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Preliminary Theoretical Considerations of Some Nonlinear Aspects of Hypersonic Panel Flutter

By

S. C. McIntosh, Jr.

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List of Principal Symbols

Symbols are listed alphabetically

$\vec{A}$	Acceleration
A_i	Components of $\vec{A}$ with respect to base vectors in B_0
a_{ik}	Amplitudes in trial-function expansions for the v_i
B	Deformed elastic body
B_0	Undeformed elastic body
$\vec{B}$	Body force per unit mass in B
B_i	Components of $\vec{B}$ with respect to base vectors in B_0
C	Boundary curve of surface S
C_0	Boundary curve of surface S_0
E	Young's modulus
e_i	Relative change in magnitude of differential line element along θ_i coordinate curve
e_{ij}	Functions of derivatives of displacement components; defined in Eqs. (2.28)
e^{ijk}	Alternating tensor: unity when $i \neq j \neq k$ and i,j,k in cyclic order; negative unity when $i \neq j \neq k$ and i,j,k in noncyclic order; zero otherwise
F_i	Sum of surface loads referred to S_0 – see Eqs. (7.20)
F_i^*	Integrated surface loads on C_0 – see Eqs. (7.21)
F_{ik}	Coefficients of the δa_{ik} in expansion for δW_e that result from surface loads referred to S_0 – see Eq. (7.35)
F_{ik}^*	Coefficients of the δa_{ik} in expansion for δW_e that result from surface loads on C_0 – see Eq. (7.36)
f_i	Generalized surface loads associated with virtual displacements δv_i

G	Determinant of metric components G_{ij}
$\vec{G}_i, \vec{G}^i$	Covariant and contravariant base vectors, respectively, of body-fixed coordinate system in B
G_{ij}, G^{ij}	Covariant and contravariant components, respectively, of metric tensor for body-fixed coordinate system in B
$g, \vec{g}_i, \vec{g}^i, g_{ij}, g^{ij}$	Same definitions as above for body-fixed coordinate system in B_0
h	Plate thickness
k	Summation index for trial-function expansions
M_{ij}	Integrated plate stresses — see Eqs. (7.27)
m_i	Sum of surface loads referred to S_0 — see Eqs. (7.20)
$\overset{*}{m}_i$	Integrated surface loads on C_0 — see Eqs. (7.21)
m_n	Surface loads m_i referred to normal coordinate n on C_0
$\overset{*}{m}_n$	Surface loads $\overset{*}{m}_i$ referred to normal coordinate n on C_0
N_{ij}	Integrated plate stresses — see Eqs. (7.27)
n	Coordinate normal to C_0
$\vec{n}$	Surface unit normal
n_i	Components of $\vec{n}$ with respect to base vectors in B
P	Point in B
$\vec{P}$	Surface load vector
P_i	Components of $\vec{P}$ with respect to base vectors in B
$\vec{R}$	Vector from origin to P
S	Surface of B

dS_i	Elemental area defined by surface θ_i = const.
s	Coordinate tangential to C_0
s^{ij}	Stresses referred to undeformed area
T	Kinetic energy
t	Time
$\vec{t}_i$	Stress vector referred to area in B
V	Volume of B
$\vec{v}$	Displacement vector
v_i	Components of $\vec{v}$ with respect to base vectors in B_0.
$\bar{v}_n$	Components of middle-surface displacement along normal to C_0
$\bar{v}_s$	Components of middle-surface displacement along **tangent** to C_0
W	Strain energy
δW_e	Virtual work of external loads
x_i	Cartesian coordinates of points in B_0
y_i	Cartesian coordinates of points in B
δ^i_j, δ_{ij}	Kronecker delta: unity if $i = j$, zero otherwise
ϵ_{ij}	Covariant components of strain tensor referred to B_0
θ_i	Coordinates of body-fixed coordinate system (**can be written** θ_i or θ^i)
λ, μ	Lamé constants of elasticity
ν	Poisson's ratio
ρ	Density

I. INTRODUCTION

This report presents a summary of the first year of research on nonlinear aspects of hypersonic panel flutter. The research was carried out under NASA Grant NGR 05-020-102, monitored technically by the Nonsteady Phenomena Branch of Ames Research Center.

The work is summarized in three principal categories. First, the full three-dimensional equations of elasticity are derived, and simplified versions of these equations are presented for different levels of approximation. The sections dealing with these equations are essentially a synthesis of the tensor derivations of Ref. 1 and the successive approximations given in Ref. 2, with the exception of the treatment of the stress tensor. It is felt that including such derivations in this report will aid both in justifying a number of assumptions made in the derivation of the panel-flutter equations and in illustrating the relationships between panel-flutter theory and the more general three-dimensional theory. Secondly, a level of approximation suitable for problems in panel flutter is chosen, and panel-flutter equations are derived with the aid of a variational formulation. A systematic manner of obtaining higher levels of approximation for these equations is also outlined. Finally, a method of solution of these equations is proposed and discussed. A concluding section then deals with plans for further research.

II. STRAIN

2.1 The Strain Tensor

Denote by B_0 an elastic body at rest relative to a fixed rectangular Cartesian axis system; the coordinates of the body are $\chi_i (i = 1,2,3)$.* Then denote by θ_i the coordinates of an arbitrary curvilinear coordinate system fixed to the body. When the body is at rest, the coordinates are related by equations of the form

$$\begin{aligned} \chi_i &= \chi_i(\theta_1, \theta_2, \theta_3) \\ \theta_i &= \theta_i(\chi_1, \chi_2, \chi_3) \end{aligned} \tag{2.1}$$

The position vector of a point P_0 in B_0 is given as

$$\begin{aligned} \vec{R}_0 &= \vec{R}_0(\chi_1, \chi_2, \chi_3) \quad \text{or} \\ \vec{R}_0 &= \vec{R}_0(\theta_1, \theta_2, \theta_3) \end{aligned} \tag{2.2}$$

Covariant base vectors are defined for the body-fixed coordinates in B_0 as

$$\vec{g}_i = \frac{\partial \vec{R}_0}{\partial \theta^i} \tag{2.3}$$

The contravariant base vectors are related to the covariant ones by the equations

$$\vec{g}^i \cdot \vec{g}_j = \delta^i_j \tag{2.4}$$

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*It will be assumed throughout this report that Latin indices take on values of one, two, or three, and that an index that appears twice in a term implies summation over these values, unless it is otherwise noted or obvious. Indices that appear more than twice will not imply summation unless the summation is explicity indicated. Also, distinction will be made where necessary between contravariant and covariant quantities by using respectively raised (a^{ij}) or lowered (a_{ij}) indices. Note in this regard that the coordinates θ_i are neither contravariant nor covariant in themselves, although there is a difference in general between contravariant and covariant differentials. The position of the coordinate index will therefore be raised or lowered as is convenient.

Finally, from the base vectors we can define the covariant and contravariant components of the metric tensor for B_0:

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j \qquad g^{ij} = \vec{g}^i \cdot \vec{g}^j \tag{2.5}$$

Then we denote by B the deformed elastic body at some instant of time, whose coordinates in the Cartesian axis system are y_i. The y_i are related to the θ_i by equations of the form

$$y_i = y_i(\theta_1,\theta_2,\theta_3,t) \qquad \theta_i = \theta_i(y_1,y_2,y_3,t) \tag{2.6}$$

Of course, these relations along with Eqs. (2.1) imply similar relations between the χ_i and the y_i. We assume that all of these relations are unique, so that every point in B_0 or B is determined by unique coordinates θ_i, χ_i, or y_i.

