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Angular Momentum Analysis of A Composite Particle Scattering Problem And Its Cross Channel Analog*

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"Angular Momentum Analysis of a ...."

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ABSTRACT

We investigate the solutions, by a determinental method, for a partial wave Bethe-Salpeter equation describing composite particle scattering and for its cross channel analog. We compare the behavior at $s = -\infty$ of the leading angular momentum singularity from the Bethe-Salpeter equation with that of the leading Regge singularity of the cross channel equation. We mention the effect of the general properties of the kernel of the Bethe-Salpeter equation on the angular momentum structure of the solution.

I. INTRODUCTION

In a previous paper [1] we derived what we called a "multi-peripheral" [2] model with continued cross channel unitarity beginning with the Bethe-Salpeter equation

$$T(p,q,s) = B(p,q,s) + \frac{1}{(2\pi)^3} \frac{d^3k}{2\,\omega_k} \frac{B(p,k,s)}{\mathscr{D}(\sigma_k)} T(k,q,s) \tag{1}$$

for composite particle scattering in the s channel. This equation treats the case of three spinless, identical particles and describes the elastic scattering of one particle from a spinless composite state of the other two. This equation is such that the composite particle scattering amplitude derived from it obeys exact two and three body unitarity in the s channel.

By using a spectral representation in momentum transfer, t, for the amplitude T and going to high t values, we derived a homogeneous equation for the discontinuity of T across the t-cut. This equation describes two elementary particles scattering into two composite particles. We found that the solution to this equation Reggeized, i.e., it was of the form

$$t^{\alpha(s)}\,\phi(p,q,s)\,.$$

This led to the following eigenvalue equation for the exponent $\alpha(s)$,

$$f(p,q,s) = \frac{g^2}{2\pi} \int_0^\infty \frac{dk}{2\omega_k \mathscr{D}(\sigma_k)} \int_0^\infty \frac{ds'}{s'-s} \; Q_\alpha \left[\frac{(\sqrt{s'} - \omega_p - \omega_k)^2 - (p^2+k^2+m^2)}{2p_k} \right]$$

$$\times f(p,k,s), \tag{2}$$

where

$$f(p,q,s) = p^{\alpha+1} \phi(p,q,s) \tag{3}$$

Thus, it is of interest to study the behavior of the exponent $\alpha(s)$ as a function of s.

In this paper we mainly do two things, we study the leading Regge singularity emerging from the above eigenvalue equation for large negative s (i.e., $s \to \infty$). We compare this with the behavior of $\alpha(s)$ coming from a determinental solution of the partial wave projection of the composite particle Bethe-Salpeter equation (Eq. 1.1). We study $\alpha(s)$ from the Bethe Salpeter equation for $s \to -\infty$ and its general features emerging from properties of the kernel of the equation.

This paper is organized as follows. In Sec. II we write down an $N_\ell(s)/D_\ell(s)$ form for the partial wave amplitude from the composite particle Bethe-Salpeter equation. In Sec. III we write down determinental power series expansions for $N_\ell(s)$ and $D_\ell(s)$ and examine boundedness conditions necessary for the convergence of the power series. In Sec. IV we study the analytic properties of $D_\ell(s)$ in the s-plane. Section V contains an examination of the angular momentum structure of the Bethe-Salpeter equation and the leading angular momentum singularity at $s = -\infty$. Finally, in Sec. VI we extract the leading Regge singularity from the "multiperipheral" eigenvalue equation for $s \to -\infty$. Section VII is a discussion of our results.

II. N/D FORM FOR THE PARTIAL WAVE AMPLITUDE

We begin with the Bethe-Salpeter equation describing composite-particle scattering which was written down in I (Fig. 1):

$$T(p,q,s) = B(p,q,s) + \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} \frac{B(p,k,s)}{\mathscr{D}(\sigma_k)} T(k,g,s) \tag{2}$$

where

$$B(p,q,s) = \frac{g^2(\omega_p + \omega_q + \omega_{p+q})}{\omega_{p+q}\left[(\omega_p + \omega_p + \omega_{p+q})^2 - s\right]} \tag{4}$$

and

$$\mathscr{D}(\sigma_k) = s - 2\sqrt{s}\, \omega_g + m^2 - m_R^2 + F(\sigma_q) \tag{5}$$

is the inverse composite particle propagator, both chosen so that T obeys exact two and three body unitarity in s.

