



Beam Optical Functions & Betatron Motion.

David Robin



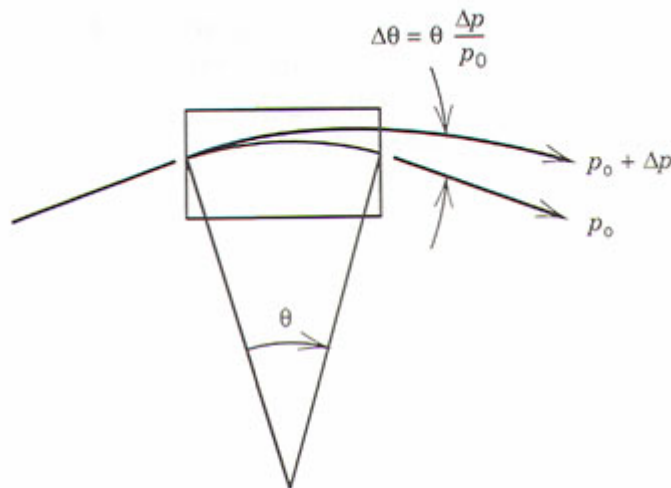
Want to touch on a number of concepts including:

- **Achromat**
- **Isochronous**
- **Weak Focusing**
- **Betatron Tune**
- **Strong Focusing**
- **Closed Orbit**
- **One-Turn Matrix**
- **Twiss Parameters and Phase Advance**
- **Dispersion**
- **Chromaticity**

Dispersive Systems



- Dispersion is the distance between the design on-energy particle and the design off energy particle divided by the relative difference in momentum between the two.**



$$x = D_x \frac{\Delta p}{p}$$

$$x' = D'_x \frac{\Delta p}{p}$$

Figure 3.15. A bending magnet deflects particles of momentum higher than that of the ideal particle through a lesser angle, leading to a variety of closed orbits for particles of differing momenta.

Achromatic Transport



- **No dispersion or dispersion slope at the beginning and end of the line**

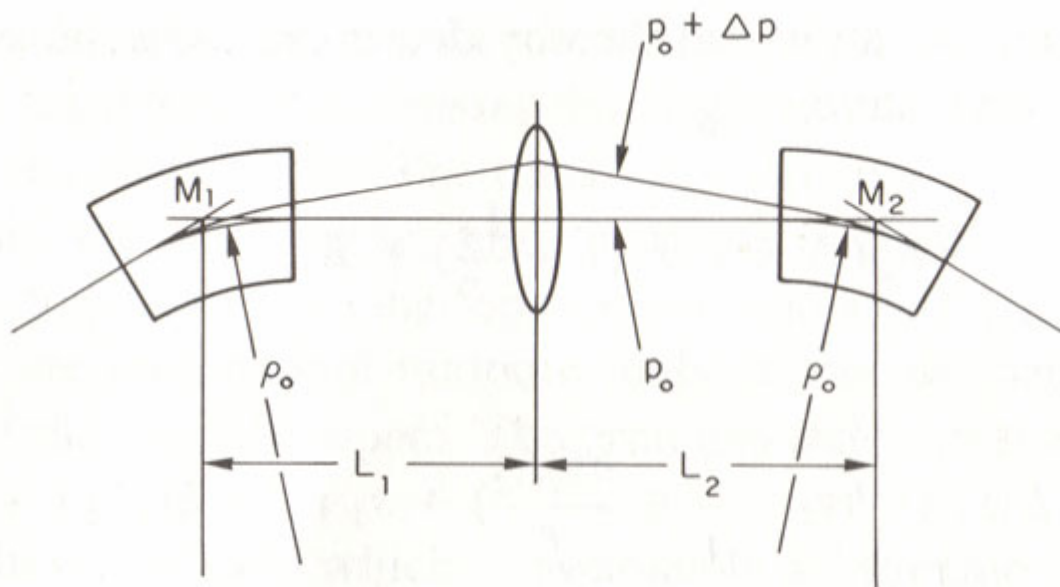


Figure 5.5 A simple achromatic system consisting of two bending magnets separated by a horizontally focusing quadrupole.

Isochronous Transport



Isochronous means “same time”

- No dispersion or dispersion slope at the end of the line
- Dispersion is negative in the central bends (cuts the corner)

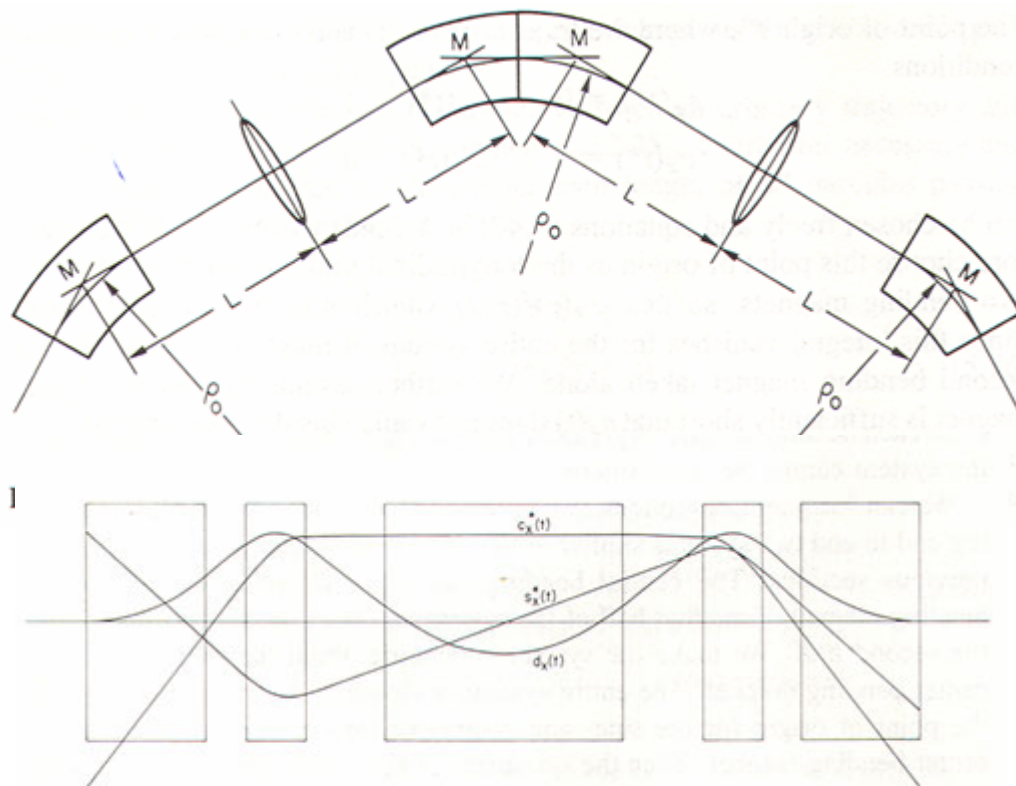
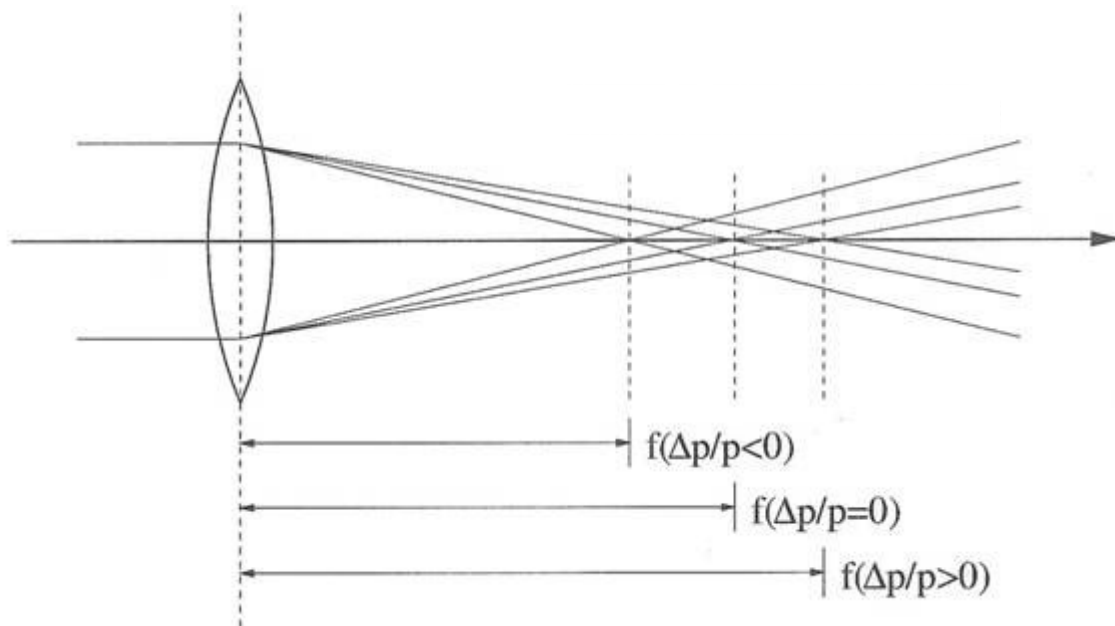


Figure 5.9 Principal trajectories in an isochronous system.