The point P_0 in B_0 has now moved to P in B. For fixed time t, we can take the first of Eqs. (2.6) and obtain the Cartesian coordinates of a curve through P by varying, say, θ_1, while holding θ_2 and θ_3 constant at the values they take on at P. This curve is then the θ_1 coordinate curve through P, and in similar fashion the θ_2 and θ_3 coordinate curves can be defined. Thus every point in B (or B_0, for that matter) has three coordinate curves associated with it, as illustrated in Fig. 2.1:

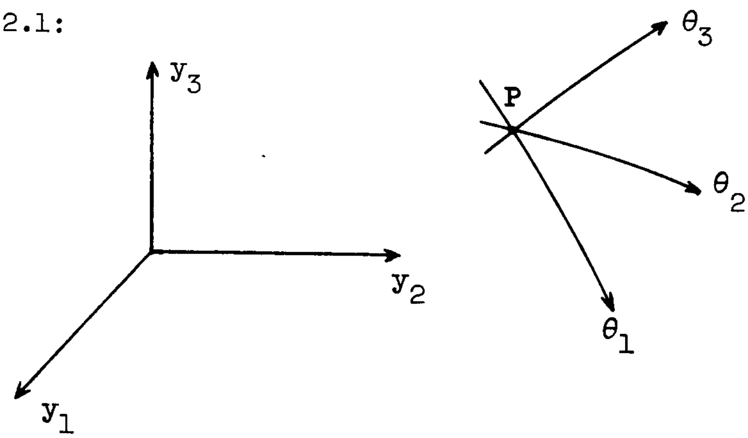

Figure 2.1. Coordinate Curves through P.

By holding one coordinate constant and varying the other two for fixed time we define coordinate surfaces through P. The coordinate surface θ_1 = constant, for example, is the surface through P containing the θ_2 and θ_3 coordinate lines.

As was done for B_0, we can define a position vector $\vec{R}(\theta_1,\theta_2,\theta_3,t)$, covariant and contravariant base vectors $\vec{G}_i$ and $\vec{G}^i$, and covariant and contravariant metric-tensor components G_{ij} and G^{ij}.

A differential line element in B_0 is given by

$$ds_0^2 = g_{ij}d\theta^i d\theta^j \tag{2.7}$$

In B this line element becomes

$$ds^2 = G_{ij}d\theta^i d\theta^j \tag{2.8}$$

The covariant components of the strain tensor are then defined as

$$\begin{aligned} ds^2 - ds_0^2 &= (G_{ij} - g_{ij})d\theta^i d\theta^j = 2\epsilon_{ij}d\theta^i d\theta^j \\ 2\epsilon_{ij} &= G_{ij} - g_{ij} \end{aligned} \tag{2.9}$$

In this manner the metric tensor for the deformed body is given in terms of the strain tensor and the metric tensor for the undeformed body.

The displacement vector is defined as

$$\vec{v}(\theta_1,\theta_2,\theta_3,t) = \vec{R} - \vec{R}_0 \tag{2.10}$$

Thus

$$\vec{G}_i = \frac{\partial\vec{R}}{\partial\theta^i} = \frac{\partial\vec{R}_0}{\partial\theta^i} + \frac{\partial\vec{v}}{\partial\theta^i} = \vec{g}_i + \frac{\partial\vec{v}}{\partial\theta^i} \tag{2.11}$$

and

$$\begin{aligned} G_{ij} &= \vec{G}_i\cdot\vec{G}_j = \left(\vec{g}_i + \frac{\partial\vec{v}}{\partial\theta^i}\right)\cdot\left(\vec{g}_j + \frac{\partial\vec{v}}{\partial\theta^j}\right) \\ &= g_{ij} + \frac{\partial\vec{v}}{\partial\theta^i}\cdot\vec{g}_j + \vec{g}_i\cdot\frac{\partial\vec{v}}{\partial\theta^j} + \frac{\partial\vec{v}}{\partial\theta^i}\cdot\frac{\partial\vec{v}}{\partial\theta^j} \end{aligned} \tag{2.12}$$

Hence

$$2\epsilon_{ij} = \frac{\partial \vec{v}}{\partial \theta^i} \cdot \vec{g}_i + \vec{g}_i \cdot \frac{\partial \vec{v}}{\partial \theta^j} + \frac{\partial \vec{v}}{\partial \theta^i} \cdot \frac{\partial \vec{v}}{\partial \theta^j} \tag{2.13}$$

We write $\vec{v}$ in terms of its components with respect to the undeformed body:

$$\vec{v} = v_m \vec{g}^m \tag{2.14}$$

Then we can write

$$\frac{\partial \vec{v}}{\partial \theta^i} = v_{m,i} \; g^m \tag{2.15}$$

where the comma denotes covariant differentiation of v_m with respect to θ_i and the metric components of B_0. The strain tensor then becomes

$$\epsilon_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i} + v^r_{,i} \; v_{r,j}) \tag{2.16}$$

Note in particular that the strain tensor is symmetric and is written with reference to the undeformed body.

There are three other geometric quantities that will be useful later; these are the elongations, shears, and relative change in volume. The elongations represent the relative changes in magnitude of line elements along the coordinate curves. A line element along a coordinate curve θ_i in B_0 can be written as

$$\vec{ds}_{0i} = \vec{g}_i \; d\theta^i \qquad \text{(i not summed)} \tag{2.17}$$

with magnitude

$$ds_{0i} = (g_{ii})^{\frac{1}{2}} \; d\theta^i \tag{2.18}$$

For the same elements in B, we substitute $\vec{G}_i$ and G_{ii} for $\vec{g}_i$ and g_{ii}. Then the elongations are defined as follows:

$$e_i = [ds_i - ds_{0i}]/ds_{0i} = [(G_{ii})^{\frac{1}{2}} - (g_{ii})^{\frac{1}{2}}]/(g_{ii})^{\frac{1}{2}}$$

$$= (G_{ii}/g_{ii})^{\frac{1}{2}} - 1 = (1 + 2\epsilon_{ii}/g_{ii})^{\frac{1}{2}} - 1 \tag{2.19}$$

Then let ψ_{ij} be the angle between two differential line elements $\vec{ds}_{0i}$ and $\vec{ds}_{0j}$, and let $\delta\psi_{ij}$ be the change in this angle as the body deforms. Then $\psi_{ij} + \delta\psi_{ij}$ is the angle between differential line elements $\vec{ds}_i$ and $\vec{ds}_j$ in B:

$$\cos(\psi_{ij} + \delta\psi_{ij}) = (\vec{ds}_i \cdot \vec{ds}_j)(ds_i ds_j)^{-1} = G_{ij}(G_{ii}G_{jj})^{-\frac{1}{2}}$$

$$= (g_{ij} + 2\epsilon_{ij})[(g_{ii} + 2\epsilon_{ii})(g_{jj} + 2\epsilon_{jj})]^{-\frac{1}{2}}$$

$$(i \neq j) \tag{2.20}$$

The $\delta\psi_{ij}$ are the shears; note that at any point there are only three independent ones.

A differential volume in B_0 is

$$dV_0 = (g)^{\frac{1}{2}}\, d\theta^1 d\theta^2 d\theta^3 \tag{2.21}$$

where g is the determinant of the covariant components g_{ij} of the metric tensor for B_0. The differential volume for B is found by replacing g with G, the determinant of the G_{ij}. Thus the relative change in volume as a result of the deformation is

$$dV/dV_0 = (G/g)^{\frac{1}{2}} \tag{2.22}$$

Also, it can be shown[1*] that g and G are related as follows:

$$G = g + e^{ijk}e^{rst}(g_{ir}g_{js}\epsilon_{kt} + 2g_{ir}\epsilon_{js}\epsilon_{kt} + \frac{4}{3}\,\epsilon_{ir}\epsilon_{js}\epsilon_{kt}) \tag{2.23}$$

2.2 The Small-Strain Assumption

Up to now there has been no restriction imposed on the magnitude of the deformations (elongations and shears). Such generality is rarely

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*Number superscripts in the text denote references listed after Chapter VIII.

necessary, however, because for the most part we are concerned with purely elastic deformations. In particular, this is undoubtedly the case for a fluttering plate that is not nearing its fatigue limit. Recourse to a stress-strain diagram for any metal commonly used in aerospace structural applications will show that the strains below the proportional limit are much smaller than unity. Therefore the first simplification will be to assume that the strains are negligible in comparison with terms of order unity.