We partial wave project this equation (with signatured amplitudes) to obtain

$$T_\ell^\pm(p,q,s) = B_\ell^\pm(p,q,s) + \frac{1}{(2\pi)^2} \int \frac{dk}{2\omega_k} \frac{B_\ell^\pm(p,k,s)\, T_\ell^\pm(k,q,s)}{\mathscr{D}(\sigma_k)}\,. \tag{6}$$

Our definition of $T^{\pm}$ and $T^{\pm}_{\ell}$ are such that

$$T^{\pm}(p,q,s) = \frac{1}{4\pi pq} \sum_{\ell} (2\ell + 1) T^{\pm}_{\ell}(p,q,s) P_{\ell}(\cos\theta) \tag{7}$$

Writing

$$K^{\pm}_{\ell}(p,k,s) = \frac{1}{(2\pi)^2} \frac{B_{\ell}(p,k,s)}{2\omega_k \mathscr{D}(\sigma_k)} \tag{8}$$

we have

$$T^{\pm}_{\ell}(p,q,s) = B^{\pm}_{\ell}(p,q,s) + \int dk\, K^{\pm}_{\ell}(p,k,s) T^{\pm}_{\ell}(k,q,s) \tag{9}$$

We now follow a procedure analogous to that of Lee and Sawyer [3]. If we consider T_{ℓ}, B_{ℓ}, and K_{ℓ} as matrix elements of integral operators in a one-dimensionsal space with operator products defined by

$$\langle p | A_{\ell}(s)\, B_{\ell}(s) | q \rangle = \int dk \langle p | A_{\ell}(s) | k \rangle \langle k | B_{\ell}(s) | q \rangle, \tag{10}$$

we have for Eq. (2.6)

$$\langle p | T^{\pm}_{\ell}(s) | q \rangle = \langle p | B^{\pm}_{\ell}(s) | q \rangle + \langle p | k^{\pm}_{\ell}(s)\, T^{\pm}_{\ell}(s) | q \rangle \tag{11}$$

or

$$\langle p | T^{\pm}_{\ell}(s) | q \rangle = \langle p | B^{\pm}_{\ell}(s) + K^{\pm}_{\ell}(s)\, T^{\pm}_{\ell}(s) | q \rangle. \tag{12}$$

We may solve for the T matrix element as

$$\langle p | T^{\pm}_{\ell}(s) | q \rangle = \langle p | \frac{B^{\pm}_{\ell}(s)}{1 - K^{\pm}_{\ell}(s)} | q \rangle \tag{13}$$

$$= - \langle p | B^{\pm}_{\ell}(s) [\frac{\partial}{\partial K^{\pm}_{\ell}} D_{\ell}(s)] \frac{1}{D_{\ell}(s)} | q \rangle \tag{14}$$

$$\text{or} \quad T^{\pm}_{\ell}(s) = \frac{N^{\pm}_{\ell}(s)}{D_{\ell}(s)} \quad \text{and} \quad \langle p \mid T^{\pm}_{\ell}(s) \mid q \rangle = \frac{\langle p \mid N^{\pm}_{\ell}(s) \mid q \rangle}{D_{\ell}(s)} \tag{15}$$

where

$$N^{\pm}(s) = -B^{\pm}(s)\left[\frac{\delta}{\partial K^{\pm}_{\ell}} D_{\ell}(s)\right] \tag{16}$$

and

$$D_{\ell}(s) = \det(1 - K_{\ell}) . \tag{17}$$

III. DETERMINENTAL SERIES EXPANSIONS OF $N^{\pm}_{\ell}(s)$ AND $D_{\ell}(s)$

Following a development due to M. Baker [4] we write

$$\begin{aligned} D_{\ell}(s) &= \det(1 - K_{\ell}) \\ &= 1 - \mathrm{tr} K_{\ell} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \int_0^{\infty} \prod_{i=1}^{n} dq_i \, K^{\ell}_{n} \begin{pmatrix} q_1 \cdots q_n \\ q_1 \cdots q_n \end{pmatrix} \end{aligned} \tag{18}$$

