On Multi-dimensional Compactons

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We study the *two and three dimensional*, $N = 2, 3$, nonlinear dispersive equation $C_N(m, a+b)$: $u_t + (u^m)_x + \left[u^a \nabla^2 u^b\right]_x = 0$ where the degeneration of the dispersion at ground state induces cylindrically and spherically symmetric compactons convected in x-direction. An initial pulse of bounded extent decomposes into a sequence of robust compactons. Colliding compactons seem to emerge from the interaction intact, or almost so.

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Introduction.- Arguably, modern nonlinear science has started in earnest with the discovery of solitons, c.f.,[1]. What started as a humble attempt to understand the FPU and KdV equations has turned into one of the pillars of nonlinear science with a wide range of applications in optics, plasma hydrodynamics and solid state physics. However, with rare exceptions which do not have scalar counterparts [3,4], the well known solitary structures (whether exactly solitonic or almost so)in the continuum are limited to one-spatial dimension, $1-D$,[4].

In spite of many remarkable mathematical advances this picture has not significantly changed in the last forty years. In fact, the stunning success of the $1-D$ solitonic theory can be only compared to its equally stunning failure in higher dimensions.

Viewing integrability as a miracle, one is tempted to metaphysical resignation. However, when one seeks robust structures, which are close to, but not exactly integrable, then the quest for localized structures in higher dimensions has a pertinent scientific validity. We surmise that the search for soliton bearing equations, like the metaphoric search for a coin under an 'enlightened' lantern, was for too long attached to 'lantern's pole', [5].

Without dwelling on exact definitions we recall that solitons are a manifestation of the balance between inertial and dispersive forces. Typically, soliton supporting equations are obtained via weakly nonlinear perturbation schemes. However in higher spatial dimensions these procedures yield equations which are non-integrable and, as a rule, are unable to support genuinely localized patterns. *While very disappointing this should not have come as a surprise* for increasing the spatial dimension *shifts the balance between inertia and dispersion;* whereas nonlinearity due to inertia plays the same role, *the increase in degrees of freedom increases the effectiveness of dispersion.* It thus stands to reason that a well balanced model in $1-D$ will be less so in a higher dimensions, or be lost entirely - and, indeed, this is what happens.

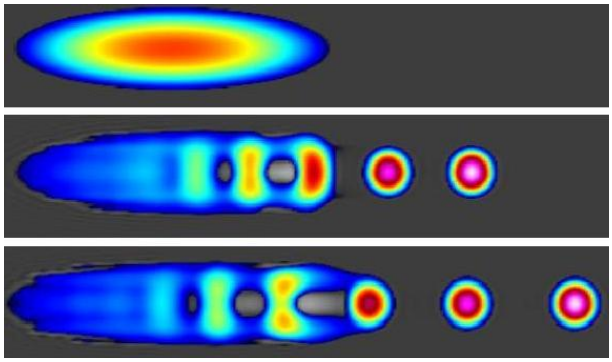

FIG. 1: Position of $C_2(3, 1+2)$ compactons support evolving out of an elongated initial support (upper panel). Their corresponding profiles are displayed on Fig.(2). Note that whenever $m = a+b$ the respective compactons have the same support. Note also that periodic boundaries were assumed.

For a genuinely localized structure to emerge in N-D, *one has either to enhance the nonlinearity or, as we shall do,[5], to properly weaken the dispersion.* This can be accomplished using nonlinear dispersion that degenerates at a ground state. The $1-D$ implication of this approach begets *compactons*, robust solitary waves with compact support, [6] (for a recent exposition see [7]). In this letter we demonstrate that this mechanism can also create N-dimensional compact structures.

The compacton, a compact solitary wave, is a non-analytical robust entity *with the singularity dependent on the nonlinearity of the degenerating dispersion but is independent of the spatial dimension.* To understand these localized dispersive structures in higher dimensions one has to probe deeper into the nonlinear domain. Our present aim is to explore the essential ingredients needed for such structures to exist.