To illustrate the process of simplification, we choose the body-fixed coordinate system so that it coincides with the Cartesian axis system when the body is at rest. Then $g^{ij} = g_{ij} = \delta_{ij}$, and there is no need to distinguish covariant components from contravariant components. From Eqs. (2.19) we find that the elongations are comparable to the strains:

$$e_i \approx \epsilon_{ii} \qquad \text{(i not summed)} \tag{2.24}$$

Also, the angles ψ_{ij} are right angles, so the shears become, from Eqs. (2.20),

$$\left.\begin{aligned} \cos(\psi_{ij} + \delta\psi_{ij}) &= \sin\,\delta\psi_{ij} \\ \sin\,\delta\psi_{ij} &\approx \delta\psi_{ij} \approx 2\epsilon_{ij} \end{aligned}\right\} \quad (i \neq j) \tag{2.25}$$

The determinant g is now unity, so with the aid of Eqs. (2.22) and (2.23) we get for the relative change in volume

$$dV/dV_0 \approx \epsilon_{ii} \tag{2.26}$$

Thus the elongations, shears, and relative change in volume are also negligible in comparison with terms of order unity. To this order of approximation, an infinitesimal volume element $d\theta_1 d\theta_2 d\theta_3$ will remain cubic during the deformation, and it will have the same volume. The body-fixed coordinates will remain orthogonal, and the body itself is for all practical purposes incompressible. However, the translation and "rigid-body" rotation of the volume element are still without limitation. The expressions for the strain components, Eqs. (2.16), can be written in terms of the usual partial derivatives:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial \theta_j} + \frac{\partial v_j}{\partial \theta_i} + \frac{\partial v_r}{\partial \theta_i}\frac{\partial v_r}{\partial \theta_j}\right) \tag{2.27}$$

As a prelude to further simplification, we define the following quantities:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial \theta_j} + \frac{\partial v_j}{\partial \theta_i}\right)$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial \theta_i} - \frac{\partial v_i}{\partial \theta_j}\right) \tag{2.28}$$

$$\cos \varphi_i = \vec{g}_i \cdot \vec{G}_i/(g_{ii}G_{ii})$$

The ω_{ij} characterize the average rotation of infinitesimal volume elements, and for small strain they can be identified directly with the rigid-body rotation of these volume elements. The angles φ_i are the rotations of the body-fixed coordinate lines as a result of the deformation. They are not the same as the rotations ω_{ij}, but they are comparable in magnitude.[2] Note that the e_{ij} are symmetric and the ω_{ij} are anti-symmetric with respect to the indices i and j, so that the e_{ij} comprise six independent quantities at any point while the ω_{ij} comprise three.

The assumption of small elongations and shears does not in itself imply that the rotations are small. However, the simultaneous occurrence of small elongations and shears and large rotations does imply that at least one characteristic dimension of the body is small in relation to the others. The obvious example of such a body is of course a thin plate.

2.3 The Assumption of Limits on the Rotations

Let us now assume some limitation on the rotations. A convenient one is to assume that the squares of the rotations are small relative to terms of order unity. Under these circumstances the ϵ_{ij} and the e_{ij} differ by terms of the order of products of the coordinate-line rotations:[2]

$$\epsilon_{ij} - e_{ij} = O(\varphi_i \varphi_j) \tag{2.29}$$

Since these rotations are of the same order of magnitude as the volume-element rotations ω_{ij}, we view Eqs. (2.29) as stating that the e_{ij} are of the order of the strains or of products of the volume-element rotations, whichever are larger, and it is for the most part less restrictive to take as larger quantities products of the rotations. We write the strain components as

$$\epsilon_{ij} = e_{ij} + \frac{1}{2} (e_{ri} - \omega_{ri})(e_{rj} - \omega_{rj}) \tag{2.30}$$

Then the products like $e_{ri}\omega_{rj}$ are third order in the rotations, and products like $e_{ri}e_{rj}$ are fourth order in the rotations. Neglecting such products in comparison with the other terms gives

$$\epsilon_{ij} = e_{ij} + \frac{1}{2} \omega_{ri}\omega_{rj} \tag{2.31}$$

Finally, assuming that the rotations are of the order of the strains gives the linear relation

$$\epsilon_{ij} = e_{ij} \tag{2.32}$$

In this case there is no distinction between the pre-deformation and post-deformation geometry.

III. COMPATIBILITY EQUATIONS

The compatibility equations are obtained from the requirement that the Riemann-Christoffel tensor for a Euclidean space be identically zero. They are useful for the most part when problems in elasticity are posed in terms of the strains or the stresses. Since problems of interest in this report will be posed directly in terms of the displacements, the compatibility equations will not be discussed further; their derivation and simplification for small strains and rotations can be found in Ref. 2.

IV. STRESS; THE EQUATIONS OF MOTION; BOUNDARY CONDITIONS

4.1 Stress

We assume that the body B_0 is deformed to B by the action of two types of forces: surface forces $\vec{P}$ per unit surface area of B and body forces $\vec{B}$ per unit mass of B. The acceleration at a point in B is denoted by $\vec{A}$.

The force exerted across any element of area ΔS in B is statically equivalent to a force $\Delta\vec{T}$ and a moment $\Delta\vec{M}$ at some point on ΔS. We assume that, as ΔS approaches zero around the point, $\Delta\vec{M}/\Delta S$ approaches zero and $\Delta\vec{T}/\Delta S$ has a finite limit $\vec{t}$, the stress vector or traction. This stress vector depends on two vectors – the position vector of the point, and the unit vector normal to the area to which $\vec{t}$ refers.

We then consider a point P in B with the three coordinate curves through it. As illustrated in Fig. 4.1, we define an elementary tetrahedron at P with the aid of the surfaces θ_i = constant through P; also shown are the appropriate base vectors and the corresponding tetrahedron at P_0 in the unstrained body B_0:

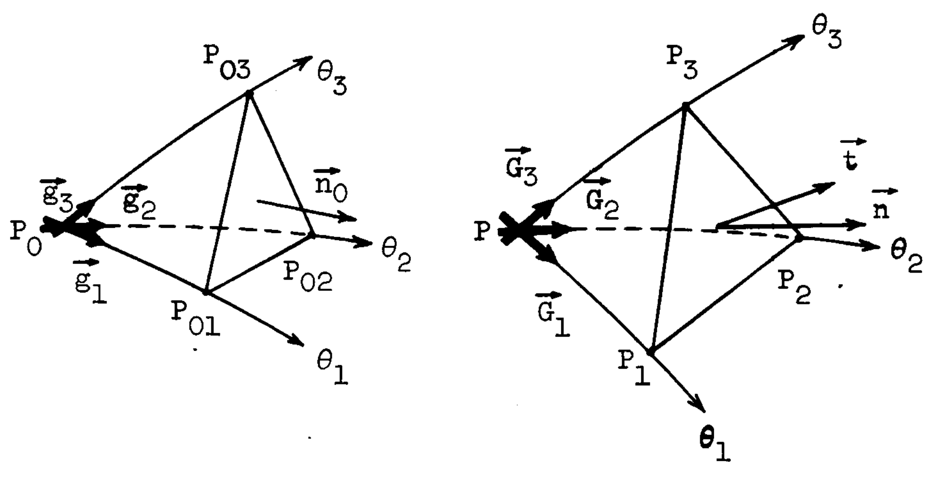

Figure 4.1. Elemental Tetrahedrons for Undeformed and Deformed Body.