where

$$K^{\ell}_{n} \begin{pmatrix} q_1 \cdots q_n \\ q_1 \cdots q_n \end{pmatrix} = \begin{vmatrix} \langle q_1 | K_{\ell} | q_1 \rangle & \cdots & \langle q_1 | K_{\ell} | q_n \rangle \\ \langle q_n | K_{\ell} | q_1 \rangle & \cdots & \langle q_n | K_{\ell} | q_n \rangle \end{vmatrix} \tag{19}$$

correspondingly we have

$$\langle p | N^{\pm}_{\ell}(s) | q \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\infty}^{\infty} \prod_{i=1}^{n} dq_i \, K^{\pm}_{n+1} \begin{pmatrix} p & q_1 \cdots q_n \\ q & q_1 \cdots q_n \end{pmatrix} . \tag{20}$$

We have developed $\langle p | N^{\pm}_{\ell}(s) | q \rangle$ and $D_{\ell}(s)$ as power series convergert for all strengths of the coupling constant, g^2, for certain boundedness conditions. The boundedness conditions which insure the convergence of the power series are

$$\mathrm{tr}\, K_\ell^\pm < \infty \quad \text{and } \mathrm{tr}\,[K_\ell^\pm(s)]\,[K_\ell^\pm(s)]^+ < \infty \tag{21}$$

that is

$$\int_0^\infty dq\ K_\ell^\pm(q,q,s) < \infty \quad \text{and} \tag{22}$$

$$\int_0^\infty dq_1\, dq_2\, |K_\ell^\pm(q_1, q_2, s)|^2 < \infty\ . \tag{23}$$

Now

$$K_\ell^\pm(p,q,s) = \frac{1}{(2\pi)^2}\ \frac{B_\ell^\pm(p,q,s)}{2\omega_q\, \mathscr{D}(\sigma_q)} \tag{24}$$

$$= K_\ell^{\pm(1)}(p,q,s) + K_\ell^{\pm(2)}(p,q,s) \tag{25}$$

where

$$B_\ell^\pm(p,q,s) = B_\ell^{\pm(1)}(p,q,s) + B_\ell^{\pm(2)}(p,q,s) \tag{26}$$

with

$$B_\ell^{\pm(1)}(p,q,s) = \pm\frac{\sqrt{pq}}{2\ell+1}\ \frac{(\omega_p+\omega_q)}{s-(\omega_p+\omega_q)^2}\left[\frac{\sqrt{(p+q)^2+m^2}-\sqrt{(p-q)^2+m^2}}{2\sqrt{pq}}\right]^{2\ell+1} \tag{27}$$

and

$$B_\ell^{\pm(2)}(p,q,s) = \pm\,\zeta_\ell\left\{ Q_\ell\left[\frac{(\omega_p+\omega_q+\sqrt{s})^2-(p^2+q^2+m^2)}{2pq}\right] + Q_\ell\left[\frac{(\omega_p+\omega_q-\sqrt{s})^2-(p^2+q^2+m^2)}{2pq}\right]\right\}\ . \tag{28}$$

Then

$$\int dq\ K_\ell^\pm(q,p,s) = \frac{1}{2\ell+1}\int dq\left[\frac{\sqrt{4q^2+m^2}-m}{2q}\right]^{2\ell+1}$$

$$\times \frac{q}{s-4\omega g^2} \frac{1}{\mathscr{D}(\sigma_q)} \quad . \tag{29}$$

This integral is bounded for all ℓ, provided $s \neq 4\omega_q^2$ and s is not such that $\sigma = m_R^2$ in $\mathscr{D}(\sigma_q)$ (this will be dealt with further in the next section). The same holds for

$$\int dq_1\, dq_2 \left| K_\ell^{\pm(1)}(q_1, q_2, s) \right|^2$$

as can be easily checked.

We also have

$$\int dq\, K_\ell^{\pm(2)}(q,q,s) = \zeta_\ell \int \frac{dq}{2\omega_q \mathscr{D}(\sigma_q)}$$
$$\times \left\{ Q_\ell \left[\frac{(2\omega_q + \sqrt{s})^2 - (2q^2 + m^2)}{2q^2} \right] + Q_\ell \left[\frac{(2\omega_q - \sqrt{s})^2 - (2q^2 + m^2)}{2q^2} \right] \right\}. \tag{30}$$

