A Power-Law Formulation of Laminar Flow in Short Pipes

Max Sherman
Indoor Environment Program
Applied Science Division
Lawrence Berkeley Laboratory,
University of California
Berkeley, California 94720

ABSTRACT

This report develops a theoretical description of the hydrodynamic relationship based on a power-law representation between the air flow and applied pressure for laminar flow in short pipes. It is found that short pipes can be described with a simple power law dependence on pressure, but that the exponent of the power law is itself a function of pressure. The entry length of the flow is derived based on a formulation for short, sharp-edged pipes. The theoretical formulation is compared to measured data. A dimensionless quantity, S, is defined to account for the power law behavior and maps simply to the flow exponent. The S number can be used to infer many of the characteristics of the flow and may prove useful in the inverse problem of determining flow geometry from fluid properties and the measured pressure and flow.

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INTRODUCTION

When applying physical laws to complex situations, laws with different scaling properties may come into play simultaneously. The analysis of these situations is often made easier by using power law descriptions of the phenomena. The dimensionless numbers and exponents associated with such a description can be used to extract the essential characteristics of the system of interest.

The system studied herein is the flow of an incompressible, viscous fluid through a short pipe in response to a pressure difference. There are two distinct loss mechanisms in operation: the energy required to accelerate the fluid from rest and the energy required to overcome the viscous losses associated with laminar flow.

There are three types of problems to consider: flow prediction, viscometry, and systems characterization. In *flow prediction* the pipe characteristics and fluid properties are known and the flow vs. pressure response is desired. In *viscometry* the flow vs. pressure response and pipe characteristics are known and the fluid properties are desired. In *system characterization* the fluid properties and flow vs. pressure response are known and the pipe characteristics are desired.

Power laws are often used as an empirical relationship for physical phenomena, when all that is known is that there is a smooth monotonic relationship between to measurable variables. Power laws are often used in to describe turbulent flow situations (e.g., pipe-network[1] or skin-resistance[2] solutions). In general power-law approaches to the solution of laminar flow problems are not used. This report can be used to help bridge the gap in formalism between laminar and turbulent flow by providing a rigorous power-law treatment of laminar flow.

One area in which laminar flow has been treated using a power-law approach is that of building physics. The techniques for measuring the air leakage of building envelopes[3-5] all treat the flow as a power-law function of the applied pressure. Justification for such treatment has been empirical, but as shown herein, a power-law formulation is a good description of the physical phenomena.

BACKGROUND

Regardless of the flow regime it may be possible to cast the *mean velocity* (hereafter referred to as the velocity) as a power law in the pressure drop:

$$v \propto \Delta P^n \tag{1}$$

If the power law were absolute, the exponent would be constant and completely independent of the applied pressure. If, however, the exponent is slowly-varying in pressure, the power law can be used in an approximate form where both the exponent and coefficient are slowly-varying functions of pressure.

The exponent is only a good descriptor if it does not vary too much. A good criterion for determining the usefulness of the power law description is

$$\left| \delta P \; \frac{dn}{d\Delta P} \right| \ll 1 \tag{2}$$

where δP is the range of pressure of interest around ΔP.

If this criterion is met then the power law is a reasonable description. Regardless of the true functional relationship between the pressure and velocity, the following expression can be used to define the local exponent:

$$n \equiv \frac{\Delta P}{v} \frac{dv}{d\Delta P} \tag{3}$$

Laminar Entry Flow

The problem of laminar flow in short pipes has been investigated for over a century. Several authors[6-8] have elected to treat it by linearizing the Navier-Stokes equation and all have come up with an equation of the following form:

$$\Delta P = \frac{32\,\mu\, l\, v}{d^2} + m\,\tfrac{1}{2}\rho v^2 \tag{4}$$

The first term can be recognized as the Hagen-Poiseuille equation for (fully-developed) laminar flow. As described in detail in the references, the second term results from acceleration of the fluid into its exit profile and the excess viscous losses resulting therefrom.

The factor m can be calculated from the linearized theory and has also been measured. For pipes long enough for the exit profile to be parabolic, the parameter m can be treated as a constant; and the literature contains a range of values for it, $2.16<m<2.41$, which depend on the details of the linearization. The estimate of m=2.28 by Langharr[8] will be used herein as being most representative.