Chromatic Aberration



Focal length of the lens is dependent upon energy

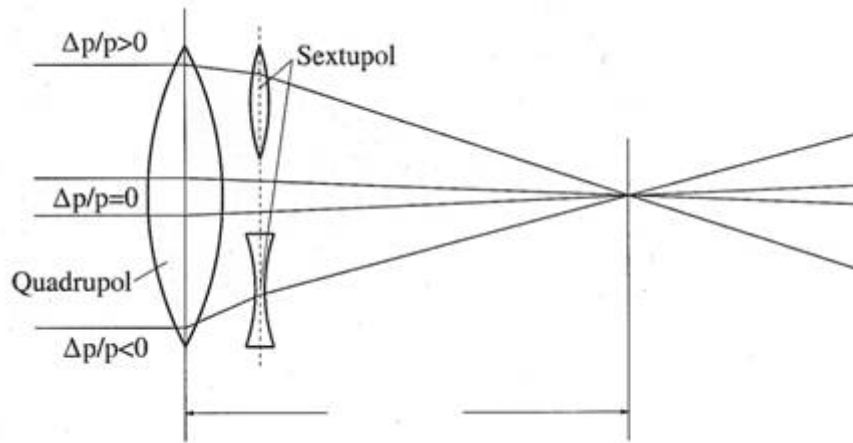


Larger energy particles have longer focal lengths

Chromatic Aberration Correction



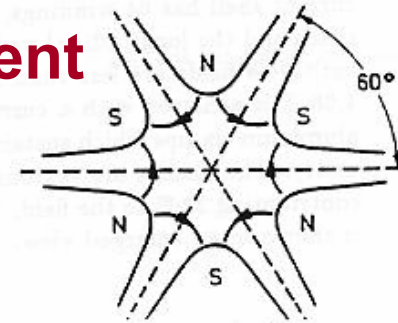
By including dispersion and sextupoles it is possible to compensate (to first order) for chromatic aberrations



The sextupole gives a position dependent
Quadrupole

$$B_x = 2Sxy$$

$$B_y = S(x^2 - y^2)$$





**Particles may need to circulate for many
(Billions) of turns**

Storage Rings are a Periodic Systems

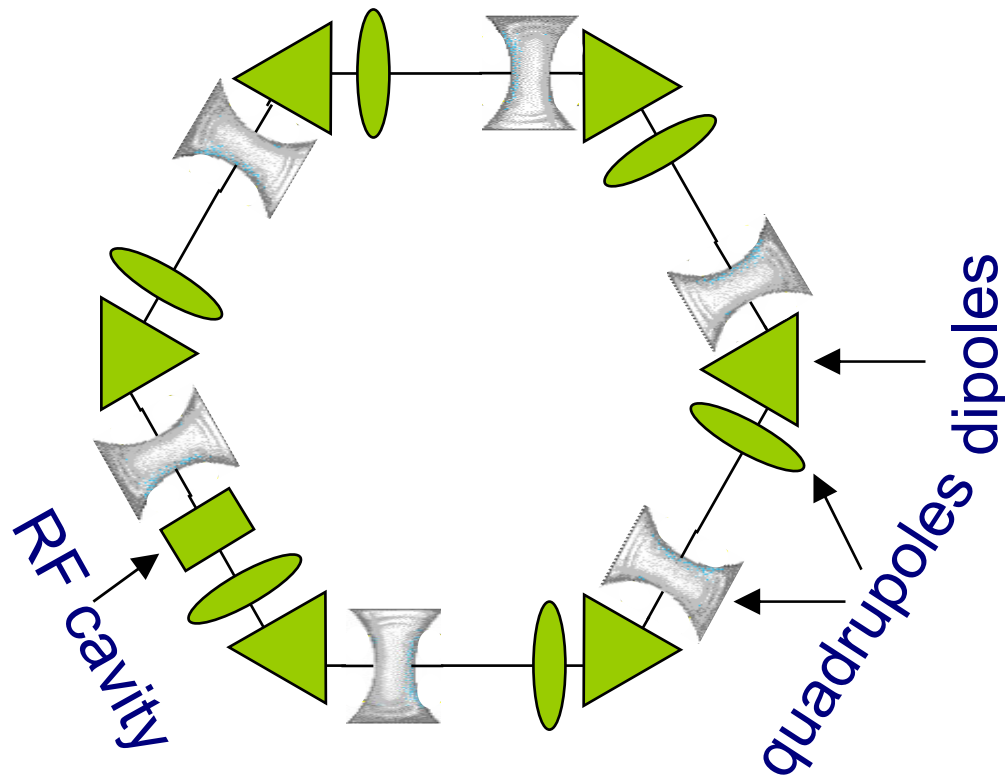
Special properties

Particle Storage Ring

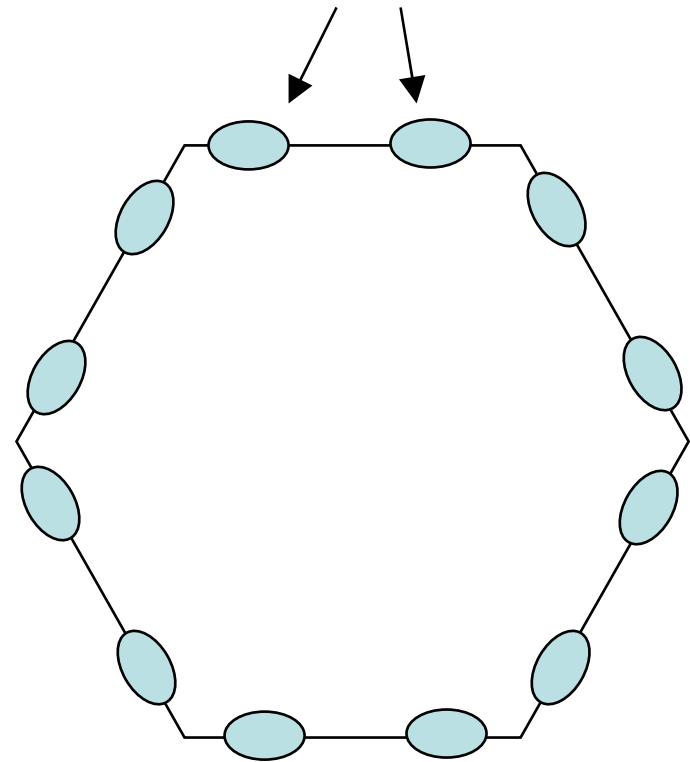


In a particle storage rings, charged particles circulate around the ring in bunches for a large number of turns.

Optics elements



Particle bunches





Weak Focusing

- V. Veksler and E. M. McMillan around 1945



Strong Focusing

- Christofilos (1950),
Courant, Livingston, and
Snyder (1952)



Christofilos



Courant



Livingston



Snyder

Weak-Focusing Synchrotrons



Weak Focusing

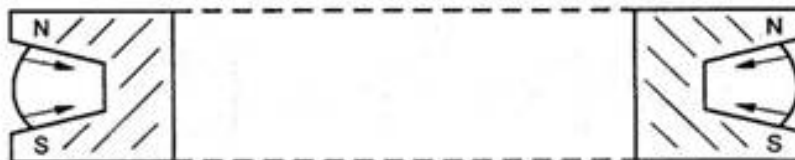


Figure 3.3. Cross section of weak focusing circular accelerator.

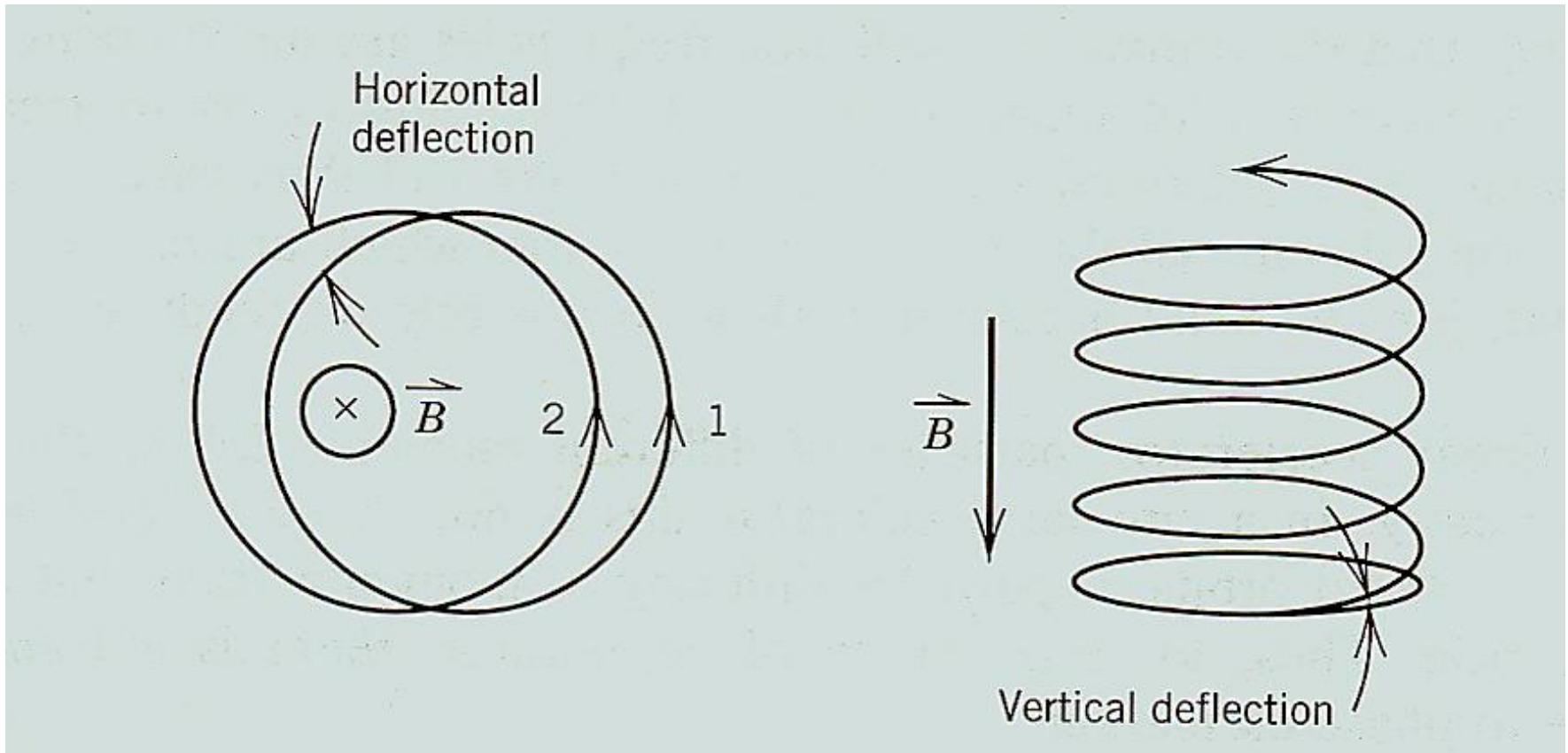
The first synchrotrons were of the so called weak-focusing type.

- The vertical focusing of the circulating particles was achieved by sloping magnetic fields, from inwards to outwards radii.
- At any given moment in time, the average vertical magnetic field sensed during one particle revolution is larger for smaller radii of curvature than for larger ones.