From the behavior of $Q_\ell(x)$ at $x = \infty$ and the logarithmic singularities of $Q_\ell(x)$ at $x = \pm 1$, the integral converges for $\ell > -3/2$. A similar result holds for

$$\int dq_1\, dq_2 \left| K_\ell^{\pm(2)}(q_1, q_2, s) \right|^2 .$$

IV. ANALYTIC PROPERTIES OF $\mathscr{D}_\ell(s)$ IN g

$\mathscr{D}_\ell(s)$ is given by trace integrals two of which were given in the previous section. Let us consider the integral

$$\int dq\, K_\ell^{\pm}(q,q,s) = \int dq\, [K_\ell^{\pm(1)}(q,q,s) + K_\ell^{\pm(2)}(q,q,s)] \quad . \tag{31}$$

For $K_\ell^{\pm(1)}$ we have to examine

$$\frac{1}{2\ell+1}\int dq \left[\frac{\sqrt{4q^2+m^2}-m}{2q^2}\right]^{2\ell+1} \frac{q}{\mathscr{D}(\sigma_q)} \quad \frac{1}{s-4\omega_q^2} .$$

Singularities in the variable s can arise in two ways: i) a zero of the $\frac{1}{s-4\omega_q^2}$ term may pinch the q integration contour. This leads to a cut in s from $s = 4m^2$ to $s = \infty$. ii) A zero of $\mathscr{D}(\sigma_q)$ may pinch the q integration contour. This happens when σ_q is at a resonance threshold, we have $\sigma_q = s - 2\omega_q \sqrt{s} + m^2 = m_R^2$. This implies that $\sqrt{s} = \omega_q \pm \sqrt{q^2 + m_R^2}$. This leads to a cut in s from $s = (m + m_R)^2$ to $s = \infty$. There is also a pseudo-threshold at $s = (m_R - m)^2$.

For $K_\ell^{\pm(2)}$ we have to consider the integral

$$\int dq \quad Q_\ell\left[\frac{(2\omega_q+\sqrt{s})^2 - (2q^2 + m^2)}{2q^2}\right]\frac{1}{2\omega_q\mathscr{D}(\sigma_q)}$$

Again the singularities in the variable s can occur in two ways: i) a zero of $\mathscr{D}(\sigma_q)$ may pinch the q integrate contour, leading to a cut in s from $(m_R + m)^2$ to ∞, as before, ii) branch points of Q_ℓ may pinch the q integration contour. This leads to a cut in s from $s = m^2$ to $s = \infty$.

V. THE LEADING ANGULAR MOMENTUM SINGULARITY IN THE BETHE SALPETER EQUATION

The angular momentum poles of the problem are determined from the roots of the equation $D_\ell(s) = 0$. Now $D_\ell(s)$ is given by the power series expansion in Eq. (18). Since the kernel of our original integral equation satisfies the appropriate boundedness conditions (Eq. 21), provided we are in the s plane, cut as indicated in Sec. IV, we approximate the expansion of $D_\ell(s)$ with the first two terms.

Therefore, the determinental condition for the angular momentum poles is

$$1 - \mathrm{tr}\, K_\ell = 0$$

or

$$\int \frac{dq}{s - 2\omega_q\sqrt{s} + m^2 - m_R{}^2 + F(\sigma_q)} \left\{ \frac{1}{2\omega_q} \zeta_\ell \left(Q_\ell\left[\frac{(2\omega_q+\sqrt{s})^2-(2q^2+m^2)}{2q^2}\right] + Q_\ell\left[\frac{(2\omega_q-\sqrt{s})^2 - (2q^2+m^2)}{2q^2}\right]\right) + \frac{1}{2\ell+1}\left[\frac{\sqrt{4q^2+m^2} - m}{2q}\right]^{2\ell+1} \frac{g}{s-4\omega_q^2} \right\}$$

$$-1 = 0 \tag{32}$$

This is a transcendental equation for ℓ which has, in general, an infinite number of solutions.