Inherent in the derivation of Eq. 4 are assumptions regarding the inlet flow and outlet flow conditions. As Prandtl and Tietjens[9] — among others— have pointed out, these assumptions can become suspect when pipe length become shorter than the entry length. Schiller[7] introduced a correction to m for very short pipes with bell-mouthed inlets. A correction in the opposite sense, however, is required to account for flow contraction due to separation at (a sharp-edged) inlet.

The experiments and theory used in refs 6-8, assumed a square inlet profile with no flow separation. For such a case, the corrections such as Schiller's are needed, but is often difficult to keep separation from occurring— especially outside a well-controlled laboratory situation. In this report we shall assume m need not be corrected and is a constant even for very short pipes.

POWER-LAW EXPONENT AS A CHARACTERISTIC

Equation 4 can be solved to yield the mean velocity as a function of the applied pressure:

$$v = \frac{32\,\mu\, l}{m\,\rho\, d^2}\left(\sqrt{1+\frac{m\,\rho\, d^4 \Delta P}{512\mu^2 l^2}} - 1\right) \tag{5.1}$$

or, equivalently,

$$Q = \frac{8\pi\,\nu\,l}{m}\left(\sqrt{1+\frac{m\,\rho\,A^2\Delta P}{64\pi^2\mu^2 l^2}} - 1\right) \tag{5.2}$$

To recast Eq. 5 in terms of a power law we must be able to find the power law *exponent* using Eq. 3:

$$n = ½\left[1 + \left(1+\frac{m\,\rho\,d^4\Delta P}{512\mu^2 l^2}\right)^{-½}\right] \tag{6}$$

Eq. 2 can be evaluated to show that the exponent is slowly varying enough to be a useful concept:

$$\left|\Delta P\ \frac{dn}{d\Delta P}\right| = n\ (1-n)\ (2n-1) < 0.1 \tag{7}$$

A pipe is characterized by its geometry (i.e., length, diameter, and perhaps the shape of the entry), but the system (of the pipe and the fluid) is more complex. The flow exponent as defined in Eq. 6 is a characteristic of the system. Comparison of its definition with expression for the mean velocity allows us to use the exponent in place of the pressure (or the velocity) to express many of the quantities of interest:

The *mean (volumetric) flow* can be found by eliminating the pressure from Eqs. 5 and 6.

$$Q = \frac{8\pi\,\nu\,l}{m}\ \frac{1-n}{n-½} \tag{8}$$

The standard *Reynolds number* (i.e., based on diameter and fluid velocity) can similarly be calculated:

$$\mathrm{Re} = \frac{32\ l}{m\ d}\ \frac{1-n}{n-½} \tag{9}$$

The *discharge coefficient* is often used to describe the actual flow in terms of the equivalent perfect nozzle.

$$C_d \equiv v\left(\frac{2\Delta P}{\rho}\right)^{-½} \tag{10}$$

This expression can also be rewritten in terms of the exponent:

$$C_d = \sqrt{\frac{1-n}{m\ n}} \tag{11}$$

From this expression it is clear that the value of m may have to change slightly for very short pipes (i.e., $n \rightarrow ½$) to reflect the shape of the inlet.

The *friction factor*, (defined as $\lambda \equiv (d/l)(\Delta P/½\rho v^2)$ relates the shear at the walls to energy loss in the fluid.

$$\lambda = m\frac{d}{l}\ \frac{n}{1-n} \tag{12}$$

The friction factor is often used in the study of turbulence, because it is independent of the flow; in the laminar regime, however, it is not.

Entry Length

The concept of *entry length* is often used to determine if the flow has reached its steady-state behavior. For lengths less than the entry length the flow is said to be developing, for lengths greater than the entry length the flow is developed (because the exit profile is parabolic) and for pipe length much greater than the entry length the flow is said to be fully developed because entry effects can be ignored.

One common approach to defining the entry length is as that length at which the profile becomes (approximately) parabolic, leading to an entry length of $d\,\mathrm{Re}/16$. An examination of Eq. 4, however, suggests a slightly different value; namely, we choose the entry length such that the the frictional losses due to laminar flow of a pipe of length equal to the entry length is equal to the entry loss (i.e., the two terms in Eq. 4 are equal). Thus,

$$l_e \equiv m \frac{\mathrm{Re}}{64} d \tag{13}$$

This definition leads to a somewhat smaller value for the entry length than the conventional definition and it can be expressed in terms of the exponent as follows:

$$\frac{l_e}{l} = \frac{1-n}{2n-1} \tag{14}$$

Thus the exponent and the length of the pipe relative to its entry length are uniquely related. As can be seen from the right-hand axis of Fig. 1, the entry length is quite sensitive to the exponent near the limits of its range.