The points P_i, P_{0i} are located on their respective coordinate curves an infinitesimal distance from P or P_0:

$$\vec{ds}_i = \vec{PP}_i = \vec{G}_i d\theta^i \qquad \text{(i not summed)} \qquad (4.1)$$
$$\vec{ds}_{0i} = \vec{P_0P}_{0i} = \vec{g}_i d\theta^i$$

The surface θ_1 = constant through P is the surface defined by the θ_2 and θ_3 coordinate curves, and its unit normal at P is $\vec{G}^1(G^{11})^{-\frac{1}{2}}$, the reciprocal or contravariant base vector divided by its magnitude. The area of the side of the tetrahedron defined by the surface θ_1 = constant is given vectorially as one-half the area of the parallelogram defined by $\vec{ds}_2$ and $\vec{ds}_3$, or as $(\vec{ds}_2 \times \vec{ds}_3)/2$:

$$2\vec{dS}_1 = (\vec{ds}_2 \times \vec{ds}_3) = (\vec{G}_2 \times \vec{G}_3)d\theta^2 d\theta^3 \qquad (4.2)$$
$$\vec{G}_2 \times \vec{G}_3 = \vec{G}^1 (G)^{1/2}$$

Thus we can say in general

$$\vec{dS}_i = \vec{G}^i (G^{ii})^{-\frac{1}{2}} dS_i \qquad (4.3)$$

where

$$dS_i = [(GG^{ii})^{\frac{1}{2}}/2]d\theta^j d\theta^k \quad (\text{i not summed, } i \neq j \neq k) \qquad (4.4)$$

The corresponding areas $\vec{dS}_{0i}$ and dS_{0i} for B_0 are found by substituting the proper metric components for B_0. Then we can define another useful quantity, the ratio of the area magnitudes:

$$dS_i/dS_{0i} = (GG^{ii}/gg^{ii})^{\frac{1}{2}} \qquad (4.5)$$

Denote by $\vec{n}$ the unit normal to the area $P_1P_2P_3$, or dS, as illustrated in Fig. 4.1; $\vec{n}_0$ is the corresponding normal for dS_0. Let $\vec{t}$ be the stress vector associated with dS and $\vec{n}$. These areas are related vectorially to the other sides of the tetrahedrons as follows:

$$\vec{n}\, dS = \sum_{i=1}^{3} \vec{G}^i (G^{ii})^{-\frac{1}{2}} dS_i \qquad (4.6)$$
$$\vec{n}_0 dS_0 = \sum_{i=1}^{3} \vec{g}^i (g^{ii})^{-\frac{1}{2}} dS_{0i}$$

Then, denoting by n_i and n_{0i} the components of $\vec{n}$ and $\vec{n}_0$ with respect to base vectors $\vec{G}^i$ and $\vec{g}^i$, we get the scalar relations

$$\begin{aligned} n_i(G^{ii})^{\frac{1}{2}}dS &= dS_i \\ n_{0i}(g^{ii})^{\frac{1}{2}}dS_0 &= dS_{0i} \end{aligned} \tag{4.7}$$

Associated with the surfaces θ_i = constant of the tetrahedron in B are stress vectors $-\vec{t}_i$. The equation of motion for this tetrahedron reduces to an equation of static equilibrium, because the inertia term is of higher order in the differential limit. We find then

$$\vec{t}dS = \vec{t}_i dS_i \tag{4.8}$$

By substituting for dS_i from the first of Eqs. (4.7) we obtain

$$\vec{t} = \sum_{i=1}^{3} n_i(G^{ii})^{\frac{1}{2}}\,\vec{t}_i \tag{4.9}$$

The stress vector $\vec{t}$ associated with a surface normal $\vec{n}$ in B is invariant under coordinate transformation if $\vec{n}$ is fixed. The n_i are covariant components of the unit vector $\vec{n}$. Therefore the stress vectors $(G^{ii})^{\frac{1}{2}}\vec{t}_i$ must transform according to the contravariant transformation laws, and from them we can define the contravariant components of a stress tensor:

$$(G^{ii})^{\frac{1}{2}}\,\vec{t}_i = \sigma^{ij}\vec{G}_j \tag{4.10}$$

We say then that σ^{ij} is the j^{th} contravariant component of the stress vector $(G^{ii})^{\frac{1}{2}}\vec{t}_i$ associated with the surface θ_i = constant. Note that the σ^{ij} are referred to the deformed body B — their dimensions are force per unit area in B, and the indices i and j refer to coordinate directions in B. A more convenient formulation is obtained by basing the stress vectors on the pre-deformation, rather than the post-deformation, element of area. This can be achieved by recasting Eq. (4.8) in terms of the original areas dS_0 and dS_{0i} with the aid of the second of Eqs. (4.6) and Eqs. (4.5). We thereby obtain

$$\vec{t}(dS/dS_0)dS_0 = \vec{t}_0 dS_0 = \sum_{i=1}^{3} \vec{t}_i (GG^{ii}/gg^{ii})^{\frac{1}{2}} dS_{0i} = \vec{t}_{0i} dS_{0i}$$
$$\vec{t}_0 = \sum_{i=1}^{3} n_{0i} (g^{ii})^{\frac{1}{2}} \vec{t}_{0i} \tag{4.11}$$

where

$$\vec{t}_0 = (dS/dS_0)\vec{t}$$
$$\vec{t}_{0i} = (GG^{ii}/gg^{ii})^{\frac{1}{2}} \vec{t}_i \tag{4.12}$$

It is important to understand that Eqs. (4.11) still represent the equation of motion of the tetrahedron in B; they are merely rewritten so that the reference area is the corresponding area in B_0. Note, however, that the components n_{0i} are the components of the surface normal in B_0 referred to base vectors in B_0. We can write the stress vector in terms of components as before:

$$(g^{ii})^{\frac{1}{2}} \vec{t}_{0i} = s^{ij} \vec{G}_j \tag{4.13}$$

The relation between the s^{ij} and the σ^{ij} is deduced from Eqs. (4.13), the second of Eqs. (4.12), and Eqs. (4.10):

$$s^{ij} = (G/g)^{\frac{1}{2}} \sigma^{ij} \tag{4.14}$$

We can say then that s^{ij} is the j^{th} component of the stress vector $(g^{ii})^{\frac{1}{2}} \vec{t}_{0i}$ referred to the surface θ_i = constant of the tetrahedron in B, whose reference area, however, is the surface area of the same side of the tetrahedron in its undeformed state. We cannot state in general that the s^{ij} are the components of a tensor.

4.2 The Equations of Motion

Let us now expand the tetrahedron so that it becomes an infinitesimal curvilinear parallelepiped with faces θ_i = constant, $\theta^i + d\theta^i$ = constant, as shown in Fig. 4.2:

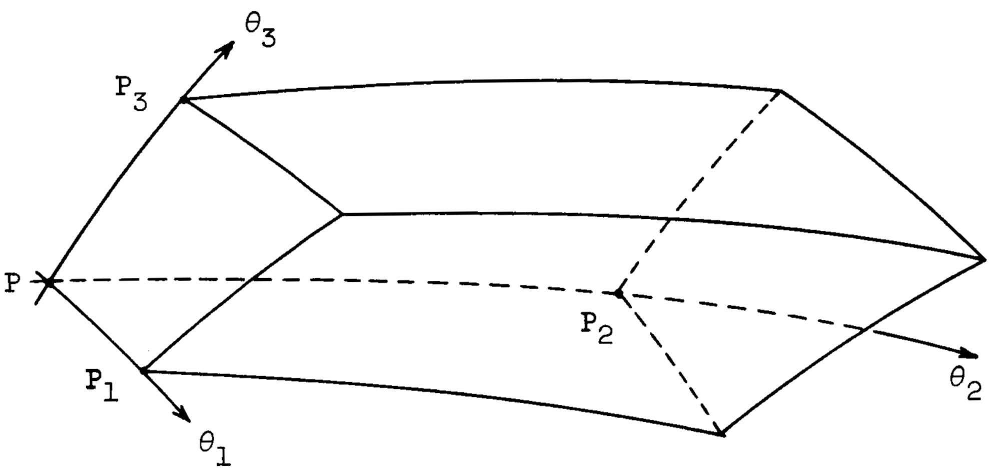

Figure 4.2. Elemental Parallelepiped in Deformed Body

The areas of the faces θ_i = constant are to first order just twice the areas of the corresponding faces of the tetrahedron, so we have for the forces on these faces

$$-\vec{t}_i dS_i = -\vec{t}_{0i}(gg^{ii})^{\frac{1}{2}} d\theta^j d\theta^k \quad \text{(i not summed, } i \neq j \neq k) \tag{4.15}$$