If we look at the pie ce of the kernel which gives rise to the second term in Eq. (32), we have

$$K_\ell^{(2)\pm}(p,q,s)$$

$$= \frac{1}{2\omega_q \mathscr{D}(\sigma_q)} \frac{1}{2\ell+1}\left[\frac{\sqrt{(p+q)^2+m^2} - \sqrt{(p-q)^2+m^2}}{2\sqrt{pq}}\right]^{2\ell+1} \frac{\sqrt{pq}\,(\omega_p+\omega_q)}{s-(\omega_p+\omega_q)^2} \tag{33}$$

In the vicinity of $\ell = -1/2$, this is

$$K_\ell^{\pm(1)}(p,q,s)$$

$$= \frac{1}{2\omega_q \mathscr{D}(\sigma_q)} \quad \frac{1}{2\ell+1} \quad \frac{\sqrt{pq}(\omega_p + \omega_q)}{s - (\omega_p + \omega_q)^2} \tag{34}$$

Therefore, the residue of the pole of the kernel at $\ell = -1/2$ is not a separable function of the variables p and q, i.e., the residue is not of finite rank [5]. This leads to an infinite accumulation of Fredholm poles in T near $\ell = -1/2$.

Nevertheless, we can examine the behavior of the determinental equation for s large and negative (or s large positive in the cut plane). We do this by separating out the largest singular part of the $Q_\ell(x)$ in Eq. (32) to obtain

$$\int \frac{dq}{s-2\omega_q\sqrt{s}+m^2-m_R^2+F(\sigma_q)} \left\{ \frac{1}{2\omega_q} \frac{2}{\ell+1} + \frac{1}{2\ell+1}\left[\frac{\sqrt{4q^2+m^2}-m}{2q}\right]^{2\ell+1} \frac{q}{s-4\omega_q^2} \right\} -1 = 0 \tag{35}$$

As $|s| \to \infty$, the first term inside the brackets in the integral dominantes and we find out that the leading singularity is at $\ell = -1$. That is, at $s = -\infty$, the Regge pole begins at $\ell = -1$. For $s = 4m^2$ or $s = (m+m_R)^2$ the integral in Eq. (32) is not bounded. Our approximation to the determinental equation fails for s at these values. As s approaches

$4m^2$, the second term in Eq. (32) becomes increasingly important and we have a transcendental equation with an infinite number of solutions.

VI. THE LEADING REGGE SINGULARITY FROM THE "MULTIPERIPHERAL" EQUATION

In paper I, we derived the eigenvalue equation for α given below for high t values from a "multiperipheral" model with continued cross channel unitarity. We found

$$f(p,q,s) = \frac{g^2}{2\pi} \int_0^\infty \frac{dk}{2\omega_k \mathscr{D}(\sigma_k)} \int_{s_0}^\infty \frac{ds'}{s'-S'} \; Q_\alpha\left[\frac{\sqrt{s'} - \omega_p - \omega_k)^2 - (p^2+k^2+m^2)}{2p_k}\right]$$

$$\times\, f(p,k,s). \tag{36}$$

This is an eigenvalue equation of the form $f = K_\alpha f.$ (37)

The eigenvalue condition is then $\det(1-K_\alpha) = 0$. (38)

Again, following Baker [4], we write

$$D_\alpha = \det(1 - K_\alpha)$$

$$= 1 - \mathrm{tr}\, K_\alpha + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \int_0^\infty \prod_{i=1}^{n} dq_i \; K_n^\alpha \begin{pmatrix} q_1 \ldots q_n \\ q_1 \ldots q_n \end{pmatrix} \tag{39}$$

where $K_n^\alpha \begin{pmatrix} q_1 \ldots q_n \\ q_1 \ldots q_n \end{pmatrix}$ is given in Eq. 19. The same boundedness conditions pertain as in Sec. III.

Therefore if we keep the first two terms in the expansion of det $(1 - K_\alpha)$ we have as our eigenvalue condition

$$D_\alpha = 1 - \mathrm{tr}\, K_\alpha = 0 \tag{40}$$

or

$$D_\alpha = \frac{g^2}{2\pi} \int \frac{dq}{2\omega_q \mathscr{D}(\sigma_q)} \int_{s_0}^{\infty} \frac{ds'}{s'-S}\, Q_\alpha \left[\frac{(\sqrt{s'} - 2\,\omega_q)^2 - (2q^2 + m^2)}{2q^2} \right] - 1 = 0 \tag{41}$$

The second integral in this expression is analogous to one encountered for Yukawa potentials [3] when the exchanged particle is replaced by a continuous distribution of masses. The integral encountered is

$$\int_{s_0}^{\infty} ds' \quad Q_\alpha \left(1 + \frac{s'}{2q^2}\right) \sigma(s') \ .$$

The important point is the behavior of the spectral function $\sigma(s')$ at infinity. We consider the case

$$\lim_{s' \to \infty} \sigma(s') = C\, s'^{-1+\eta} \tag{42}$$

If $\eta = 1$ the integral above does not converge for any α. For $0 < \eta < 1$, D_α will have a fixed simple pole at $\alpha = \eta - 1$. For $\eta \leq 0$, the first α in D_α is at $\alpha = -1$.