Comparison with Measured Data

To determine whether our power-law formulation is justified, we can use measured data. Kreith[10] has measured the flow characteristics of pipes of different dimensions (e.g., $0.45 < l/d < 17.25$) Figure 1 contains a plot of the original data overlaid with the theoretical curve. The measured data relates the exponent to the dimensionless length of the pipe.

$$\frac{l/d}{\mathrm{Re}} = \frac{m}{64}\,\frac{l}{l_e} \tag{15}$$

The left axis is the data as presented in the source; the right axis is the dimensionless length from Eqs. 14 and 15. The last two curves demonstrate that the data agrees with the theory for all reasonable values of m and that this dataset cannot be used to refine the value of m.

The measurements were made for square inlet capillary tubes, rather than the bell-mouthed inlets assumed in refs. 6-8. Our theoretical curve gives reasonable agreement to within the precision of the data. It is especially important to note that there is good agreement for low exponent values in short pipes, suggesting that m may be taken as constant. A more detailed experimental investigation, however, could determine the value of m and its dependence on length more precisely.

S NUMBER

When the length of the pipe is equal to its entry length (i.e., the pressure drops due to the two terms of Eq. 4 are equal), the pressure drop is equal to a critical value:

$$P_c \equiv \frac{512\pi^2\mu^2 l^2}{m\,\rho\,A^2} \tag{16}$$

At pressures above this value the losses will be principally from the entry and at pressures below this the losses will be principally by steady-state viscous friction.

This critical pressure value suggests that there is a non-dimensionalization of the pressure appropriate for *s*hort pipes. We define *S* number as follows:

$$S \equiv \frac{\Delta P}{P_c} \tag{17}$$

Using this definition, the fluid flow, exponent, and entry length can be expressed in terms of the S number.

$$Q = \frac{8\pi\,\nu\,l}{m}\left(\sqrt{1+8S}\;-1\right) \tag{18}$$

$$n = \tfrac{1}{2}\left(1+\left(1+8S\right)^{-\frac{1}{2}}\right) \tag{19}$$

$$\frac{l_e}{l} = \frac{\sqrt{1+8S}\;-1}{2} \tag{20}$$

Although Eq. 8 defines the flow in terms of the power-law exponent, it is not a power law. Our definitions of the flow exponent and S number can be used to reformulate Eq. 4 as a power law.

$$Q = \frac{16\pi\,\nu\,l}{m}\,\phi\,S^n \tag{21}$$

where the power-law factor, ϕ, is defined as follows:

$$\phi \equiv (2/n)^n\,(1-n)^{1-n}\,(2n-1)^{2n-1} \tag{22}$$

This factor varies between one and two as a function of exponent.

Eq. 21 is not a true power law because both the exponent, n, and the coefficient, specifically ϕ, have a pressure dependence, albeit small. As shown by Eq. 7, however, it is locally a power-law and may be treated as such within a restricted range of pressures.

The power-law factor is an artifact of the equation of interest not being a true power law. When the S number is near unity, the power-law factor is also unity and is slowly-varying. As the S number deviates significantly from unity, ϕ begins to slowly increase and approaches the limit for either laminar or inertial flow.

The S number, which is a non-dimensionalized pressure, is a good indicator of the shortness of the pipe. For large values of S the pipe is very short and can be characterized as an orifice in the limit. At small values of S the pipe is very long and is well-described by viscous flow equations. Fig. 2 shows the S-number dependence of the (dimensionless) length, discharge coefficient, and power-law factor.

DISCUSSION

The fundamental equation, Eq. 4, was derived in ref. 6-8 assuming a flat inlet profile and a parabolic exit profile. The parameter *m* could then be modified for pipes too short to establish fully-developed laminar flow. Kreith's data, however, fits the model using a constant *m*, when ostensibly the pipe was too short. Even in the limit of zero length (i.e., an orifice) our expression for the discharge coefficient (Eq. 11) yields a quite reasonable value (0.66) compared to that for a sharp-edged circular orifice (0.60). This formulation has also allowed a definition of entry length (Eq. 13) that is clearer and more directly tied to the physics than others.

