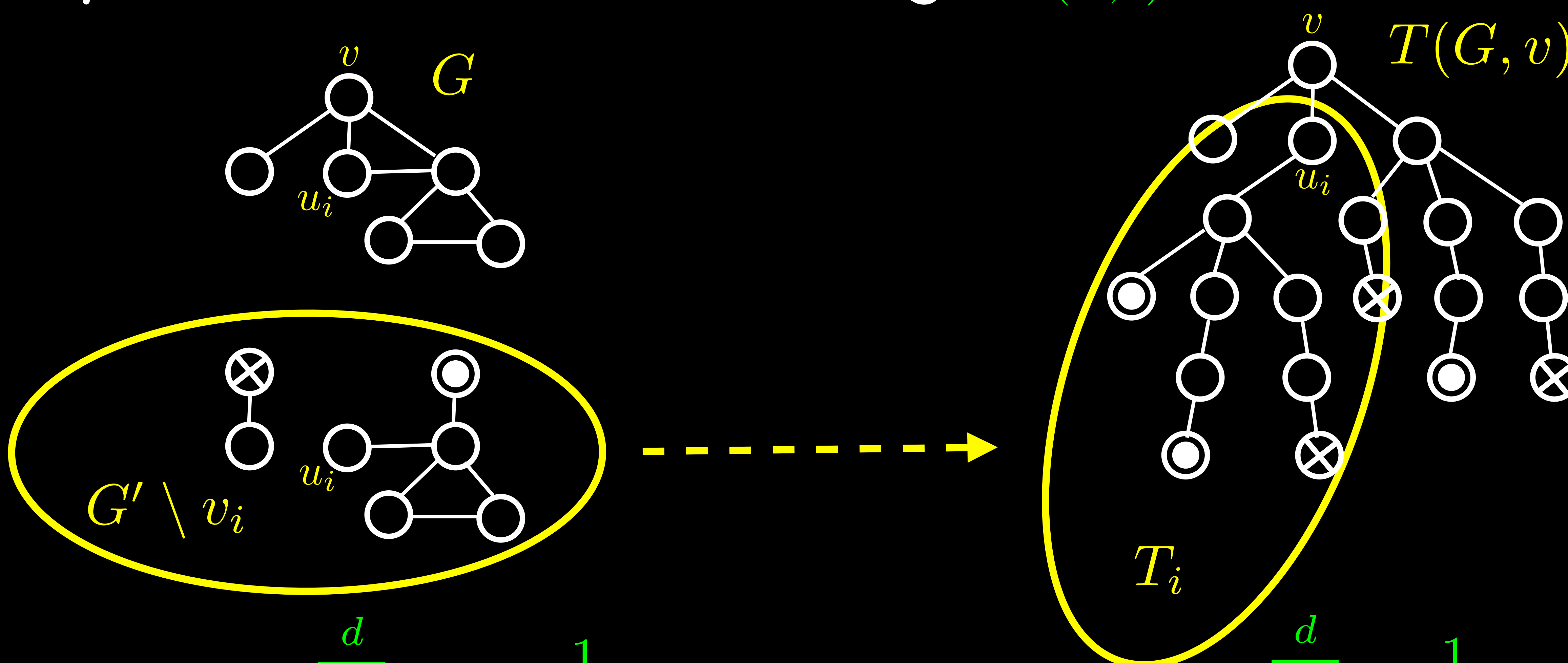
$$R^{\tau_i}_{G',v_i} = \frac{\lambda^{1/d}}{1 + R^{\tau_i}_{(G' \setminus v_i),u_i}}$$

$$\Downarrow$$

$$R_{G,v} = \lambda \prod_{i=1}^{d} \frac{1}{1 + R^{\tau_i}_{(G' \setminus v_i),u_i}}$$

$$R_{G,v} = R_{T(G,v)}$$

The procedure for calculating $R_{G,v}$ makes exactly the same calculations as the tree procedure for calculating $R_{T(G,v)}$.