The class of model equations in this letter is perhaps the simplest attempt to make a solitonic break through the $1-D$ barrier. We believe that the seasoned reader

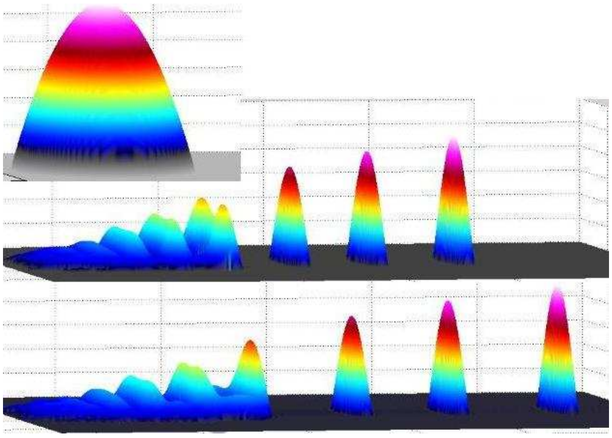

FIG. 2: Temporal snapshots of of $C_2(3, 1+2)$ compactons evolving out of an elongated initial pulse (upper panel).

should have no difficulty to see how the essentials of our discussion apply to phenomena for which a very mathematical description will apply.

The model Equation.- To demonstrate the viability of our arguments, we now present an $N-D$ model wherein the unidirectional convection is balanced by a N-dimensional dispersive force, [5,9],

$$\mathcal{C}_\mathcal{N}(m, a+b) : u_t + (u^m)_x + \frac{1}{b}\left[u^a(\nabla^2 u^b)\right]_x = 0, \quad (1)$$

where $m \geq max(1, a-1)$ and $b > 0$. As in the $1-D$ model, a particular choice of exponents (m, a and b) reflects a specific physical mechanism. Notably, when $a = b = 1$ and $m = 2$, Eq.(1) describes the sedimentation of particles in a dilute dispersion, though at the time neither the multidimensional feature of the model nor its ability the generate compact structures was realized,[8].

Travelling Compactons: We assume spherically symmetric compactons traversing in x-direction and define

$$s = x - \lambda t \quad and \quad R = \sqrt{s^2 + y^2 + z^2}. \quad (2)$$

Integrating in a travelling frame yields

$$u^a\Big[-\lambda u^{1-a} + u^{m-a} + \frac{1}{bR^{N-1}}\frac{d}{dR}R^{N-1}\frac{d}{dR}u^b\Big] = 0. \quad (3)$$

Note that λ may be scaled out in terms of $u = \lambda^{m-1}U\left[\lambda^{\frac{m-n}{2(m-1)}}R\right]$,where $n \equiv a+b$. Therefore structure's width scales as $\sim \lambda^{\frac{n-m}{2(m-1)}}$ and as the wave's speed increases its width shrinks (swells), for $m > n$ ($n > m$). When $m = n$, convection and dispersion are in a detailed balance and compactons width is fixed.

Though existence of compactons for Eq.(3) is easily asserted, for $N > 1$ we have identified only two types of explicit solutions, [5,9]:

a) $C_N(m = 1+b, 1+b)$: The compacton solution is

$$u = \lambda^{\frac{1}{b}}\Big[1 - \frac{F(R)}{F(R_*)}\Big]^{\frac{1}{b}}, \quad 0 < R \leq R_*, \quad (4)$$

and vanishes elsewhere. In the planar case: $F(R) = J_0(\sqrt{b}R)$ and in $3-D$: $F(R) = \sin(\sqrt{b}R)/\sqrt{b}R$. In each case integration constant assures that u vanishes at R_*, the first trough of $F(R)$, where it is compactified.