Stability of transverse oscillations



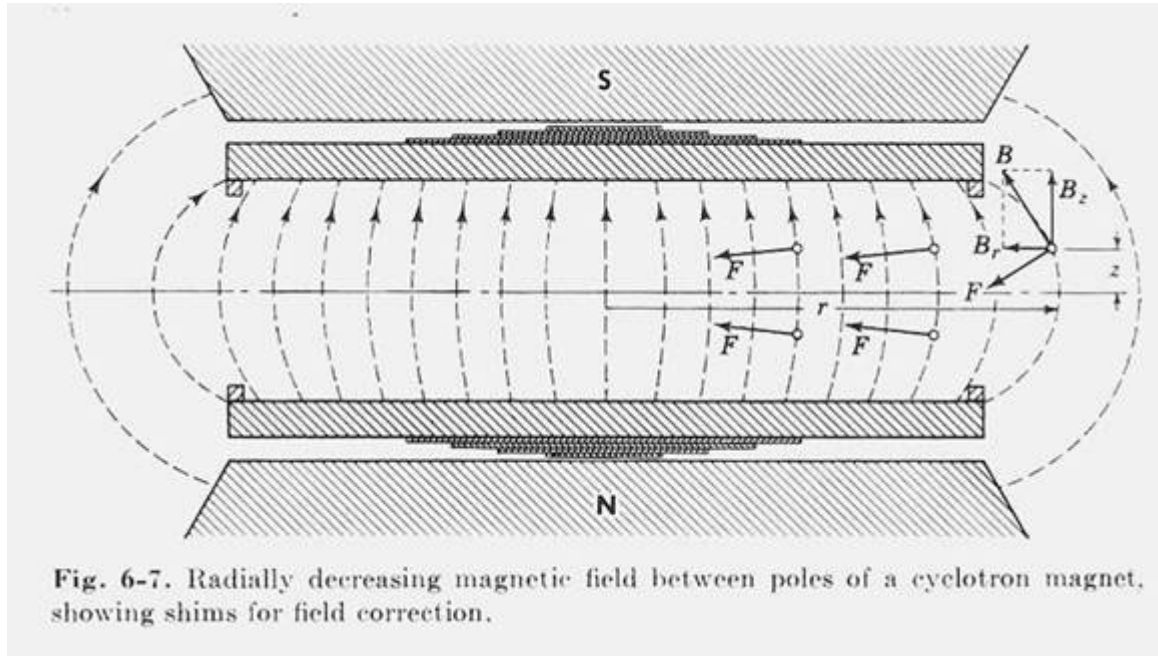
Uniform field is focusing in the radial plane but not in the vertical plane



Weak focusing



Focusing in both planes if field lines bend outward



$$n = - \frac{dB / B}{dr / r}$$

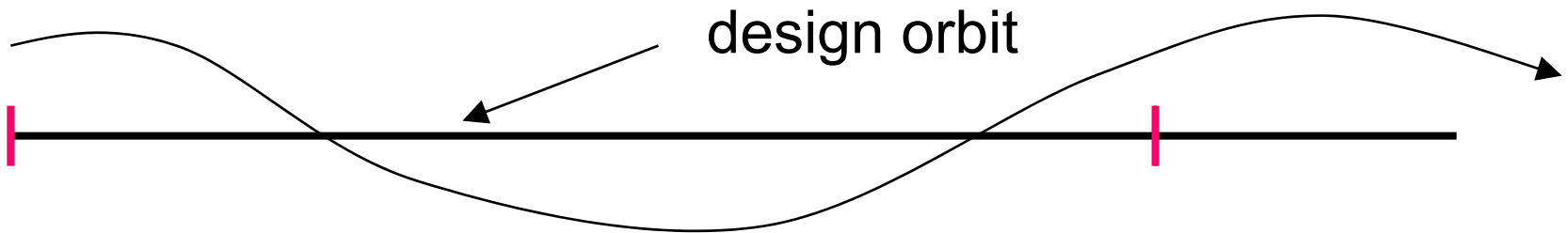
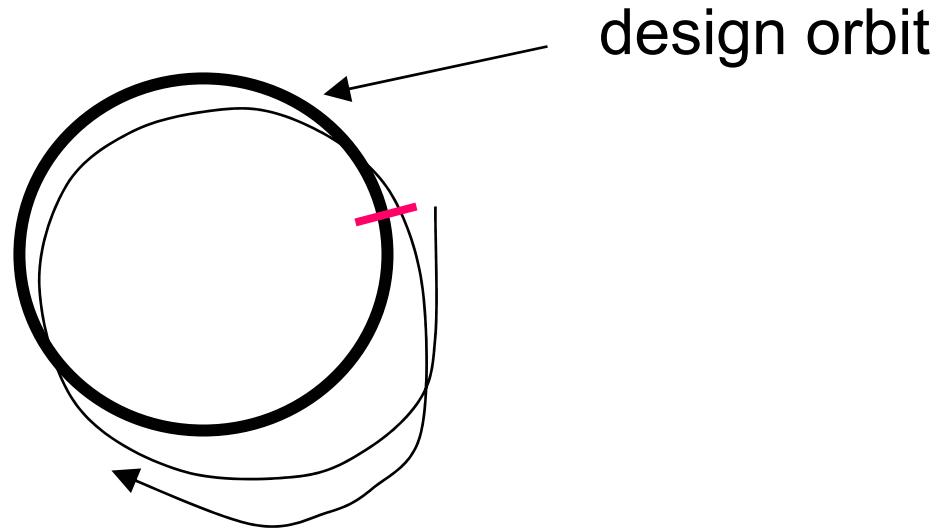
Stability in BOTH PLANES requires that $0 < n < 1$

Vertical focusing is achieved at the expense of horizontal focusing

Tune



The number of oscillations about the design orbit in one turn



Weak focusing



Expressing these results in terms of derivatives measured along the equilibrium orbit

$$x'' + \frac{(1-n)x}{R_0^2} = 0,$$

$$y'' + \frac{ny}{R_0^2} = 0$$

where ' is a derivative with respect to the design orbit

The particle will oscillate about the design trajectory with the number of oscillations in one turn being

$$\sqrt{1-n} \quad \text{radially}$$

$$\sqrt{n} \quad \text{vertically}$$

The number of oscillations in one turn is termed the tune of the ring.

Stability requires that $0 < n < 1$

For stable oscillations the tune is less than one in both planes.

Disadvantages of weak focusing



Disadvantage

- **Tune is small (less than 1)**
 - **As the design energy increased so does the circumference of the orbit**
 - **As the energy increases the required magnetic aperture increases for a given angular deflection**
 - **Because the focusing is weak the maximum radial displacement is proportional to the radius of the machine.**
- The result is that the scale of the magnetic components of a high energy synchrotron become unreasonably large and costly**

Weak-Focusing Synchrotrons



Cosmotron

- The first synchrotron of this type was the Cosmotron at the Brookhaven National Laboratory, Long Island. It started operation in 1952 and provided protons with energies up to 3 GeV.
- In the early 1960s, the world's highest energy weak-focusing synchrotron, the 12.5 GeV Zero Gradient Synchrotron (ZGS) started its operation at the Argonne National Laboratory near Chicago, USA.
- The Dubna synchrotron, the largest of them all with a radius of 28 meters and with a weight of the magnet iron of 36,000 tons



Solution

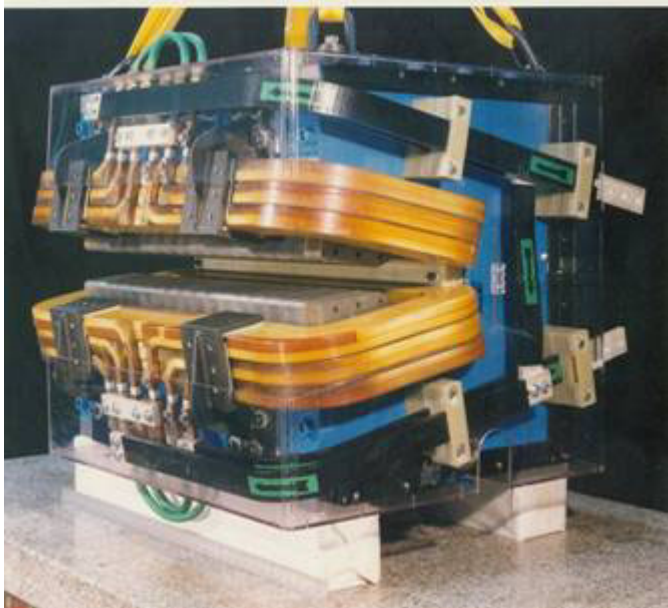
Strong focusing

Use strong focusing and defocusing elements
($|n| \gg 1$)

Strong Focusing



One would like the restoring force on a particle displaced from the design trajectory to be as strong as possible.



ALS Bend (n~25)

- In a strong focusing lattice there is a sequence of elements that are either strongly focusing or defocusing.
- The overall lattice is “stable”
- In a strong focusing lattice the displacement of the trajectory does not scale with energy of the machine
- The tune is a measure of the amount of net focusing.

Strong-Focusing Synchrotrons



In 1952 Ernest D. Courant, Milton Stanley Livingston and Hartland S. Snyder, proposed a scheme for strong focusing of a circulating particle beam so that its size can be made smaller than that in a weak-focusing synchrotron.

- In this scheme, the bending magnets are made to have alternating magnetic field gradients; after a magnet with an axial field component decreasing with increasing radius follows one with a component increasing with increasing radius and so on.
- Thanks to the strong focusing, the magnet apertures can be made smaller and therefore much less iron is needed than for a weak-focusing synchrotron of comparable energy.
- The first alternating-gradient synchrotron accelerated electrons to 1.5 GeV. It was built at Cornell University, Ithaca, N.Y. and was completed in 1954.



Christofilos



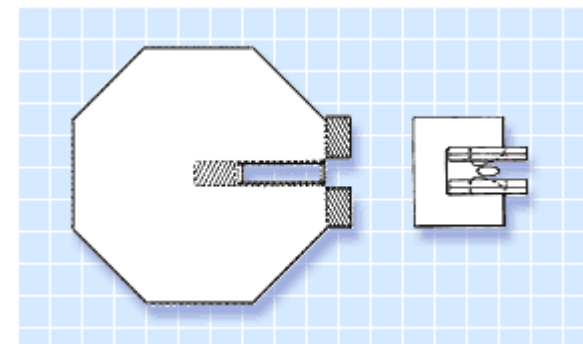
Courant



Livingston



Snyder



Size comparison between the Cosmotron's weak-focusing magnet (L) and the AGS alternating gradient focusing magnets



Describing the Motion

In principle knowing both the magnetic lattice and the initial coordinates of the particles in the particle beam is all one needs to determine where all the particles will be in some future time.

Ray-tracing each particle is a very time consuming → especially for a storage ring where the particles go around for billions of turns.