The forces on the faces $\theta^i + d\theta^i$ = constant are given to first order by

$$\left\{\vec{t}_{0i}(gg^{ii})^{\frac{1}{2}} + \frac{\partial}{\partial\theta^i}[\vec{t}_{0i}(gg^{ii})^{\frac{1}{2}}]d\theta^i\right\} d\theta^j d\theta^k \qquad (i \neq j \neq k) \tag{4.16}$$

The body force and inertia force of the parallelepiped are given by

$$\begin{aligned} \rho(G)^{\frac{1}{2}}\, \vec{B} d\theta^1 d\theta^2 d\theta^3 &= \rho_0(g)^{\frac{1}{2}}\, \vec{B} d\theta^1 d\theta^2 d\theta^3 \\ \rho(G)^{\frac{1}{2}}\, \vec{A} d\theta^1 d\theta^2 d\theta^3 &= \rho_0(g)^{\frac{1}{2}}\, \vec{A} d\theta^1 d\theta^2 d\theta^3 \end{aligned} \tag{4.17}$$

Here we have used the equation of continuity:

$$\rho(G)^{\frac{1}{2}} = \rho_0(g)^{\frac{1}{2}} \tag{4.18}$$

Applying the force equation of motion to the parallelepiped gives

$$\sum_{i=1}^{3} \frac{\partial}{\partial \theta^i} [\vec{t}_{0i}(gg^{ii})^{\frac{1}{2}}] + \rho_0(g)^{\frac{1}{2}} \vec{B} = \rho_0(g)^{\frac{1}{2}} \vec{A} \tag{4.19}$$

Summing moments about some point in the parallelepiped gives the symmetry property of the stress-tensor components:

$$\sigma^{ij} = \sigma^{ji} \tag{4.20}$$

Thus from Eqs. (4.14) we see that the components s^{ij} have the same symmetry property.

Let us now assume that the body-fixed coordinate system coincides with the fixed Cartesian axis system when the body is undeformed. Then the metric-tensor components become very simple, and Eq. (4.19) reduces to the following:

$$\frac{\partial}{\partial \theta^i} (s^{ij} \vec{G}_j) + \rho_0 \vec{B} = \rho_0 \vec{A} \tag{4.21}$$

Here we have written the $\vec{t}_{0i}$ in component form with the aid of Eqs. (4.13). In order to obtain the scalar form of this equation, we resolve it in terms of the fixed base vectors $\vec{g}_i$. This is accomplished by relating the pre- and post-deformation Cartesian coordinates to each other with the aid of the displacement components and then by using the proper transformation laws. The coordinates are related as follows:

$$y_i = \theta_i + v_i(\theta_1, \theta_2, \theta_3, t) \tag{4.22}$$

These equations are viewed as defining coordinate curves in B that were parallel to the Cartesian axes in B_0. For example, we can obtain the Cartesian coordinates in B of the line that coincided with the θ_3 axis in B_0 by setting $\theta_1 = \theta_2 = 0$ in Eqs. (4.22). In this same sense we can resolve the base vectors for the body-fixed coordinates in B in terms of those for the Cartesian coordinates by using the appropriate transformation law:

$$\vec{G}_j = \frac{\partial y_s}{\partial \theta^j} \vec{g}_s = \left(\delta_{sj} + \frac{\partial v_s}{\partial \theta^j}\right) \vec{g}_s \tag{4.23}$$

Thus Eq. (4.21) becomes

$$\frac{\partial}{\partial\theta^i}\left[s^{ij}\left(\delta_{sj} + \frac{\partial v_s}{\partial\theta^j}\right)\vec{g}_s\right] + \rho_0\vec{B} = \rho_0\vec{A} \tag{4.24}$$

Since the $\vec{g}_s$ are fixed vectors, the scalar form of this equation is easily written as

$$\frac{\partial}{\partial\theta^i}\left[s^{ij}\left(\delta_{sj} + \frac{\partial v_s}{\partial\theta^j}\right)\right] + \rho_0 B_s = \rho_0 A_s \tag{4.25}$$

where B_s and A_s are components of $\vec{B}$ and $\vec{A}$ with respect to the fixed base vectors $\vec{g}_s$.

We then proceed to simplify these equations as was done for the strain-displacement relations. In terms of the quantities e_{ij} and ω_{ij}, Eqs. (4.25) become

$$\frac{\partial}{\partial\theta^i}\left[s^{ij}(\delta_{sj} + e_{sj} - \omega_{sj})\right] + \rho_0 B_s = \rho_0 A_s \tag{4.26}$$

The small-strain assumption (elongations and shears negligible in comparison with terms of order unity) does not permit any direct simplification of Eqs. (4.26), although their meaning in physical terms is considerably simplified. The body-fixed coordinates in B can now be considered orthogonal, so the equations express the equation of motion of a rectangular parallelepiped with arbitrary translation and rotation from its original state. The components s^{ij} (or, now, s_{ij}) become indistinguishable from the tensor components σ^{ij} (or σ_{ij}). With the assumption that squares of the rotations are small compared with terms of order unity, the e_{sj} in Eqs. (4.26) become of higher order and can be neglected; we have then

$$\frac{\partial}{\partial\theta_i}\left[s_{ij}(\delta_{sj} - \omega_{sj})\right] + \rho_0 B_s = \rho_0 A_s \tag{4.27}$$

Finally, we obtain the linear equations when the rotations themselves are assumed much smaller than terms of order unity:

$$\frac{\partial s_{is}}{\partial\theta_i} + \rho_0 B_s = \rho_0 A_s \tag{4.28}$$

4.3 Boundary Conditions

At any point on the boundary where the displacements are not given, the conditions to be satisfied express the requirement that the stress vector at the boundary surface be equal to the surface loading — that is, that $\vec{P} = \vec{t}$. We use Eqs. (4.11) and Eqs. (4.13), where now the unit normal $\vec{n}_0$ is the normal to the boundary surface of B_0:

$$\vec{P}(dS/dS_0) = \vec{P}_0 = \sum_{i=1}^{3} n_{0i}(g^{ii})^{\frac{1}{2}} \vec{t}_{0i}$$
$$\vec{P}_0 = n_{0i}\, s^{ij}\, \vec{G}_j \tag{4.29}$$

It has been tacitly assumed, as before, that the most desirable form of these equations is one where the reference areas are those in B_0. If the surface loading vector is given per unit area of B, then the ratio dS/dS_0 is needed. This ratio depends in a complicated manner on the elemental areas dS_{0i}, the vector components n_{0i}, and the strains. It will not be written explicitly here; it is given in Ref. 2 for Cartesian coordinates.

As was done with the equations of motion, we resolve the second of Eqs. (4.29) into components along the fixed Cartesian axes. We let P_{0s} be the components of P_0 with respect to the base vectors $\vec{g}_s$ and use Eqs. (4.23); this gives

$$P_{0s} = n_{0i}\, s^{ij}\left(\delta_{sj} + \frac{\partial v_s}{\partial \theta_j}\right) \tag{4.30}$$

In terms of the quantites e_{ij} and ω_{ij}, we have

$$P_{0s} = n_{0i}\, s^{ij}(\delta_{sj} + e_{sj} - \omega_{sj}) \tag{4.31}$$

The process of simplification follows closely that for the equations of motion. The small-strain assumption permits no change in the form of Eqs. (4.31), but simplification does result because the area ratios dS_i/dS_{0i} and dS/dS_0 are approximately unity. The e_{sj} are neglected when squares of the rotations are small in comparison with terms of order unity:

$$P_{Os} = n_{Oi} s_{ij} (\delta_{sj} - \omega_{sj}) \tag{4.32}$$

And, finally, the linear equations are obtained when the rotations are assumed negligibly small:

$$P_{Os} = n_{Oi} s_{is} \tag{4.33}$$

V. STRESS-STRAIN RELATIONS; STRAIN ENERGY

5.1 Stress-Strain Relations

We observed earlier that the problems of greatest interest in this study would be those where the proportional limit of the material is not exceeded. In other words, we consider problems that are geometrically nonlinear but elastically linear, and for the time being we restrict ourselves to isotropic materials. The effects of any internal dissipative processes are neglected. The stress-strain relations are then linear, and we can use Hooke's law:

$$s_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \tag{5.1}$$

where λ and μ are the Lamé constants of elasticity. In terms of Young's modulus E and Poisson's ratio ν, Eqs. (5.1) become

$$s_{ij} = [E\nu/(1+\nu)(1-2\nu)]\epsilon_{kk}\delta_{ij} + [E/(1+\nu)]\epsilon_{ij} \tag{5.2}$$