For the s' integral in Eq. (41),

$$\sigma(s') = \frac{1}{s' - s} \tag{43}$$

Therefore

$$\lim_{s'\to\infty} \sigma(s') = \frac{1}{s'} = s'^{-1} \quad (\text{i.e.}, \eta = 0) \tag{44}$$

Therefore the first singularity in D_α is at $\alpha = -1$, or, in other words, the leading Regge singularity begins at $\alpha = -1$ at $s = -\infty$.

V. DISCUSSION

We have made a comparative study of the angular momentum properties of a Bethe-Salpeter equation describing composite particle scattering and an eigenvalue equation for the crossed channel process at high energy, which we call a "multiperipheral" model with continued cross channel unitarity. Provided that certain boundedness conditions hold, we can solve for the amplitude in the Bethe-Salpeter equation in an N_ℓ/D_ℓ form where N_ℓ and D_ℓ are written as determinental power series expansions. A similar determinental solution can be developed for the cross channel problem. The same determinental equation locates the leading angular momentum singularity in one case (Bethe-Salpeter) and the leading Regge pole in the other ("multiperipheral").

We found the behavior of the angular momentum singularities at $s = -\infty$ to be the same in both cases, i.e., $\ell = -1$ at $s = -\infty$. For s finite we found that the determinental condition for the eigenvalue in the Bethe-Salpeter case led to a transcendental equation for ℓ with an infinite number of solutions. This was linked to the non-separability of the kernel for the Bethe-Salpeter equation for finite s.

The fact that the value for ℓ at $s = -\infty$ is the same as that for ladder models [3] is not surprising, since for $s = -\infty$, we are very far from the threshold for the breakup of the composite in the composite particle channel; it acts like a bound state (or "elementary particle"). Thus, we may say that we see internally the "ladder" part of the model for large, negative s. The effects of three body unitarity are felt for s near or greater than the threshold for composite breakup where the problem is very difficult to solve.

We feel that further work on this problem necessitates numerical calculations on the computer. Nevertheless, we feel that a framework has been given in this paper and in Paper I for the study of high energy composite particle amplitudes and, as a result, for the study of many related problems. Examples are particle production and triple Regge limits since a two to three production amplitude, a two to four production amplitude, and a three to three amplitude can be constructed from our composite particle amplitudes via the attachment of appropriate composite particle propagators and dissociation vertex functions to the composites. Also we may study effects of unitarity on Reggeon-Reggeon particle vertices if we Reggeize the composites. Work on these aspects is continuing.

Authoress wishes to thank Drs. B. W. Lee and A. R. White (of Fermilab Theory Group) for helpful conversations.

APPENDIX

PARTIAL WAVE PROJECTION OF THE BORN TERM

We have for the potential

$$B(p,q,s) = \frac{g^2(\omega_p + \omega_q + \omega_{p+q})}{\omega_{p+q}\,[(\omega_p + \omega_q + \omega_{p+q})^2 - s]} \tag{A.1}$$

where $\omega_p = (\ell^2 + m^2)^{\frac{1}{2}}$ and $\omega_{p+q}(p^2 + q^2 + m^2 + 2pqz)^{\frac{1}{2}}$; $p = |\underset{\sim}{p}|$. (A.2)