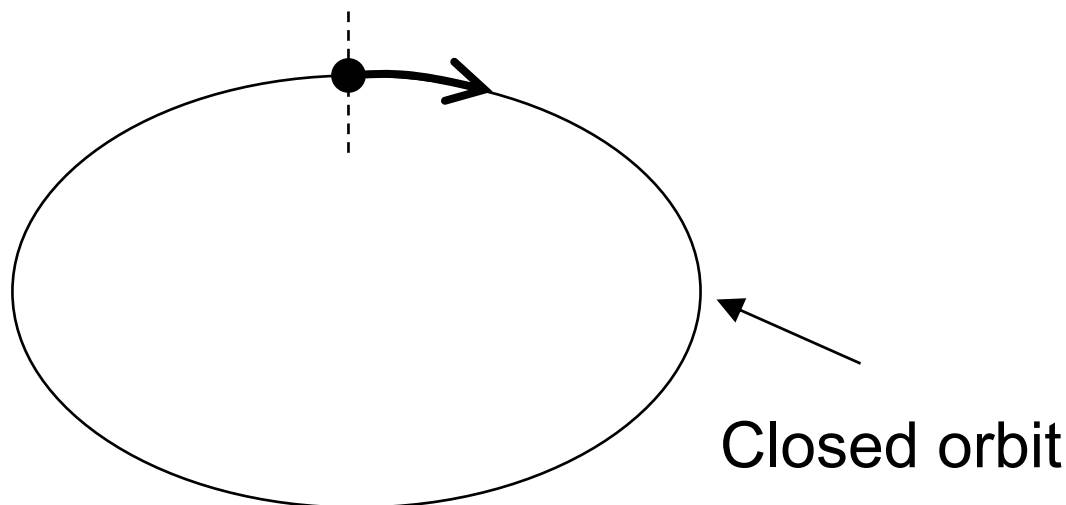
Can do much more

Want to understand the characteristics of the ring → Maps

Find the Closed Orbit



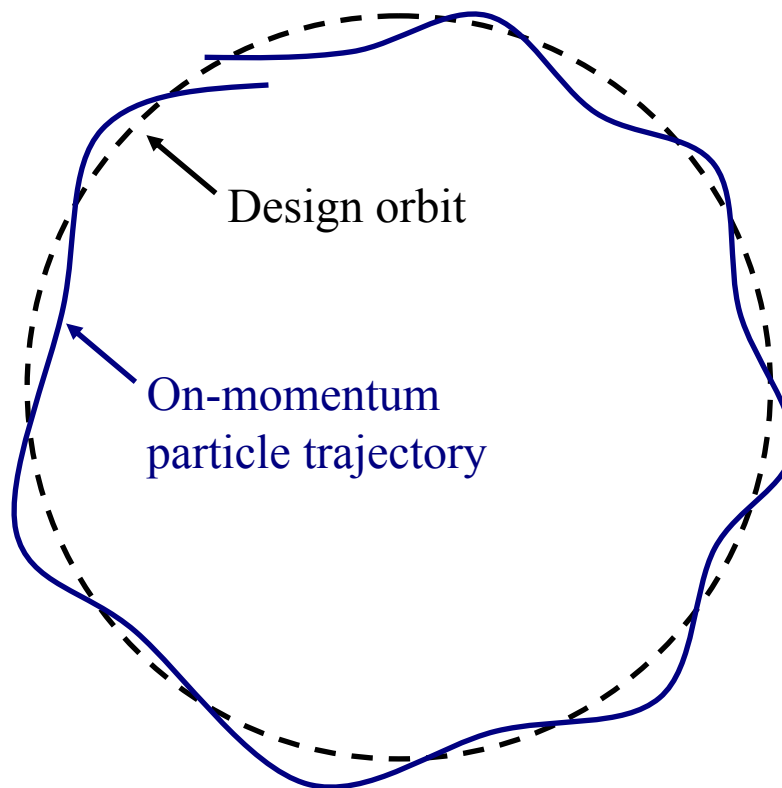
A closed orbit is defined as an orbit on which a particle circulates around the ring arriving with the same position and momentum that it began.



In every working storage ring there exists at least one closed orbit.



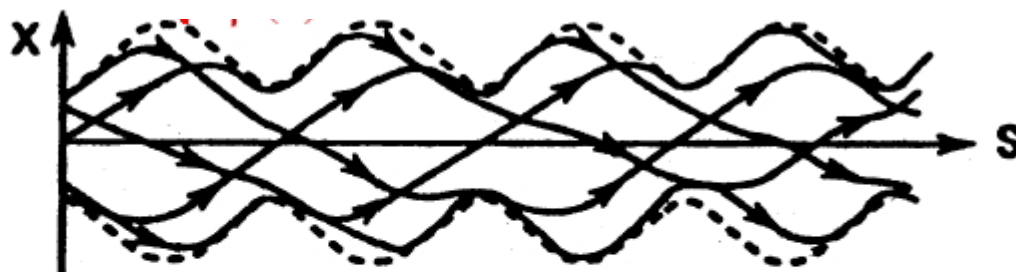
- **Tune is the number of oscillations that a particle makes about the design trajectory**



Beam Envelope



- Some parts of the ring the beam is large and in others it is small



Two approaches



There are two approaches to introduce the motion of particles in a storage ring

- 1. The traditional way in which one begins with Hill's equation, defines beta functions and dispersion, and how they are generated and propagate, ...**
- 2. The way that our computer models actually do it**

I will begin with the first way

First approach – traditional one



This approach provides some insights but is limited

Begin with on-energy no coupling case. The beam is transversely focused by quadrupole magnets. The horizontal linear equation of motion is

$$\frac{d^2 x}{ds^2} = -K(s)x,$$

where $k = \frac{B_T}{(B\rho)a}$, with

B_T being the pole tip field

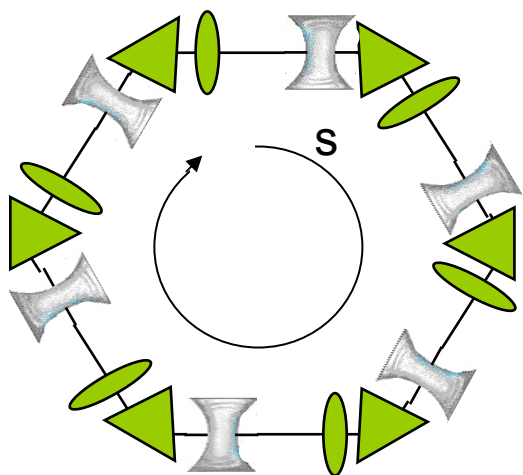
a the pole-tip radius, and

$$B\rho[\text{T-m}] \approx 3.356 p[\text{GeV}/c]$$

Piecewise Focusing



Assume that in a strong focusing synchrotron the focusing varies “piecewise around the ring



$$\mathcal{M} = \mathcal{M}_{10} \dots \mathcal{M}_5 \mathcal{M}_4 \mathcal{M}_3 \mathcal{M}_2 \mathcal{M}_1$$

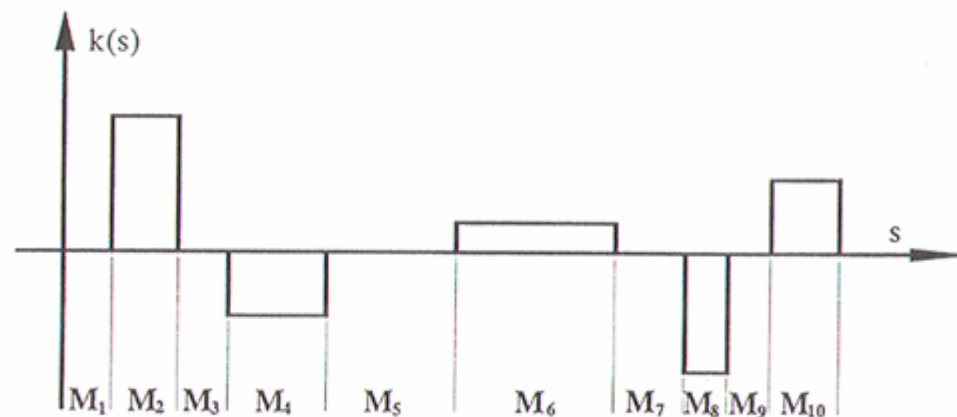


Fig. 5.3. Example of a beam transport line (schematic)

$$x'' + K_x(s)x = 0$$

$$y'' + K_x(s)y = 0$$

Case of Hill's Equation



Illustration in the simple case of Hill's Equation – on-energy

- *Analytically solve the equations of motion*
- **Generate map**
- **Analyze map**

$$x'' + K_x(s)x = 0$$

In a storage ring

$$y'' + K_y(s)y = 0$$

with periodic solutions

$$K_x(s) = K_x(s + C), \quad K_y(s) = K_y(s + C)$$

Solution of Hill's Equation



The general solution of $u'' + k(s)u = 0$

Can be written as

$$u(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\varphi(s) - \varphi(0))$$

There are two conditions that are obtained

$$\frac{1}{2} \left(\beta \beta'' - \frac{1}{2} \beta'^2 \right) - \beta^2 \varphi'^2 + \varphi^2 k = 0$$

$$\beta' \varphi' + \beta \varphi'' = 0$$



Solution of the second condition

$$\beta' \psi' + \beta \psi'' = 0$$

$$\Rightarrow \beta \psi' = \text{const}$$

If we select the integration constant to be 1: $\beta \psi' = 1$
then

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)} + \psi(0)$$

Knowledge of the function $\beta(s)$ along the line allows
to compute the phase function

Hills equation



The solution can be parameterized by a pseudo-harmonic oscillation of the form

$$x_{\beta}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\varphi(s) + \varphi_0)$$

$$x'_{\beta}(s) = -\sqrt{\varepsilon} \frac{\alpha}{\sqrt{\beta(s)}} \cos(\varphi(s) + \varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \sin(\varphi(s) + \varphi_0)$$

where $\beta(s)$ is the beta function,
 $\alpha(s)$ is the alpha function,
 $\varphi(s)$ is the betatron phase, and
 ε is an action variable

$$\varphi = \int_0^s \frac{ds}{\beta}$$

Twiss Parameters



**Define the Betatron or twiss or lattice functions
(Courant-Snyder parameters)**

$$\begin{aligned} \beta(s) & \\ \alpha(s) & \equiv -\frac{1}{2} \frac{d\beta(s)}{ds} \\ \gamma(s) & \equiv \frac{1 + \alpha^2(s)}{\beta(s)} \end{aligned}$$