The ability of our constant-*m* model to work well for very short, sharp-edged pipes is largely fortuitous. The energy loss caused by acceleration of the fluid at the (entrance) contraction happens to approximate the extra energy required to produce the parabolic profile for a longer pipe. Thus, the formulation of this report can be used for all flow lengths provided that very short pipes have sharp-edged inlets.

System Characterization

When the S number is small (i.e., the pipe is long), formulations such as Eq. 4 may prove the most straightforward to use— as in the case of viscometry. When *S* is very large, inertial forces dominate and the system can be treated as an orifice. For non-extreme values of *S* the power-law formulation may provide more insight into the problem. Such is especially the case when the problem is to characterize the pipe geometry based on measured system performance (i.e. pressure and flow).

The local exponent, *n*, of the power-law characterizes the pipe. The exponent is uniquely related to the dimensionless length (Eq. 14) and similarly to the dimensionless pressure (Eq. 19) and discharge coefficient (Eq. 11). If one wished to know the properties of the leak (as in building physics), a measurement of the exponent and coefficient of the power law would be sufficient. In this inverse problem the flow data can be collected over a narrow range of pressures and fitted to a power law:

$$Q = \mathbf{K}\,\Delta P^{n} \tag{23}$$

The S number can be found directly from the exponent:

$$S = \frac{1}{8}\,\frac{(1-n)\,n}{(n-\frac{1}{2})^{2}} \tag{24}$$

From Eqs. 8, 16, 22, 23 the area and length of the pipe can be solved for uniquely, using the intermediates P_c and ϕ:

$$A = \mathbf{K}\,P_c^{\,n-\frac{1}{2}}\,\frac{\sqrt{2m\rho}}{\phi} \tag{25.1}$$

$$l = \frac{m\,\mathbf{K}\,P_c^{\,n}}{16\pi\phi\nu} \tag{25.2}$$

Having solved for the geometric properties of the pipe, any of Eqs. 4, 8, or 21 could be used to predict the flow at a given pressure.

Bridge to Turbulence

The results derived herein are only applicable to flow in the laminar regime. However, by representing the two problems in parallel forms, similarities appear.

The friction factor is often used to characterize the losses of fluids in pipes. Eq. 12 can be cast in more conventional terms as a function of Reynold's number:

$$\lambda = 16\phi^2 \left(\frac{m}{64} \frac{d}{l} \right)^{1/n-1} \mathrm{Re}^{1/n-2} \tag{26}$$

As can be quickly verified, this expression reduces to the well-known laminar and orifice limits for n=1 and n=½, respectively.

This formulation can be compared to that for turbulent flow at moderate Reynold's numbers. The Blasius formula[11] gives the friction factor in smooth pipes for turbulent flow at $4x10^3$<Re<10^5:

$$\lambda_{Blasius} = 0.316\ \mathrm{Re}^{-1/4} \tag{27}$$

Note that in this regime the flow follows a power-law of n=4⁄7.

This similarity would allow networks of laminar pipes to be solved by the same approaches that Jeppson[1] uses for turbulent ones. Other similarities are suggested, but have not yet been pursued.

Other Geometries

The expressions derived here were based on straight pipes of circular cross-section. It may be desired to analyze cases with different cross-sections or curved or crooked lengths. Baker[12] has derived an equation similar to Eq. 4 for the case of flat plates with and without right-angle bends. As the form of the equation is the same, a power-law formulation can be similarly derived. The work suggests that the value of m could be increased to account for bends. Further modifications of Eqs. 5, 6, 16 would also prove necessary to account for different cross sections.

The application to building physics, mentioned in the introduction, involves system characterization, the solution of series/parallel networks, and non-circular cross-sections. To the extent that this problem could be treated as a single, equivalent circular pipe, the results of this report are applicable. The exact solution of this problem, however, requires more development.