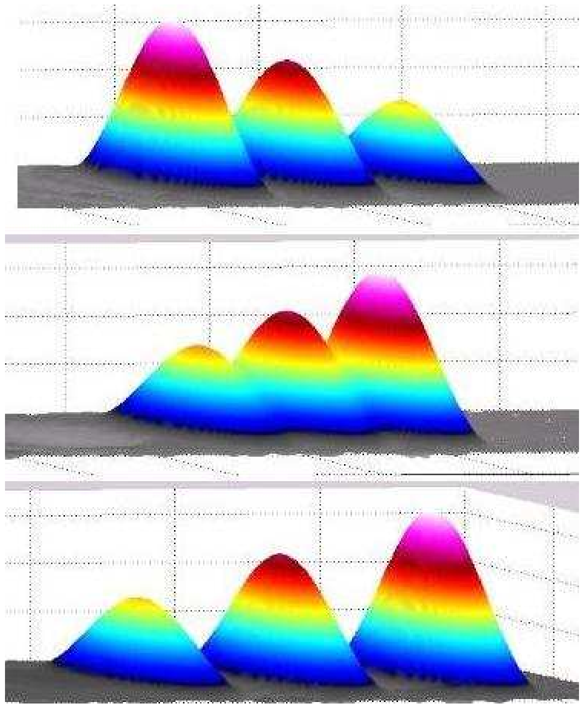

FIG. 3: Collision of three $C_2(2, 0+2)$ compactons with speed $\lambda = 2, 3/2$ and 1. As in 1-D, the interaction is very clean and the compactons seem to emerge intact leaving in the perimeter of the domain of interaction a small ripple.

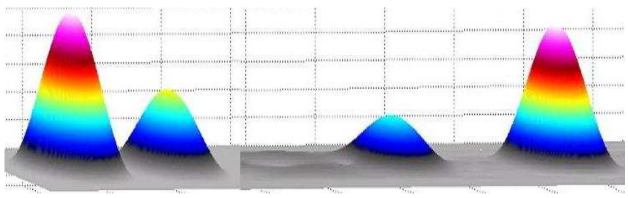

FIG. 4: Collision of two $C_2(2, 1+1)$ compactons. Note the loss of mass of the smaller compacton after the collision. In 2-D this effect is more pronounced than in 1-D.

b) $C_N(m = 2, a+b = 3)$: The solution is a parabola

$$u = \kappa_N\Big[\lambda A_N - bR^2\Big], \quad 0 < R \leq R_* \equiv \sqrt{\lambda A_N/b}. \quad (5)$$

and

$$C_N(2, 0+3) : A_N = \frac{3}{2}(4+N)^2, \quad \kappa_N^{-1} = 6(4+N). \quad (6)$$

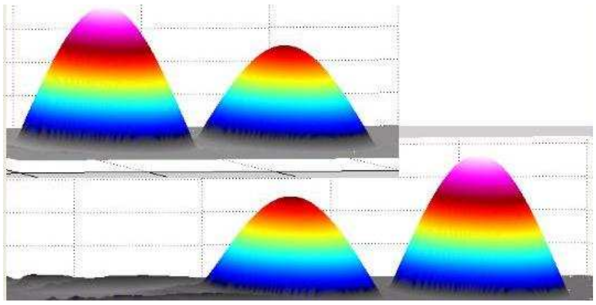

FIG. 5: Display of two colliding $C_2(3, 0+3)$ compactons. As in the $C_2(2, 0+2)$ case, see Fig.(3), the two compactons re-emerge intact after the collision without a measurable loss of mass.

$$C_N(2, 1+2): A_N = 2(2+N)^2, \quad \kappa_N^{-1} = 4(2+N). \quad (7)$$

Note that unlike case (a) now $R_* \sim \sqrt{\lambda}$.

We note the following property of solutions of Eq.(3): given a compacton solution $u(R)$ of $C_N(m, a+b)$, then $V(R) = u^\kappa(R)$ is a solution of $C_N(m_*, a_* + b_*)$ where

$$m_* = 1+\kappa(m-1), \quad a_* = 1+\kappa(a-1) \ \textit{and} \ b_* = \kappa b. \quad (8)$$

Thus every solution $u(R)$ generates a $\kappa - parameter$ family of solutions.

Numerical studies.- The results of our numerical experiments which have a vastly richer dynamics than in $1-D$, are best seen as movies at *http : //math.lanl.gov/ mac/compacton*. Every figure is a snapshot of a corresponding movie. In what follows we provide a brief exposition of the essential features of formation and interaction of compactons in $2-D$. Figs.(8) and (9) that describe a $3-D$ interaction and evolution are a prelude to a more detailed exposition,[10].

The essence of our studies is to demonstrate *the ability of degenerating nonlinear dispersion to induce robust compact patterns in higher dimensions.* Extensive numerical studies which focused on the emergence of compactons out of initial data and their collision, allow us to present a number of meaningful conclusions. It is natural to seek the impact on the the dynamics of the:

(A.)Dimensionality and convection exponent m.