5.2 Strain Energy

The strain energy for an elastic body can be generally written as follows:

$$W = \iiint_V \int_0^{\epsilon_f} s^{ij} d\epsilon_{ij} \, dV \tag{5.3}$$

Here the innermost integral represents the strain energy per unit volume, obtained by integrating with respect to the strain from a state of zero strain to the final state of strain, represented by ϵ_f. When the body is purely elastic and the stress-strain relations are linear, this integral can be evaluated, and the strain energy becomes

$$W = \frac{1}{2}\iiint_{V_0} \sum_{i=1}^{3} s_{ij}\epsilon_{ij}(gg^{ii})^{\frac{1}{2}} \, d\theta^1 d\theta^2 d\theta^3 \tag{5.4}$$

Note that the integration is over the volume of the undeformed body. When the body-fixed coordinate system in B_0 is identified with the fixed Cartesian system, Eq. (5.4) becomes simply

$$W = \frac{1}{2}\iiint_{V_0} s_{ij}\epsilon_{ij} d\theta_1 d\theta_2 d\theta_3 \tag{5.5}$$

VI. SUMMARY OF THREE-DIMENSIONAL APPROXIMATIONS

We conclude the work in three dimensions by summarizing the levels of approximation that have been enumerated above. There are essentially three. One level is characterized by the assumption that the strains do not exceed the elastic limit and are therefore negligible with respect to quantities of order unity, with no restriction placed on the rotations. The applicable equations are Eqs. (2.30) for the strain-displacement relations, Eqs. (4.26) for the equations of motion, and Eqs. (4.31) for the boundary conditions. The second level is obtained by introducing the additional assumption that squares of the rotations are negligible with respect to terms of order unity. In the same order as for the first level, the applicable equations are Eqs. (2.31), Eqs. (4.27), and Eqs. (4.32). The third level is given by restricting the rotations to be of the order of strains. The applicable equations are then the linear ones: Eqs. (2.32), Eqs. (4.28), and Eqs. (4.32), again in the same order as before. The equations are grouped together in Table 1:

Table 1. Summary of Levels of Approximation in Three Dimensions

Assumption: Small strains

Strain-displacement: $\epsilon_{ij} = e_{ij} + \frac{1}{2} (e_{ri}-\omega_{ri})(e_{rj}-\omega_{rj})$

Eqs. of motion: $\frac{\partial}{\partial\theta^i} [s^{ij}(\delta_{sj}+e_{sj}-\omega_{sj})] + \rho_0 B_s = \rho_0 A_s$

Stress boundary condition: $P_{0s} = n_{0i} s^{ij}(\delta_{sj}+e_{sj}-\omega_{sj})$

Assumptions: Small strains plus small products and squares of rotations

Strain-displacement: $\epsilon_{ij} = e_{ij} + \frac{1}{2} \omega_{ri}\omega_{rj}$

Eqs. of motion: $\frac{\partial}{\partial\theta_i} [s_{ij}(\delta_{sj}-\omega_{sj})] + \rho_0 B_s = \rho_0 A_s$

Stress boundary condition: $P_{0s} = n_{0i} s_{ij}(\delta_{sj}-\omega_{sj})$

Assumptions: Small strains and rotations

continued

Strain-displacement: $\epsilon_{ij} = e_{ij}$

Eqs. of motion: $\dfrac{\partial s_{is}}{\partial \theta_i} + \rho_0 B_s = \rho_0 A_s$

Stress boundary condition: $P_{0s} = n_{0i} s_{is}$

VII. PANEL-FLUTTER EQUATIONS AND BOUNDARY CONDITIONS; METHOD OF SOLUTION

7.1 Derivation of the Euler Equations and Boundary Conditions

We now consider a thin, isotropic, initially flat plate of constant thickness h. Let the fixed Cartesian coordinate system be located so that the X_3 axis is normal to the plate and so that the plane $X_3 = 0$ coincides with the middle surface of the plate in its unstrained state.

Then we must choose the proper level of approximation for the three-dimensional equations. It seems clear from physical considerations that for the purpose of this study the second or intermediate level is appropriate. White it is evident that significant rotations must be taken into account, it is equally evident that the added complication of accounting for unrestricted rotations is unnecessary. This level is reasonably consistent with the assumptions used in deriving the von Kármán plate equations, which have been used by some authors (Dowell[5], for example) to study the panel-flutter problem. There are however additional restrictive assumptions involved in the development of the von Kármán equations; these will be taken up after the equations appropriate to this study have been derived.

The next step is to approximate the displacements by expanding them about the middle surface in powers of the lateral coordinate:

$$v_i(\theta_1,\theta_2,\theta_3,t) = \bar{v}_i(\theta_1,\theta_2,t) + \theta_3\hat{v}_i(\theta_1,\theta_2,t) + \cdots \tag{7.1}$$

These series are truncated at two terms each, and two geometric assumptions are used to write the $\hat{v}_i$ in terms of the $\bar{v}_i$. The middle-surface displacements $\bar{v}_i$ are then the unknowns of the problem. This procedure will be followed in the development that follows, and afterwards a process for going on to higher approximations will be described.

The first of the geometric assumptions is the Love-Kirchhoff hypothesis — that fibers normal to the middle surface before the deformation remain normal and unextended after the deformation. We express this assumption mathematically by requiring that $\epsilon_{23} = \epsilon_{13} = \epsilon_{33} = 0$ at the middle surface. The other assumption is that effects of rotation about an axis normal to the plane of the plate — given by the quantity ω_{12} — are negligible in comparison with effects of rotations about axes lying in the plane of the plate — given by the quantities ω_{13} and ω_{23}.

Applying the second of the above assumptions to Eqs. (2.31) gives for the strains

$$
\begin{aligned}
\epsilon_{11} &= e_{11} + \omega_{13}^2/2 \\
\epsilon_{22} &= e_{22} + \omega_{23}^2/2 \\
\epsilon_{33} &= e_{33} + (\omega_{13}^2 + \omega_{23}^2)/2 \\
\epsilon_{12} &= e_{12} + (\omega_{13}\omega_{23})/2 \\
\epsilon_{13} &= e_{13} \\
\epsilon_{23} &= e_{23}
\end{aligned}
\tag{7.2}
$$

We write the quantities e_{13}, e_{23}, e_{33}, ω_{13}, and ω_{23} in terms of the displacement components by using Eqs. (7.1) and Eqs. (2.28). We obtain then for ϵ_{13}, ϵ_{23}, and ϵ_{33}

$$
\begin{aligned}
2\epsilon_{13} &= \hat{v}_1 + \frac{\partial \bar{v}_3}{\partial \theta_1} + \theta_3 \frac{\partial \hat{v}_3}{\partial \theta_1} \\
2\epsilon_{23} &= \hat{v}_2 + \frac{\partial \bar{v}_3}{\partial \theta_2} + \theta_3 \frac{\partial \hat{v}_3}{\partial \theta_2} \\
2\epsilon_{33} &= 2\hat{v}_3 + \frac{1}{4}\left[\left(\frac{\partial \bar{v}_3}{\partial \theta_1} + \theta_3 \frac{\partial \hat{v}_3}{\partial \theta_1} - \hat{v}_1\right)^2 + \left(\frac{\partial \bar{v}_3}{\partial \theta_2} + \theta_3 \frac{\partial \hat{v}_3}{\partial \theta_2} - \hat{v}_2\right)^2\right]
\end{aligned}
\tag{7.3}
$$