Courant-Snyder invariant



- Eliminating the angles by the position and slope we define the **Courant-Snyder invariant**

$$\gamma u^2 + 2\alpha uu' + \beta u'^2 = \epsilon$$

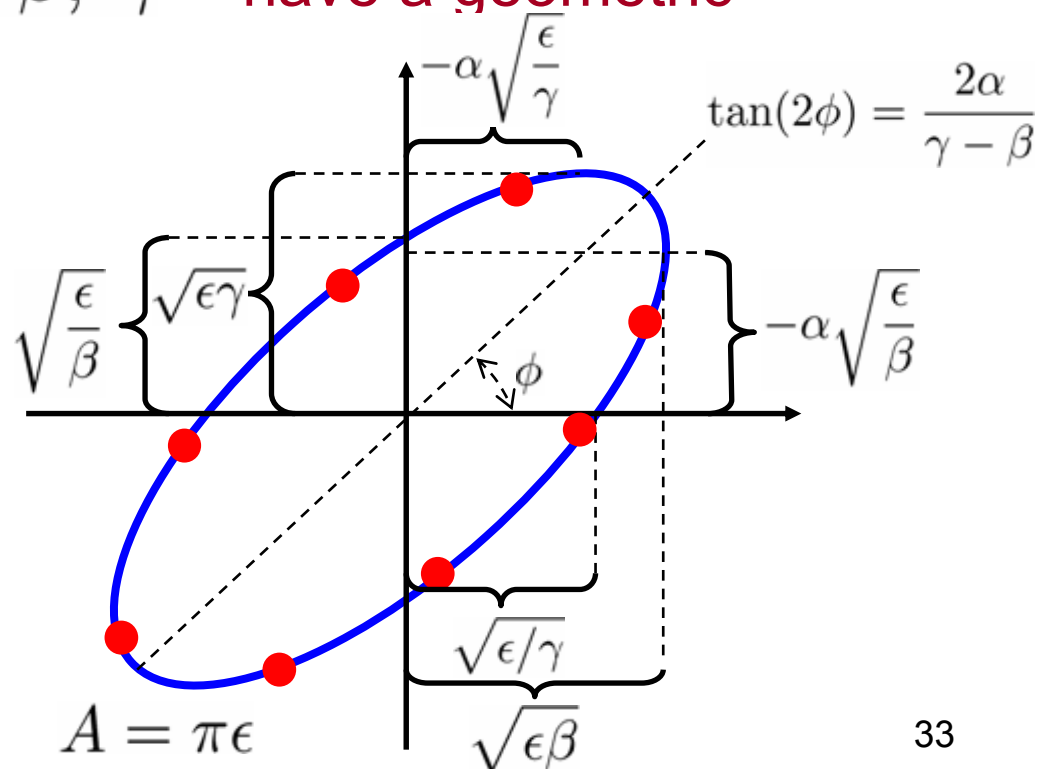
- This is an ellipse in phase space with area $\pi\epsilon$
- The twiss functions α, β, γ have a geometric meaning

- The beam envelope is

$$E(s) = \sqrt{\epsilon\beta(s)}$$

- The beam divergence

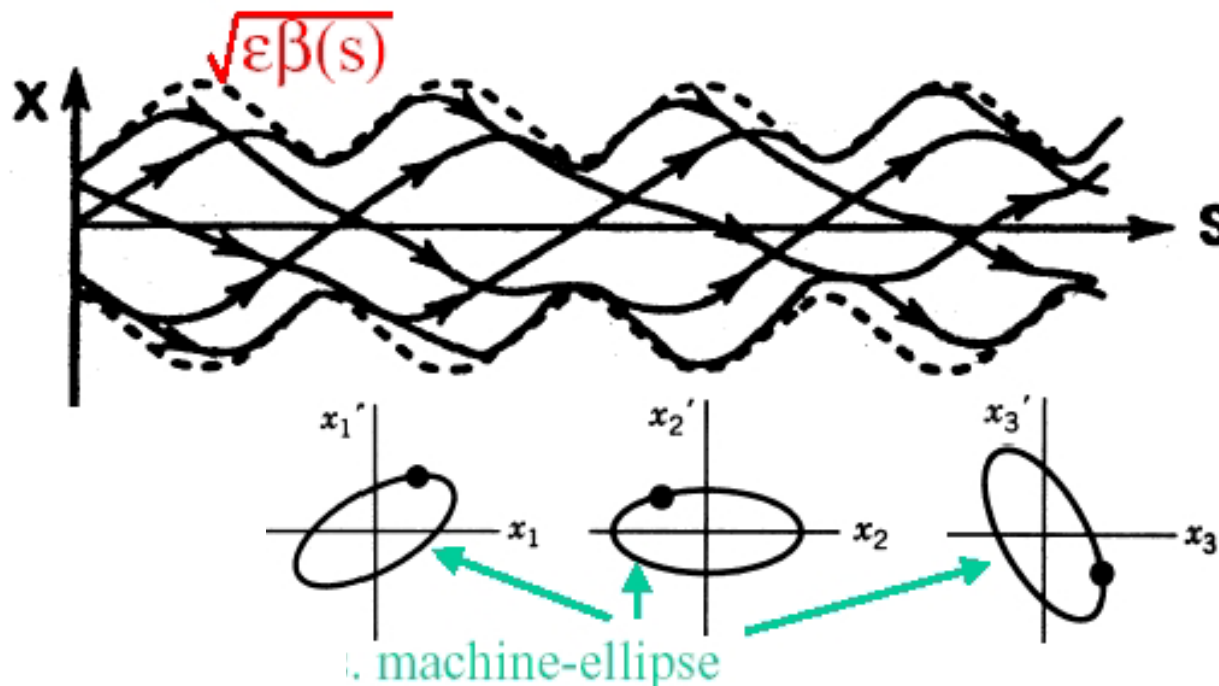
$$A(s) = \sqrt{\epsilon\gamma(s)}$$



The Beam Behavior



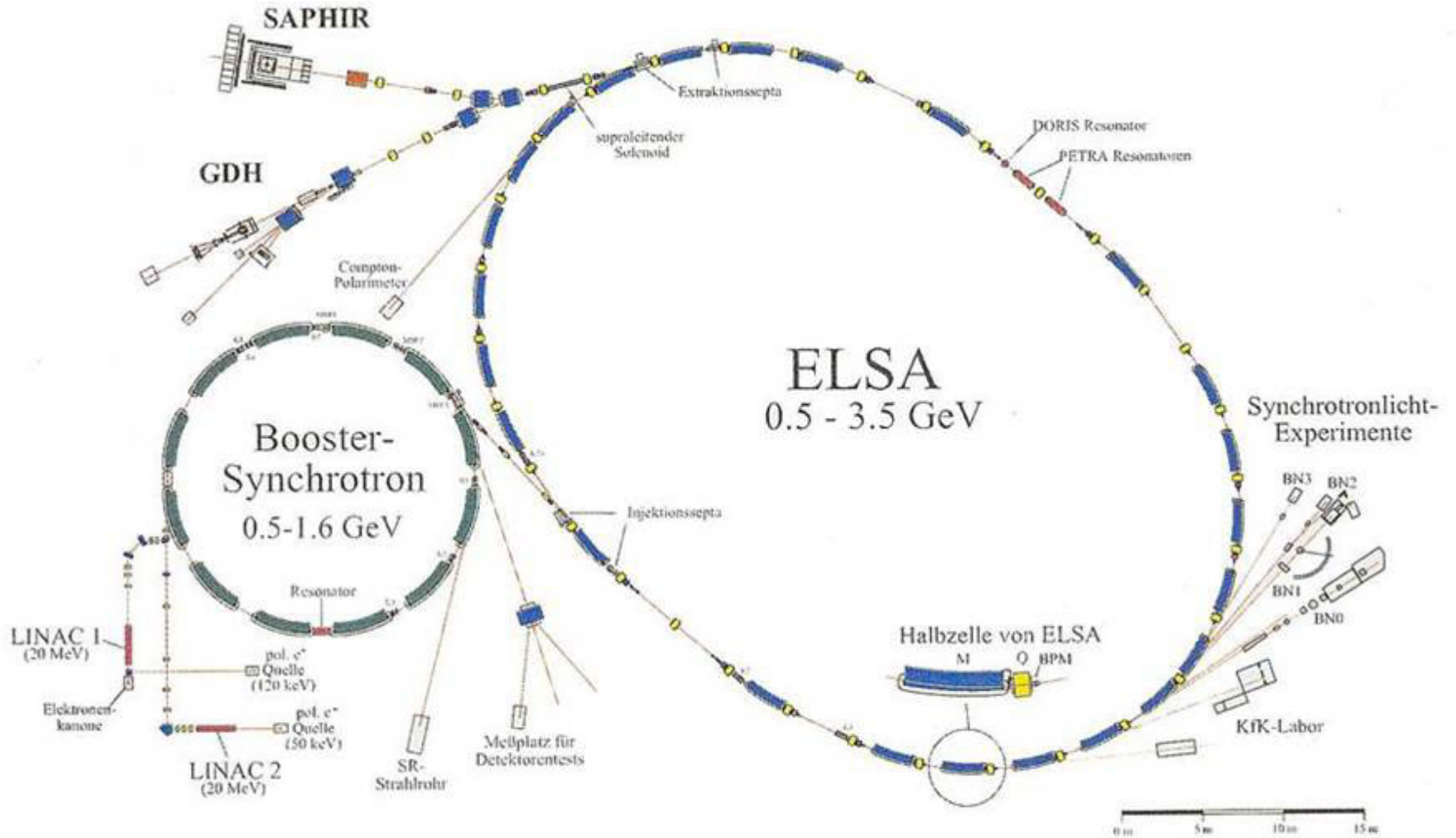
Meaning of Beam Envelope and Beta Function and Emittance



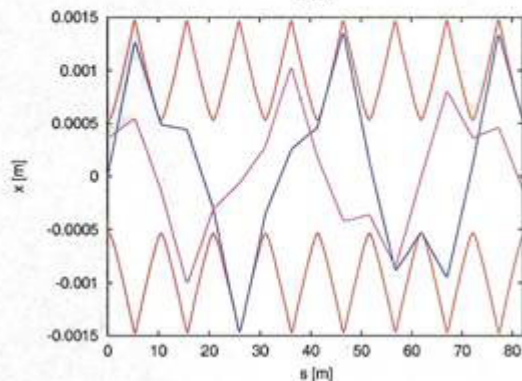
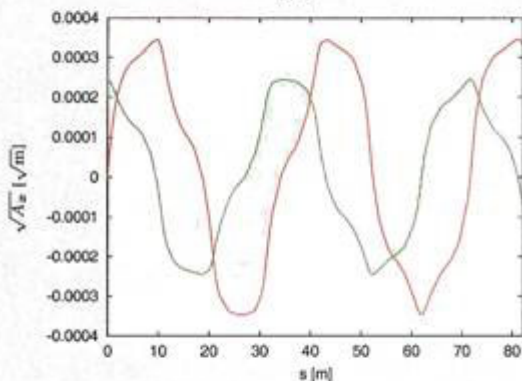
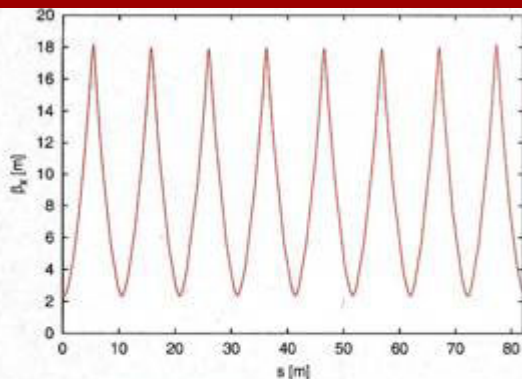
Area of ellipse the same everywhere (emittance)