This may be written

$$B(p,q,s) = g^2(\omega_p + \omega_q + \omega_{p+q}) \left\{ \frac{C_1}{\omega_{p+q}} + \frac{C_2}{(\omega_p + \omega_q - \sqrt{s})} + \frac{C_3}{(\omega_p + \omega_q + \sqrt{s})} \right\} \tag{A.3}$$

where

$$C_1 = \frac{1}{(\omega_p + \omega_q)^2 - s} ,$$

$$C_2 = \frac{-1}{2\sqrt{s}(\omega_p + \omega_q - \sqrt{s})} \tag{A.4}$$

$$C_3 = \frac{1}{2\sqrt{s}(\omega_p + \omega_q + \sqrt{s})} .$$

Rationalizing the denominators in the last two terms, we obtain

$$B(p,q,s) = g^2 \left\{ C_1 + \frac{C_1(\omega_p + \omega_q)}{\omega_{p+q}} \right.$$

$$+ \frac{C_2 \,[(\omega_p + \omega_q)^2 - \omega^2_{p+q} - \sqrt{s}\,(\omega_p + \omega_q + \omega_{p+q})]}{(\omega_p + \omega_q - \sqrt{s})^2 - \omega^2_{p+q}}$$

$$+ \frac{C_3 \,[(\omega_p + \omega_q)^2 - \omega^2_{p+q} + \sqrt{s}\,(\omega_p + \omega_q + \omega_{p+g})]}{(\omega_p + \omega_q + \sqrt{s})^2 - \omega^2_{p+g}} \qquad \text{(A. 5)}$$

$$= C_1 A + C_2 B + C_3 C \qquad \text{(A. 6)}$$

To partial wave project these terms we need two things--signatured amplitudes and an alternative definition of the Legendre function, $P_\ell(\cos\,\theta)$.

A. Signatured Amplitudes

Keeping only the dependence on the scattering angle, z, explicit, let

$$B(z) = B^R(z) + B^L(z) \qquad \text{(A. 7)}$$

where $B^R(z)$ contains only the right hand (RH) singularities of B(z) in z and $B^L(z)$ contains only the left hand (LH) ones. We then define amplitudes of definite signature

$$B^{\pm}(z) = B^R(z) \pm B^L(-z) \qquad \text{(A. 8)}$$

each of which has only right hand singularities in z. For all s, the C_1 term in Eq. (A. 5) has only left hand singularities in z. For s large and negative, the C_2 and C_3 terms in Eq. (A. 5) have only left hand singularities. Then our signatured terms are

$$B^{\pm}(z) = \pm\,[C_2\, B(z) + C_3\, C(-z) + C_1\, A(-z)] \qquad \text{(A. 9)}$$

B. Alternative Definition of the Legendre Function

In terms of the hypergeometric function $P_\ell(\cos\theta)$ may be written [6]

$$P_\ell(\cos\theta) = F(\ell+1, -\ell, 1, \sin^2\theta/2). \tag{A.10}$$

But

$$P_\ell(\cos\theta) = (-1)^\ell P_\ell(\cos\theta); \tag{A.11}$$

Thus

$$P_\ell(\cos\theta) = \frac{1}{(-1)^\ell} P_\ell(-\cos\theta) = \frac{1}{(-1)^\ell} F(\ell+1, -\ell, 1, \cos^2\theta/2) \tag{A.12}$$

So

$$f_\ell(p,q) = \frac{1}{2}\int d(\cos\theta)\, P_\ell(\cos\theta)\, f(\cos\theta, p, q)$$

$$= \frac{1}{2}\int d(\cos\theta)\, F(\ell+1, -\ell, 1, \cos^2\theta/2) f(\cos\theta, p, q) \tag{A.13}$$

$$= -\frac{2}{\rho}\int dx\ x\ J_\mu(\rho b)\ J_0(x\,b)\, db\ \ f(\cos\theta, p, q) \tag{A.14}$$

where $\rho = 2\sqrt{pq}$, $x = \rho\sin\theta/2$ and $\mu = 2\ell + 1$ (A.15)

C. Projection of the Born Term

We consider each term separately.

i) $C_1 A(z)$

$$C_1 A(z) = \frac{C_1}{\omega_{p+q}} = \frac{C_1}{(p^2+q^2+m^2+2pqz)^{\frac{1}{2}}} \tag{A.16}$$

$$C_1 A(-z) = \frac{C_1}{(p^2+q^2+m^2-2pqz)^{\frac{1}{2}}} \tag{A.17}$$

$$= \frac{C_1}{[(p-q)^2 + m^2 + 4pq \sin^2 \theta/2]^{\frac{1}{2}}} = \frac{C_1}{[a^2 + x^2]^{\frac{1}{2}}} \tag{A.18}$$