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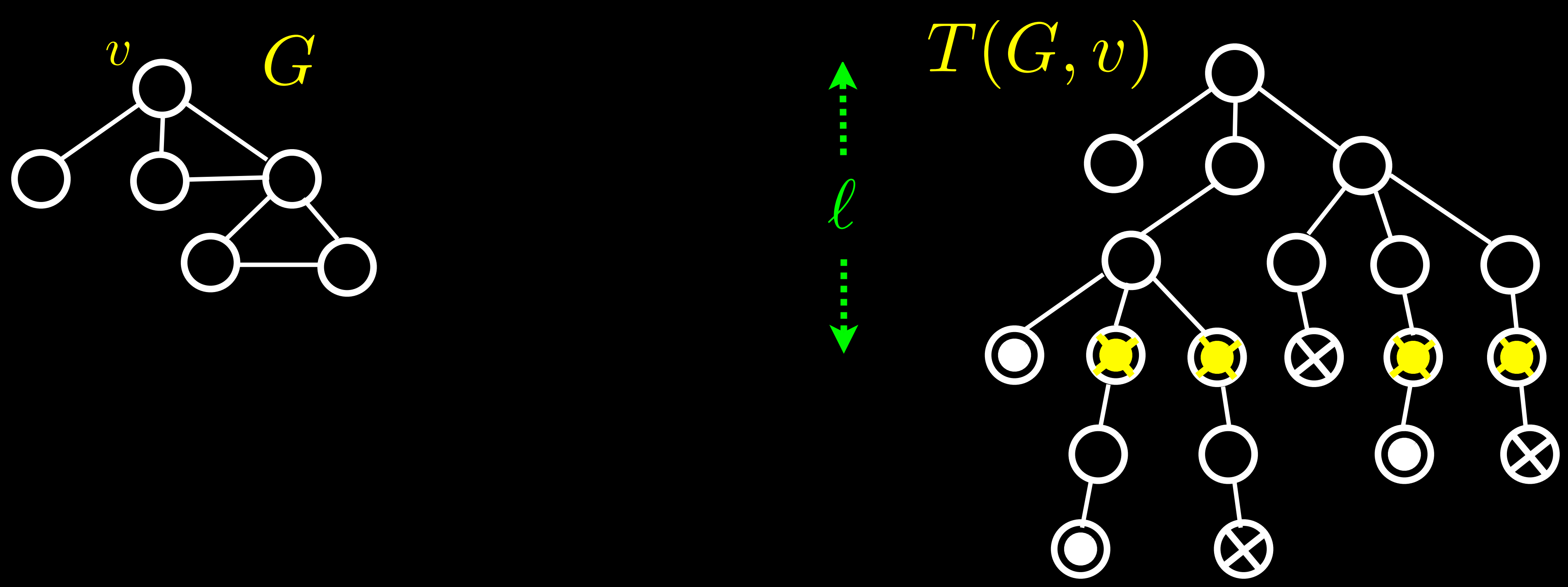
Approximation Algorithm

- Run the previous recursive procedure, but if the stack of the recursion is ℓ levels deep return trivial lower and upper bounds.



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- Running time is $O((\Delta - 1)^{\ell})$.

- For $\lambda < \lambda_c$ the difference between the resulting lower and upper bounds is $\leq \exp(-\ell)$.

  $\Rightarrow$ $(1+\epsilon)$-approximation for $\Pr(v \text{ is occupied})$ in time $\text{poly}(1/\epsilon)$.

Summary

- New Tree representation for general graphs.
- Proves that the tree is the "worst-case".
- New tree-like algorithm for approximately counting independent sets (works up to the tree threshold).
- Improved bounds for specific interesting settings:
  - Uniformly weighted independent sets with $\Delta \leq 5$.
  - The square lattice $\mathbb{Z}^2$.

Open Problems

1. Tree representation is valid for any binary spin system (i.e., Ising models). Is there a tree representation for models with more than two spins (e.g., proper colorings) ?
   - [Gamarnik-Katz, Nair-Tetali]: Tree-like algorithms (branching depends on spins as well, no direct comparison with model on the tree, require stronger and unnatural forms of decay of correlations).
   - Negative result [Sly]: tree is not worst case for uniqueness.

Open Problems

2. Improve the hardness threshold for approximately counting independent sets.

   – [Mossel-W-Wormald]: Conjecture that λ_c is the threshold for the computational probelm. Provide evedince that approximation is hard above λ_c.

3. More efficient variants of the algorithm (iterative?)

4. Solve other problems using the tree representation:
   – Spin glass Ising on $\mathbb{Z}^d$.
   – SSM down to T_c for Ising on $\mathbb{Z}^d$ for d>2.

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