Orientation and shape of the ellipse different everywhere (beta and alpha function)

ELSA



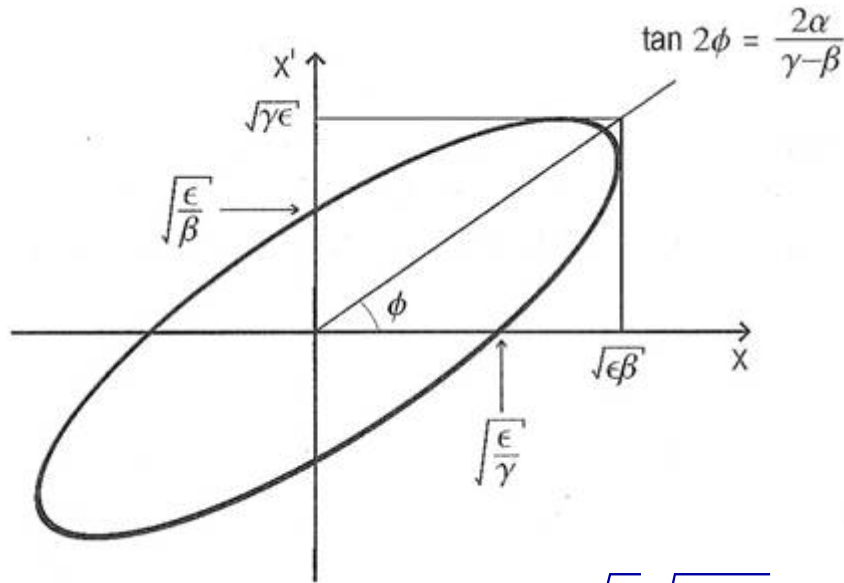
Example from ELSA



Beam Ellipse



In an linear uncoupled machine the turn-by-turn positions and angles of the particle motion will lie on an ellipse



Area of the ellipse, ε :

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

$$x_{\beta}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\varphi(s) + \varphi_0)$$

$$x'_{\beta}(s) = -\sqrt{\varepsilon} \frac{\alpha}{\sqrt{\beta(s)}} \cos(\varphi(s) + \varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \sin(\varphi(s) + \varphi_0)$$



Beam ellipse matrix

$$\sum_{beam}^x = \epsilon_x \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Transformation of the beam ellipse matrix

$$\sum_{beam,f}^x = R_{x,i-f} \sum_{beam,i}^x R_{x,i-f}^T$$

Transport of the Beam Ellipse

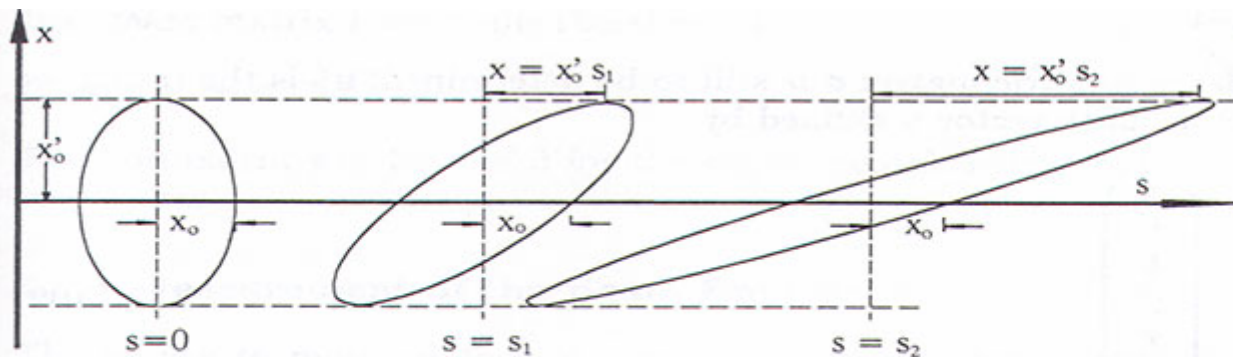


Fig. 5.23. Transformation of a phase space ellipse at different locations along a drift section

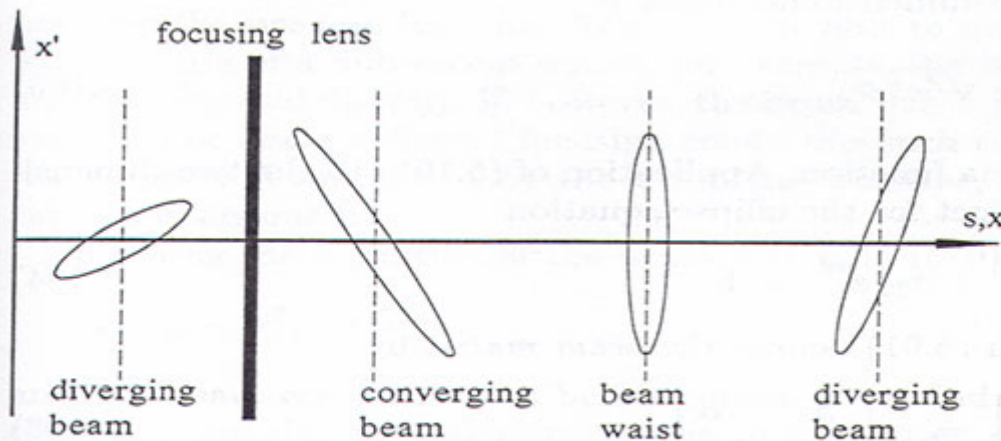


Fig. 5.24. Transformation of a phase ellipse due to a focusing quadrupole. The phase ellipse is shown at different locations along a drift space downstream from the quadrupole.

Transport of the beam ellipse



Transport of the twiss parameters in terms of the transfer matrix elements

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_f = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & 1 + C'S & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_i$$

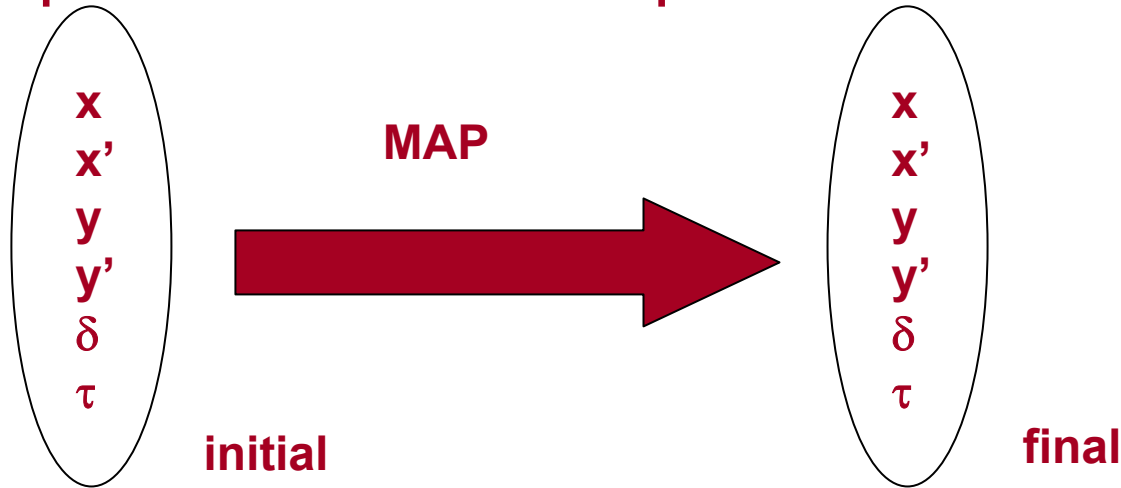
Transfer matrix can be expressed in terms of the twiss parameters and phase advances

$$R_{fi} = \begin{pmatrix} \sqrt{\frac{\beta_f}{\beta_i}} (\cos \varphi_{fi} + \alpha_i \sin \varphi_{fi}) & \sqrt{\beta_f \beta_i} \sin \varphi_{fi} \\ -\frac{1 + \alpha_i \alpha_f}{\sqrt{\beta_f \beta_i}} \sin \varphi_{fi} + \frac{\alpha_i - \alpha_f}{\sqrt{\beta_f \beta_i}} \cos \varphi_{fi} & \sqrt{\frac{\beta_i}{\beta_f}} (\cos \varphi_{fi} - \alpha_f \sin \varphi_{fi}) \end{pmatrix}$$

What is a Map?