$$= f(x, p, q) \tag{A.19}$$

where
$$x^2 = p^2 \sin^2 \theta/2; \; p^2 = 4pq \tag{A.20}$$

Thus,

$$f_\ell(p,q) = C_1\left(-\frac{2}{\rho}\right)\int dx\, x\, J_\nu(\rho b)\, J_0(xb)\, f(x,p,q) db \tag{A.21}$$

$$= C_1\left[\frac{\sqrt{(p+q)^2+m^2} - \sqrt{(p-q)^2+m^2}}{2\sqrt{pq}}\right]^\nu$$

$$= C_1\left[\frac{\sqrt{(p+q)^2+m^2} - \sqrt{(p-q)^2+m^2}}{2\sqrt{pq}}\right]^{2\ell+1} = C_1 A_\ell(p,q). \tag{A.22}$$

ii) $C_2\, B(z)$

$$C_2\, B_\ell(p,q,s) = C_2\left(-\frac{2}{\rho}\right)\int dx\, x\, J_\nu(\rho b)\, J_0(xb) db\; B(x,p,q) \tag{A.23}$$

where
$$x^2 = \rho^2 \sin^2 \theta/2 \tag{A.20}$$

Now

$$C_2 B(z) = \frac{C_2\,[(\omega_p+\omega_q)^2 - \omega_{p+q}^2 - \sqrt{s}(\omega_p + \omega_q + \omega_{p-q})]}{(\omega_p + \omega_q - \sqrt{s})^2 - \omega_{p+q}^2} \tag{A.24}$$

and

$$C_2 B(-z) = \frac{C_2\,[(\omega_p+\omega_q)^2 - \omega_{p-q}^2 - \sqrt{s}(\omega_p+\omega_q + \omega_{p-q})]}{(\omega_p + \omega_q - \sqrt{s})^2 - \omega_{p-q}^2} \tag{A.25}$$

When we project, we find

$$C_2 B_\ell(p,q,s) = \frac{1}{4pq}\, Q_\ell\left[\frac{(p^2+q^2+m^2) - (\omega_p + \omega_q - \sqrt{s})^2}{2pq}\right]\zeta_\ell \tag{A.26}$$

where $\zeta_\ell = [1 + (-1)^{\ell+1}]$ (A.27)

contributes only to odd signature trajectories.

ii) $C_3 C(z)$

Projecting

$$C_3 C_\ell(p,q,s) = -C_3\, \frac{2}{\rho}\int dx\; x\; J_\nu(\rho b) J_0(xb)\, db\; C(x,p,q) \tag{A.28}$$

where $x^2 = \rho^2 \sin^2 \theta/2.$ (A.20)

Now

$$C_3 C(z) = C_3\, \frac{\left[(\omega_p + \omega_q)^2 - \omega_{p+q}^2 + \sqrt{s}\,(\omega_p + \omega_q + \omega_{p+q})\right]}{(\omega_p + \omega_q + \sqrt{s})^2 - \omega_{p+q}^2} \tag{A.29}$$

and

$$C_3 C(-z) = C_3\, \frac{\left[(\omega_p + \omega_q)^2 - \omega_{p-q}^2 + \sqrt{s}\,(\omega_p + \omega_q + \omega_{p-q})\right]}{(\omega_p + \omega_q + \sqrt{s})^2 - \omega_{p-q}^2} \tag{A.30}$$

So

$$C_3 C_\ell(p,q,s) = \frac{1}{4pq}\, Q_\ell\left[\frac{(p^2+q^2+m^2) - (\omega_p + \omega_q + \sqrt{s})^2}{2pq}\right]\zeta_\ell \tag{A.31}$$

where again

$$\zeta_\ell = [1 + (-1)^{\ell+1}] \tag{A.27}$$

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FIGURE CAPTIONS

Fig. 1 The composite particle scattering equation.

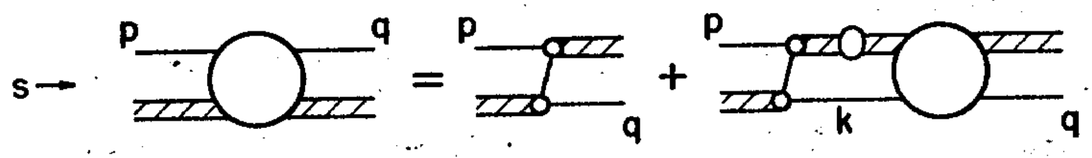

FIG. I