(B.) Interplay between a and b. Expanding the dispersive part in (3) the leading part is $u^{n-1}\nabla^2 u_x$. Although the lower order part of the dispersion

$$(u^{n-1})_x \nabla^2 u + (b-1)[u^{n-2}(\nabla u)^2]_x, \quad (9)$$

has no impact on the singularity at the edge of the compacton, it is essential for compactons propagation. Its effectiveness depends on the parameter $\omega \equiv 1+b-a$. Travelling compactons seem to emerge only if $\omega > 0$,[9]. Thus, for a fixed n, reduction of b reduces ω and the effectiveness of the lower order term to propel the motion.

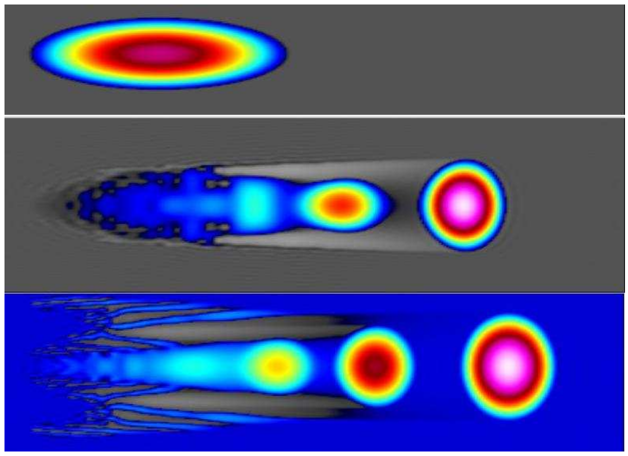

FIG. 6: Evolution of an elongated initial pulse (upper panel) at $t = 18$ for $C_2(2, 0+3)$ (lower panel) and $C_2(2, 1+2)$ (middle panel).In both cases $n = 3$, but since $b = 3$ and $b = 2$, respectively (see Eq.(8)), in the first case the evolution is faster and the emerging compactons are bigger.

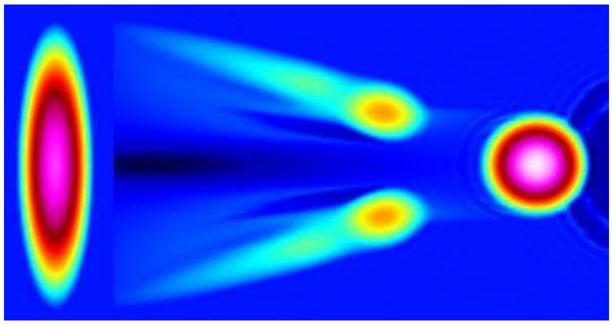

FIG. 7: Decomposition of initial vertical pulse of $C_2(2, 1+2)$ into a compacton and a pair of travelling blobs which at a later time coalesce into a second compacton. Unlike Fig.(1) where the initial support stays put while emitting compactons, here the perturbed domain as a whole is in motion.

When $\omega < 0$ evolution is confined to the initial domain. Compare $C_N(2, 0+3)$ with $C_N(2, 1+2)$ in Fig.(6): they have the same singularity at the edge but the "propelling force" of the first is stronger (see Eq.(9). Also, if you take $C_N(2, 2+1)$ where $\omega = 0$, then (5) has a solution only when $\lambda = 0$.

Emergence of compactons: See Figs.(1-2,6-7)and (9). In all cases where initial evolution was followed as an initial condition we have used

$$u = U_0 \cos^2(\alpha r), \ r \leq \frac{\pi}{2\alpha}, \ U_0, \alpha \ \textit{are constants}. \quad (10)$$

and vanishes elsewhere. Looking at the figures one notes that the number of the emerging compactons and their spatial location depends on the geometry of initial conditions and their span. In Fig.(1) the initial perturbation stays put as it emits compactons. Such scenario is typical of $1-D$ patterns. On the other hand, emission of compact pulses which are not compactons but converge into compactons as they travel; see Figs.(6) and (9), does not seem to occur in $1-D$. Also, as seen in Fig.(7), in higher dimensions we observe patterns wherein the initial pulse propagates as a whole while evolving and emitting compactons.