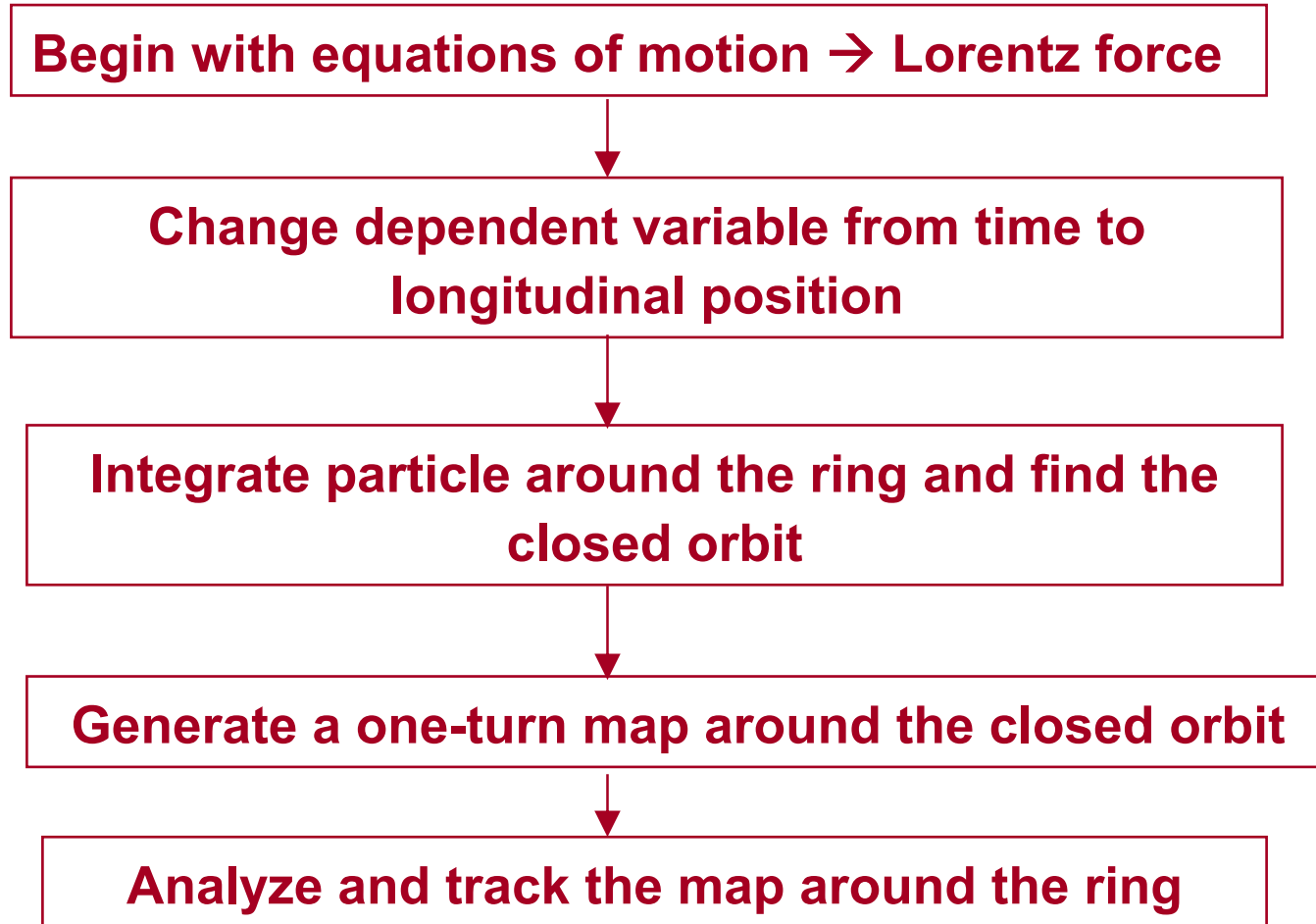


- Use a map as a function to project a particles initial position to its final position.



- A matrix is a linear map
- One-turn maps project project the particles position one turn later

Generating a Map



Generate a one-turn Map Around the Closed Orbit

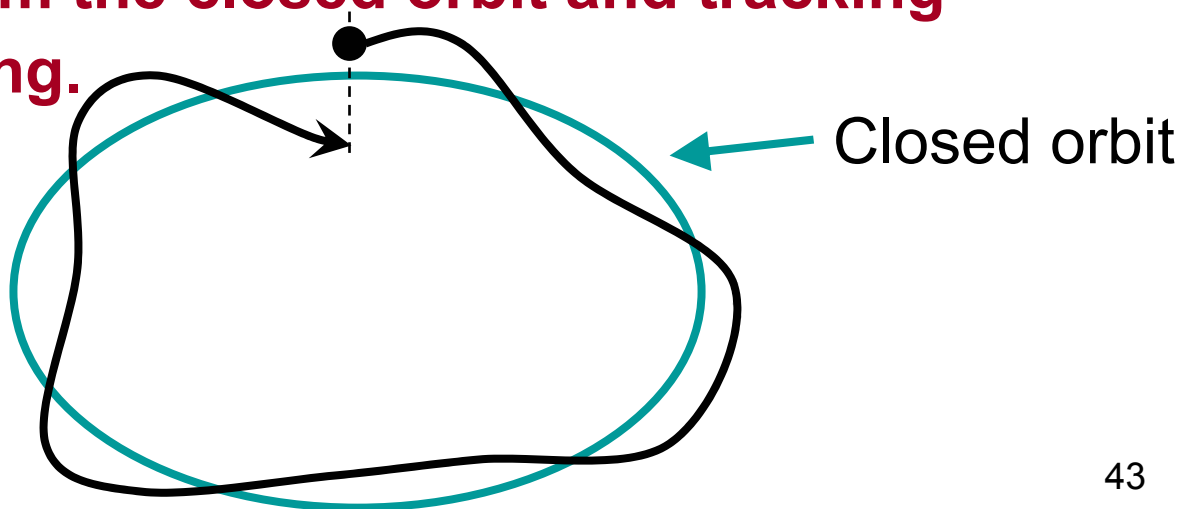


A one-turn map maps a set of initial coordinates of a particle to the final coordinates, one-turn later.

$$x_f = x_i + \frac{dx_f}{dx_i} (x_i - x_{i,co}) + \frac{dx_f}{dx'_i} (x'_i - x'_{i,co}) + \dots$$

$$x'_f = x'_i + \frac{dx'_f}{dx_i} (x_i - x_{i,co}) + \frac{dx'_f}{dx'_i} (x'_i - x'_{i,co}) + \dots$$

The map can be calculated by taking orbits that have a slight deviation from the closed orbit and tracking them around the ring.



How to Compute Twiss Parameters at one point



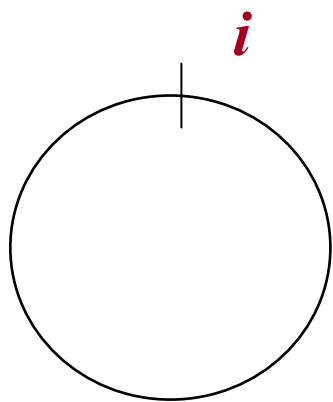
Steps

- 1. Compute the one turn transfer matrix**
- 2. Extract the twiss parameters and tunes**

One Turn Transfer Matrix



One can write the linear transformation, $R_{one-turn}$ between one point in the storage ring (i) to the same point one turn later



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

$$\text{where } R_{one-turn} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

One turn matrix



The one turn matrix (the first order term of the map) can be written

$$R_{one-turn} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos \phi + \alpha \sin \phi & \beta \sin \phi \\ -\gamma \sin \phi & \cos \phi - \alpha \sin \phi \end{pmatrix}$$

Where α, β, γ are called the Twiss parameters

and the betatron tune, $\nu = \phi / (2 * \pi)$

$$\alpha = -\frac{\beta'}{2},$$
$$\gamma = \frac{1 + \alpha^2}{\beta}$$

Emittance Conserved $\rightarrow \text{Det} |R| = 1$

For long term stability ϕ is real \rightarrow

$$|TR(R)| = |2 \cos \phi| < 2$$

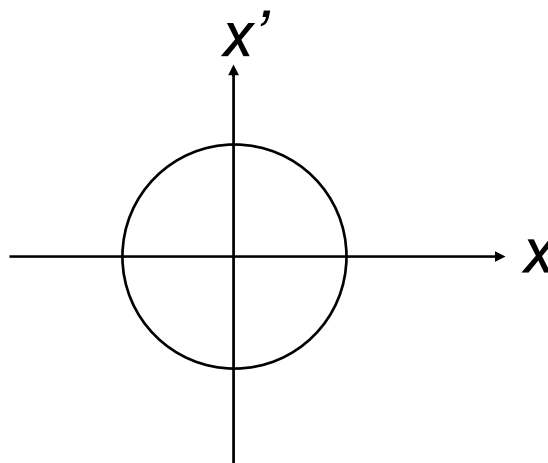
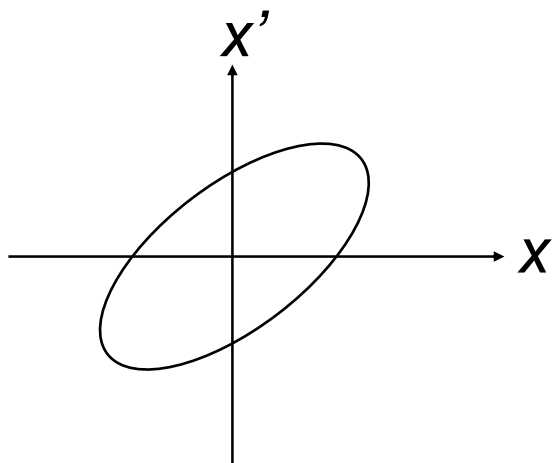
Computation of beta-functions and tunes



One can diagonalize the one-turn matrix, R

$$N_{one-turn} = A R_{one-turn} A^{-1}$$

This separates all the global properties of the matrix into N and the local properties into A .



In the case of an uncoupled matrix the position of the particle each turn in x - x' phase space will lie on an ellipse. At different points in the ring the ellipse will have the same area but a different orientation.

Computation of beta-functions and tunes



The eigen-frequencies are the tunes. A contains information about the beam envelope. In the case of an uncoupled matrix one can write A and R in the following way:

$$N_{one-turn} = AR_{one-turn}A^{-1}$$
$$\begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \cos \varphi + \alpha \sin \varphi & \beta \sin \varphi \\ -\gamma \sin \varphi & \cos \varphi - \alpha \sin \varphi \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

The beta-functions can be propagated from one position in the ring to another by tracking A using the transfer map between the initial point the final point

$$A_f = R_{fi}A_i$$

This is basically how our computer models do it.