Applying the first of the aforementioned assumptions gives

$$
\begin{aligned}
\hat{v}_1 &= - \frac{\partial \bar{v}_3}{\partial \theta_1} \\
\hat{v}_2 &= - \frac{\partial \bar{v}_3}{\partial \theta_2}
\end{aligned}
$$

continued

$$\hat{v}_3 = -\frac{1}{2}\left[\left(\frac{\partial \bar{v}_3}{\partial \theta_1}\right)^2 + \left(\frac{\partial \bar{v}_3}{\partial \theta_2}\right)^2\right]$$

and this in turn gives for the strains

$$\begin{aligned}
\epsilon_{11} &= \frac{\partial \bar{v}_1}{\partial \theta_1} + \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \theta_1}\right)^2 - \theta_3 \frac{\partial^2 \bar{v}_3}{\partial \theta_1^2} \\
\epsilon_{22} &= \frac{\partial \bar{v}_2}{\partial \theta_2} + \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \theta_2}\right)^2 - \theta_3 \frac{\partial^2 \bar{v}_3}{\partial \theta_2^2} \\
\epsilon_{12} &= \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \theta_2} + \frac{\partial \bar{v}_2}{\partial \theta_1} + \frac{\partial \bar{v}_3}{\partial \theta_1}\frac{\partial \bar{v}_3}{\partial \theta_2} - 2\theta_3 \frac{\partial^2 \bar{v}_3}{\partial \theta_1 \partial \theta_2}\right) \\
\epsilon_{13} &= \epsilon_{23} = \epsilon_{33} = 0
\end{aligned} \tag{7.5}$$

In obtaining Eqs. (7.5) we have neglected other nonlinear terms by making further use of the assumption that products of the rotations ω_{13} and ω_{23} are negligible with respect to terms of order unity. Also, the terms left over in ϵ_{13}, ϵ_{23}, and ϵ_{33} are seen to be of higher order, so these strains are taken to be effectively zero throughout the plate.

At this point it would be possible to go back to the appropriate equations of equilibrium and boundary conditions and derive the proper plate equations. We prefer however to use a variational formulation, since this is the simplest way to assure a consistent set of equations. The Euler equations and boundary conditions are derived from Hamilton's principle:

$$\int_{t_1}^{t_2} (\delta T - \delta W + \delta W_e)\, dt = 0 \tag{7.6}$$

where δT is the first variation of the kinetic energy, δW is the first variation of the strain energy, and δW_e is the virtual work done by conservative or nonconservative applied loads. The variations are taken with respect to the displacements, which are the principal unknowns of the problem.

The kinetic energy is written as follows:

$$T = \frac{1}{2}\iiint_{V_0} \rho_0(\dot{v}_1^2 + \dot{v}_2^2 + \dot{v}_3^2)d\theta_1 d\theta_2 d\theta_3 \tag{7.7}$$

where V_0 denotes the volume of the undeformed body. The velocity components are written in terms of middle-surface velocity components with the aid of Eqs. (7.1) and Eqs. (7.4). After again neglecting higher-order terms and then integrating with respect to θ_3, we find

$$T = \frac{1}{2}\iint_{S_0} \rho_0 \left\{h(\dot{\bar{v}}_1^2 + \dot{\bar{v}}_2^2 + \dot{\bar{v}}_3^2) + \frac{h^3}{12}\left[\left(\frac{\partial\dot{\bar{v}}_3}{\partial\theta_1}\right)^2 + \left(\frac{\partial\dot{\bar{v}}_3}{\partial\theta_2}\right)^2\right]\right\} d\theta_1 d\theta_2 \tag{7.8}$$

Here the integration is taken over the planform area of the plate, denoted by S_0. The first variation is then

$$\begin{aligned}\delta T = \iint_{S_0} \rho_0 &\left\{h(\dot{\bar{v}}_1\delta\dot{\bar{v}}_1 + \dot{\bar{v}}_2\delta\dot{\bar{v}}_2 + \dot{\bar{v}}_3\delta\dot{\bar{v}}_3)\right. \\ &\left.+ \frac{h^3}{12}\left[\frac{\partial\dot{\bar{v}}_3}{\partial\theta_1}\,\delta\left(\frac{\partial\dot{\bar{v}}_3}{\partial\theta_1}\right) + \frac{\partial\dot{\bar{v}}_3}{\partial\theta_2}\,\delta\left(\frac{\partial\dot{\bar{v}}_3}{\partial\theta_2}\right)\right]\right\} d\theta_1 d\theta_2\end{aligned} \tag{7.9}$$

The terms multiplied by h^3 are then integrated by parts with respect to θ_1 and θ_2 by using Green's theorem in the plane:

$$\begin{aligned}\iint_{S_0}\left[\frac{\partial\dot{\bar{v}}_3}{\partial\theta_1}\,\delta\left(\frac{\partial\dot{\bar{v}}_3}{\partial\theta_1}\right) + \frac{\partial\dot{\bar{v}}_3}{\partial\theta_2}\,\delta\left(\frac{\partial\dot{\bar{v}}_3}{\partial\theta_2}\right)\right]d\theta_1 d\theta_2 &= -\iint_{S_0}\left(\frac{\partial^2\dot{\bar{v}}_3}{\partial\theta_1^2} + \frac{\partial^2\dot{\bar{v}}_3}{\partial\theta_2^2}\right)\delta\dot{\bar{v}}_3 d\theta_1 d\theta_2 \\ &+ \int_{C_0}\left(\frac{\partial\dot{\bar{v}}_3}{\partial\theta_1}\,d\theta_2 - \frac{\partial\dot{\bar{v}}_3}{\partial\theta_2}\,d\theta_1\right)\delta\dot{\bar{v}}_3\end{aligned} \tag{7.10}$$

where C_0 is the curve bounding S_0. The contour integration is rewritten in terms of a tangent coordinate, s, positive counterclockwise, and a normal coordinate, n, positive outward. The variation of the kinetic energy thus becomes

$$\delta T = \iint_{S_0} \rho_0 \left[h(\dot{\bar{v}}_1 \delta\dot{\bar{v}}_1 + \dot{\bar{v}}_2 \delta\dot{\bar{v}}_2 + \dot{\bar{v}}_3 \delta\dot{\bar{v}}_3) - \frac{h^3}{12}\left(\frac{\partial^2 \dot{\bar{v}}_3}{\partial\theta_1^2} + \frac{\partial^2 \dot{\bar{v}}_3}{\partial\theta_2^2}\right) \delta\dot{\bar{v}}_3 \right] d\theta_1 d\theta_2$$

$$+ \frac{h^3}{12} \int_{C_0} \rho_0 \frac{\partial \dot{\bar{v}}_3}{\partial h} \delta\dot{\bar{v}}_3 ds \qquad (7.11)$$

Finally, the variation is integrated with respect to time. A further partial integration with respect to time is performed, thereby giving the final form:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \left\{ \iint_{S_0} \rho_0 \left[-h(\ddot{\bar{v}}_1 \delta\bar{v}_1 + \ddot{\bar{v}}_2 \delta\bar{v}_2 + \ddot{\bar{v}}_3 \delta\bar{v}_3) \right.\right.$$

$$\left.\left. + \frac{h^3}{12}\left(\frac{\partial^2 \ddot{\bar{v}}_3}{\partial\theta_1^2} + \frac{\partial^2 \ddot{\bar{v}}_3}{\partial\theta_2^2}\right) \delta\bar{v}_3 \right] d\theta_1 d\theta_2 - \frac{h^3}{12} \int_{C_0} \rho_0 \frac{\partial \ddot{\bar{v}}_3}{\partial h} \delta\bar{v}_3 ds \right\} dt \qquad (7.12)$$

It has been assumed, as usual, that the variations are zero at t_1 and t_2. Note also that the displacement boundary conditions are taken into account by requiring that the virtual displacements be consistent with the physical constraints of the problem; this means in particular that the virtual displacements must be zero at any point where the displacements are specified. The expression in braces in the right-hand side of Eq. (7.12) will be used for δT.