NOMENCATURE

A	Open area of pipe [m^2]
C_d	Discharge coefficient [-]
d	Diameter of pipe [m]
K	Power-law coefficient [m^3/s-Pa^n]
l	Length (along flow path) of pipe [m]
n	Power-law exponent
Q	Fluid flow through pipe [m^3/s]
Re	Reynolds number [-]
S	S number [-]
ΔP	Pressure drop across pipe [Pa]
δP	Pressure range of interest [Pa]
P_c	Critical pressure for short pipe [Pa]
v	Mean velocity of fluid [m/s]
λ	(Darcy) friction factor [-]
μ	Viscosity of fluid [kg/m-s]
ν	Kinematic viscosity (μ/ρ) of fluid [m^2/s]
ϕ	Power-law factor [-]
ρ	Density of fluid [kg/m^3]

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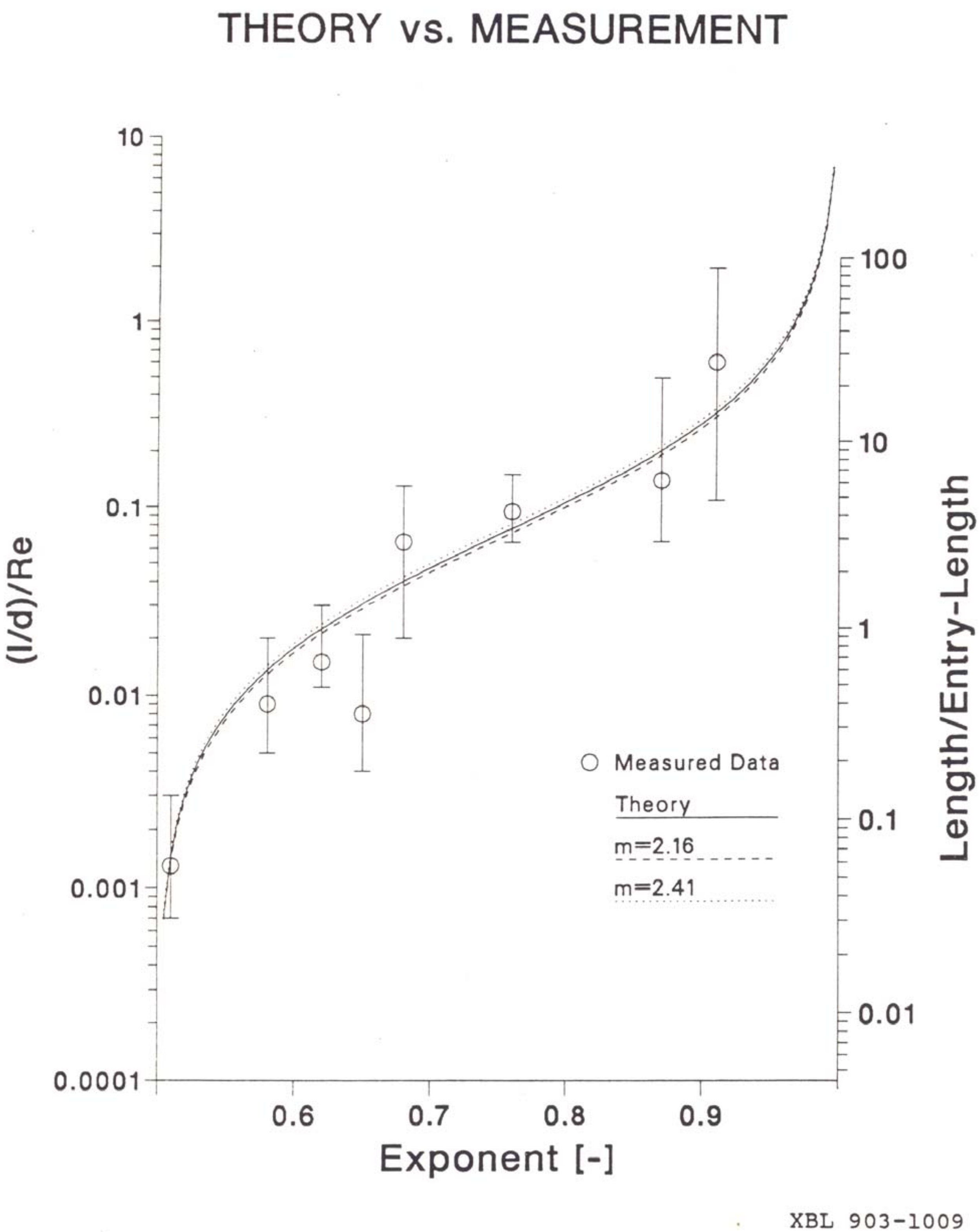

Figure 1: Kreith's data with their error bars shown as a function of exponent. Left hand axis is the normalized length from the original reference. Right hand axis is the pipe length divided by the entry length. Theoritical curve uses m=2.28 as derived in the text.