Transport of the beam ellipse



Transport of the twiss parameters in terms of the transfer matrix elements

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_f = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & 1 + C'S & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_i$$

Transfer matrix can be expressed in terms of the twiss parameters and phase advances

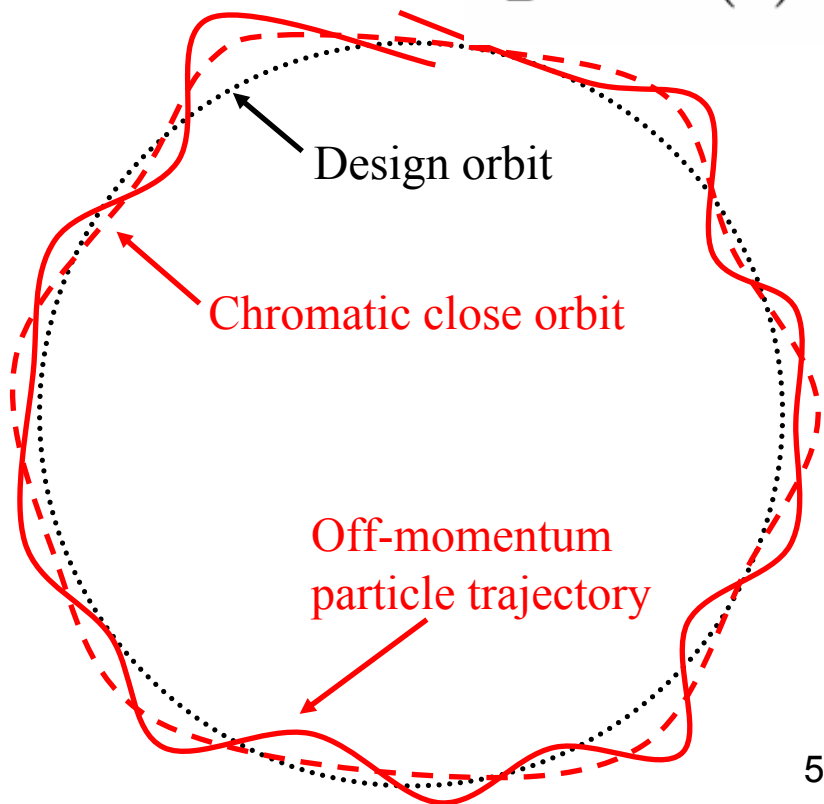
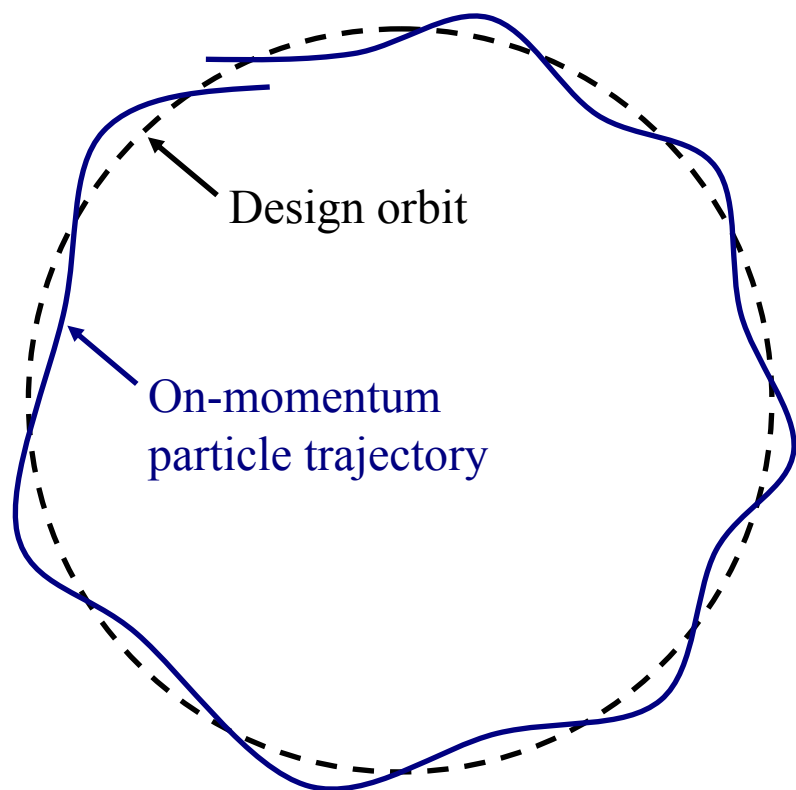
$$R_{fi} = \begin{pmatrix} \sqrt{\frac{\beta_f}{\beta_i}} (\cos \varphi_{fi} + \alpha_i \sin \varphi_{fi}) & \sqrt{\beta_f \beta_i} \sin \varphi_{fi} \\ -\frac{1 + \alpha_i \alpha_f}{\sqrt{\beta_f \beta_i}} \sin \varphi_{fi} + \frac{\alpha_i - \alpha_f}{\sqrt{\beta_f \beta_i}} \cos \varphi_{fi} & \sqrt{\frac{\beta_i}{\beta_f}} (\cos \varphi_{fi} - \alpha_f \sin \varphi_{fi}) \end{pmatrix}$$

Chromatic Closed Orbit



- **Off-momentum particles are not oscillating around design orbit, but around chromatic closed orbit**
- **Distance from the design orbit depends linearly with momentum spread and dispersion**

$$x_D = D(s) \frac{\Delta P}{P}$$

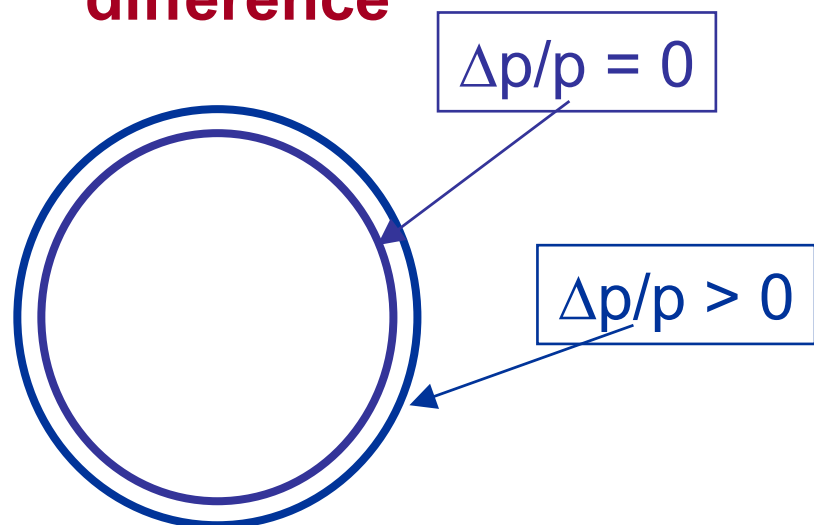


Dispersion



Assume that the energy is fixed \rightarrow no cavity or damping

- Find the closed orbit for a particle with slightly different energy than the nominal particle. The dispersion is the difference in closed orbit between them normalized by the relative momentum difference

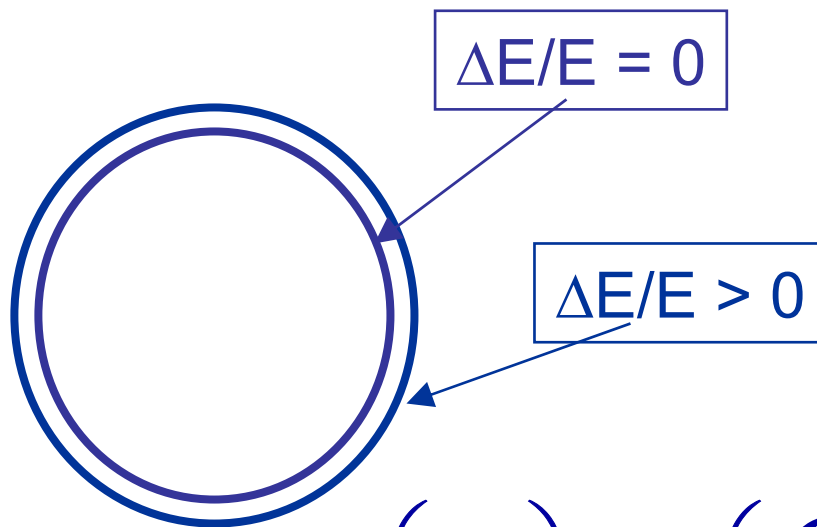


$$x = D_x \frac{\Delta p}{p}, y = D_y \frac{\Delta p}{p}$$
$$x' = D'_x \frac{\Delta p}{p}, y' = D'_y \frac{\Delta p}{p}$$

Dispersion



Dispersion, D , is the change in closed orbit as a function of energy



$$\mathbf{x} = D_x \frac{\Delta E}{E}$$

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \\ \delta \end{pmatrix}_f = \begin{pmatrix} \mathbf{C} & \mathbf{S} & D_x \\ \mathbf{C}' & \mathbf{S}' & D'_x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \\ \delta \end{pmatrix}_i$$

Possible Homework



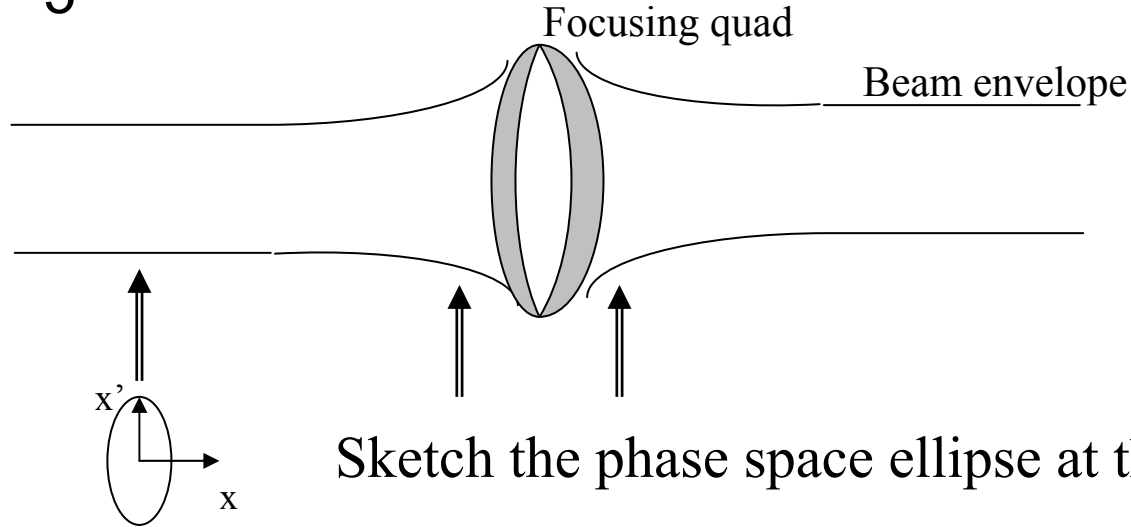
- 1. At the azimuthal position s in an proton storage ring, the Twiss parameters are $\beta_x=10$ m, $\beta_y=3$ m, and $\alpha_x=\alpha_y=0$. If the beam emittance ε is 10 nm for the horizontal plane and 1 nm for the vertical one and the dispersion function η at that location is zero for both planes, what is the rms beam size (beam envelope) and the rms beam divergence for both planes at the location s ? What will be the case for an electron beam?**
- 2. Explain what the dispersion function represent in a storage ring. Explain what is the difference between dispersion and chromaticity.**
- 3. Explain the difference between an achromat cell and an isochronous one.**
- 4. In the horizontal direction, the one-turn transfer matrix (map) for a storage ring is:**
 - Is the emittance preserved?**
 - Is the motion stable**

$$\begin{pmatrix} 1.5 & 1 \\ 0.05 & 0.7 \end{pmatrix}$$

L6 Possible Homework



5



Sketch the phase space ellipse at these locations

Isochronous and Achromatic Transport



- No dispersion or dispersion slope at the end of the line
- Dispersion is positive in the central bend but the central bend is inverted