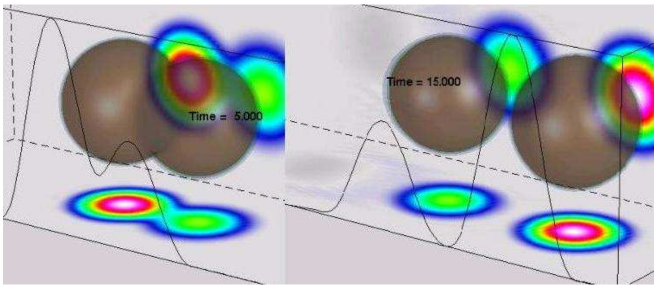

FIG. 8: Collision of two 3-dimensional $C_3(2,0{+}2)$compactons. Early stage of interaction of their supports at $t=5$ and immediately thereafter, at $t=15$. We also project compactons profile $u(R)$.

Interaction between compactons: Figs.(3-5) and (8) present the results of hard collisions which occur when the centers of colliding compactons are aligned. Soft collisions which seem more like a skirmish happen when the center of the faster compacton is off center from its prey. When the centers of softly colliding compactons are sufficiently close, *the fast and the slow compactons exchange their positions.* As a rule soft collisions seem to be less elastic then their hard counterparts,[10].

Some hard interactions appear to be much closer than others to being elastic with the cleanest interaction undoubtedly reserved to $m=n=2,3$ and $a=0$ compactons: $C_N(2,0+2)$ and $C_N(3,0+3)$. As in $1-D$,[6], we observe compactons emerging intact in both $2-D$ and $3-D$ cases, Figs. (3) and (8).

We also studied $C_2(3,1{+}3)$ and $C_2(4,1{+}3)$, not shown. Here even though $m=n$, since $a=1$, the lower order part (9) is less effective and collisions, though quite clean, are a bit less elastic than in the $m=n$, $a=0$ cases. Thus, while the larger compactons hardly loose any mass, there is a small but noticeable loss of mass in smaller compactons. Repeated interactions enhance this effect.

The interaction of 'parabolic compactons' Eq.(5), $m=2$ and $n=3$: $C_2(2,0{+}3)$ and $C_2(2,0{+}3)$ is far less robust. In collisions the smaller compactons re-emerge greatly diminished, occasionally accompanied with chunks of splitting off pieces of mass. A more comprehensive discussion with detailed $3-D$ displays will be presented in [10].

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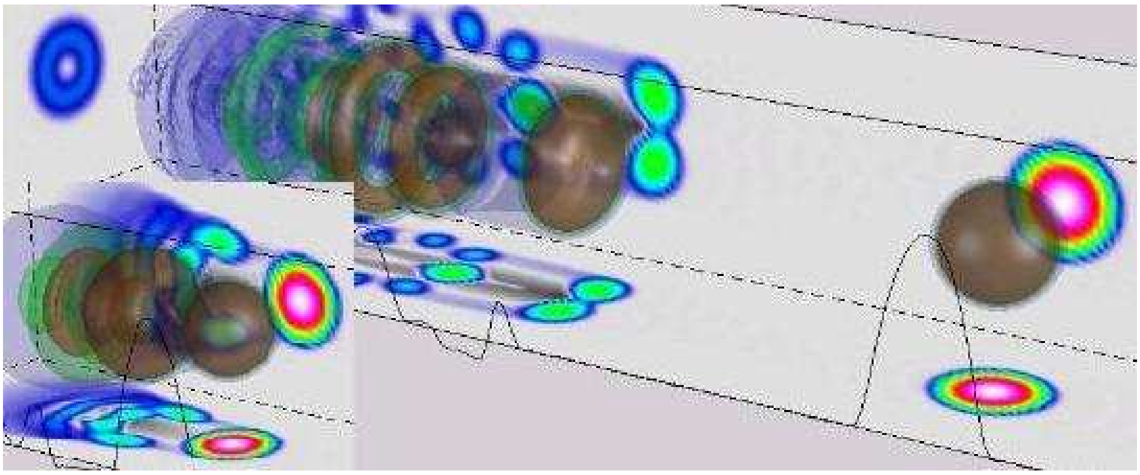

FIG. 9: Emergence of 3-D $C_3(2, 0+3)$ compactons out of an initial 3-D ball which breaks into a sequence of toroidal supports displayed at $t \sim 90$, each of which first turns into a travelling doughnut and later into a compacton. The doughnut following the compacton has just emerged. It will converge into a ball at $t \sim 140$. Insert: evolution at $t \ \sim 25$.