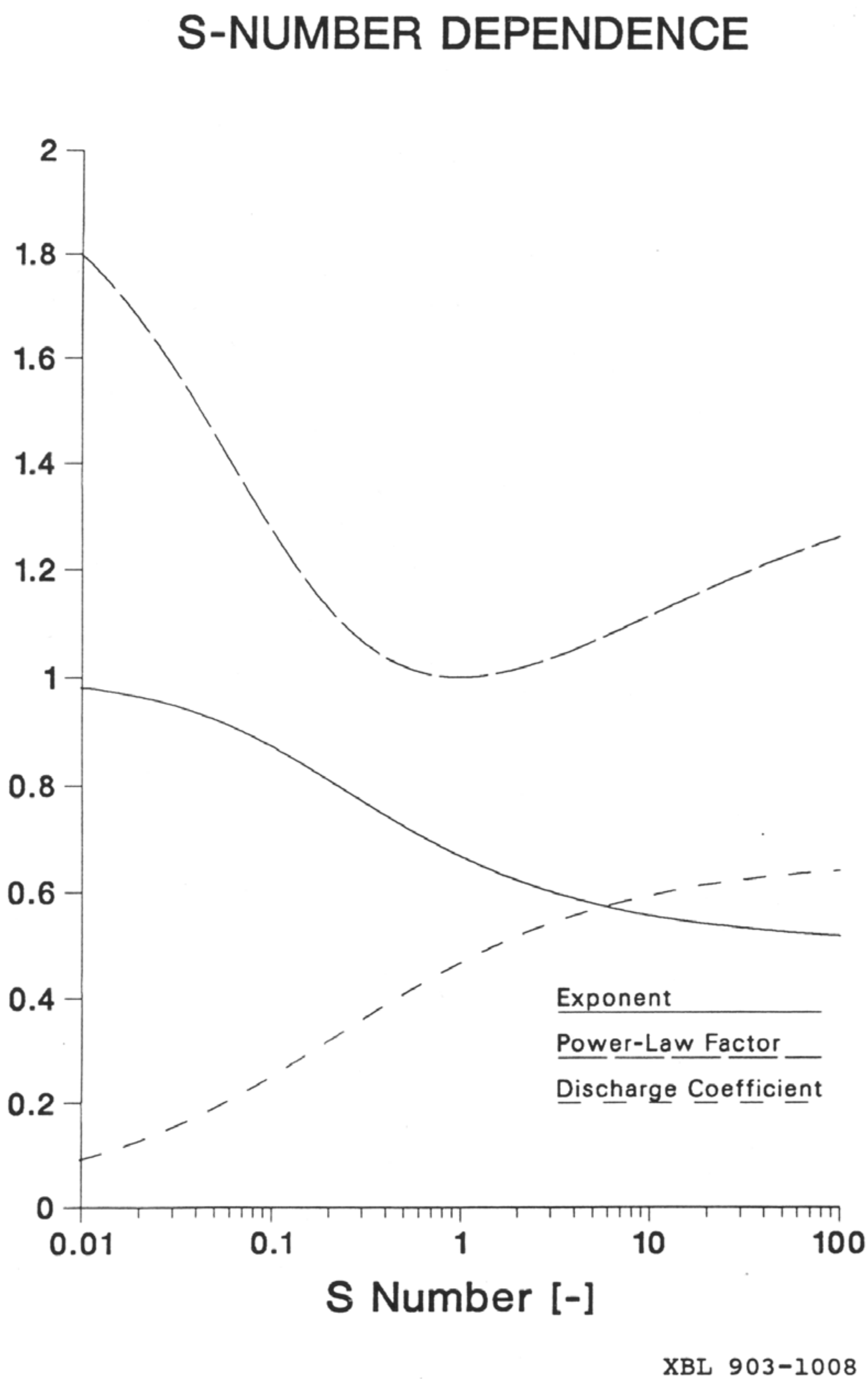

Figure 2: Dependence of length, discharge coefficient, and the power-law factor with S number.