We take Eqs. (7.5) and insert them in Eq. (5.5) in order to obtain the strain energy in terms of stresses and displacements. We then calculate the variation of the strain energy, recalling that the stresses also depend on the displacements; we find

$$\delta W = \iiint_{V_0} \Big\{ s_{11}\delta\Big[\frac{\partial\bar{v}_1}{\partial\theta_1} + \frac{1}{2}\Big(\frac{\partial\bar{v}_3}{\partial\theta_1}\Big)^2 - \theta_3 \frac{\partial^2\bar{v}_3}{\partial\theta_1^2}\Big] + s_{22}\delta\Big[\frac{\partial\bar{v}_2}{\partial\theta_2} + \frac{1}{2}\Big(\frac{\partial\bar{v}_3}{\partial\theta_2}\Big)^2 - \theta_3 \frac{\partial^2\bar{v}_3}{\partial\theta_2^2}\Big]$$

$$+ s_{12}\delta\Big(\frac{\partial\bar{v}_1}{\partial\theta_2} + \frac{\partial\bar{v}_2}{\partial\theta_1} + \frac{\partial\bar{v}_3}{\partial\theta_1}\frac{\partial\bar{v}_3}{\partial\theta_2} - 2\theta_3 \frac{\partial^2\bar{v}_3}{\partial\theta_1\partial\theta_2}\Big)\Big\} d\theta_1 d\theta_2 d\theta_3 \qquad (7.13)$$

Integrating with respect to θ_3, we find

$$\delta W = \frac{1}{2}\iint_{S_0} \Big\{ N_{11}\delta\Big[\frac{\partial\bar{v}_1}{\partial\theta_1} + \frac{1}{2}\Big(\frac{\partial\bar{v}_3}{\partial\theta_1}\Big)^2\Big] - M_{11}\delta\Big(\frac{\partial^2\bar{v}_3}{\partial\theta_1^2}\Big) + N_{22}\delta\Big[\frac{\partial\bar{v}_2}{\partial\theta_2} + \frac{1}{2}\Big(\frac{\partial\bar{v}_3}{\partial\theta_2}\Big)^2\Big]$$

$$- M_{22}\delta\Big(\frac{\partial^2\bar{v}_3}{\partial\theta_2^2}\Big) + N_{12}\delta\Big(\frac{\partial\bar{v}_1}{\partial\theta_2} + \frac{\partial\bar{v}_2}{\partial\theta_1} + \frac{\partial\bar{v}_3}{\partial\theta_1}\frac{\partial\bar{v}_3}{\partial\theta_2}\Big) - 2M_{12}\delta\Big(\frac{\partial^2\bar{v}_3}{\partial\theta_1\partial\theta_2}\Big)\Big\} d\theta_1 d\theta_2$$

$$(7.14)$$

where

$$N_{ij} = \int_{-h/2}^{h/2} s_{ij} d\theta_3$$

$$(i,j = 1,2) \qquad (7.15)$$

$$M_{ij} = \int_{-h/2}^{h/2} \theta_3 s_{ij} d\theta_3$$

As was done for the kinetic energy, we integrate by parts, rewriting the line integrals around the edges of the plate in terms of the tangential and normal coordinates s and n. The N_{ij} and M_{ij} are resolved in the tangential and normal directions, as are the in-plane displacements $\bar{v}_1$ and $\bar{v}_2$. We have then

$$\delta W = \iint_{S_0} \Big\{ -\Big(\frac{\partial N_{11}}{\partial\theta_1} + \frac{\partial N_{12}}{\partial\theta_2}\Big)\delta\bar{v}_1 - \Big(\frac{\partial N_{22}}{\partial\theta_2} + \frac{\partial N_{12}}{\partial\theta_1}\Big)\delta\bar{v}_2$$

continued

$$-\left[\frac{\partial}{\partial\theta_1}\left(N_{11}\frac{\partial\bar{v}_3}{\partial\theta_1}+N_{12}\frac{\partial\bar{v}_3}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_2}\left(N_{22}\frac{\partial\bar{v}_3}{\partial\theta_2}+N_{12}\frac{\partial\bar{v}_3}{\partial\theta_1}\right)+\frac{\partial^2 M_{11}}{\partial\theta_1^2}\right.$$

$$\left.+\,2\frac{\partial^2 M_{12}}{\partial\theta_1\partial\theta_2}+\frac{\partial^2 M_{22}}{\partial\theta_2^2}\right]\delta\bar{v}_3\Bigg\}\,d\theta_1 d\theta_2+\int_{C_0}\Bigg[N_{nn}\delta\bar{v}_n+N_{ns}\delta\bar{v}_s$$

$$+\left(N_{nn}\frac{\partial\bar{v}_3}{\partial n}+N_{ns}\frac{\partial\bar{v}_3}{\partial s}+2\frac{\partial M_{ns}}{\partial s}+\frac{\partial M_{nn}}{\partial n}\right)\delta\bar{v}_3-M_{nn}\frac{\partial(\delta\bar{v}_3)}{\partial n}\Bigg]ds \tag{7.16}$$

where N_{nn} and M_{nn} are the integrated plate stresses referred to the normal to the edge of the plate, and N_{ns} and M_{ns} are the plate stresses referred to the tangent to the edge; normal and tangential edge displacements at the middle surface are given by $\bar{v}_n$ and $\bar{v}_s$, respectively.

For the calculation of δW_e, the virtual work of the external forces, we assume that the surface loads per unit area are given by the components f_i, which represent the proper generalized forces associated with the virtual displacements δv_i. The virtual work can therefore be written simply as

$$\delta W_e=\iint_{S_0'}(f_i\delta v_i)dS \tag{7.17}$$

where by S_0' we mean the total surface area of the plate — top, bottom, and edges. Separating the integration around the edges from the rest, we obtain

$$\delta W_e=\iint_{S_0} f_i\delta v_i d\theta_1 d\theta_2\Bigg|_{\theta_3=h/2}+\iint_{S_0} f_i\delta v_i d\theta_1 d\theta_2\Bigg|_{\theta_3=-h/2}$$

$$+\int_{C_0}\int_{-h/2}^{h/2} f_i\delta v_i d\theta_3 ds \tag{7.18}$$

As before, Eqs. (7.1) and Eqs. (7.4) are used to write the displacements in terms of middle-surface displacements, and the integrand of Eq. (7.18) becomes

$$f_i\delta v_i = f_1\left[\delta\bar{v}_1 - \theta_3\delta\left(\frac{\partial\bar{v}_3}{\partial\theta_1}\right)\right] + f_2\left[\delta\bar{v}_2 - \theta_3\delta\left(\frac{\partial\bar{v}_3}{\partial\theta_2}\right)\right]$$

$$+ f_3\left\{\delta\bar{v}_3 - \theta_3\left[\frac{\partial\bar{v}_3}{\partial\theta_1}\delta\left(\frac{\partial\bar{v}_3}{\partial\theta_1}\right) + \frac{\partial\bar{v}_3}{\partial\theta_2}\delta\left(\frac{\partial\bar{v}_3}{\partial\theta_2}\right)\right]\right\} \tag{7.19}$$

We then define surface loads referred to S_0

$$\begin{aligned} F_i &= f_i\Big|_{\theta_3=h/2} + f_i\Big|_{\theta_3=-h/2} \\ m_i &= (\theta_3 f_i)\Big|_{\theta_3=h/2} + (\theta_3 f_i)\Big|_{\theta_3=-h/2} \end{aligned} \tag{7.20}$$

and edge loads on C_0

$$\begin{aligned} F_i^* &= \int_{-h/2}^{h/2} f_i d\theta_3 \\ m_i^* &= \int_{-h/2}^{h/2} \theta_3 f_i d\theta_3 \end{aligned} \tag{7.21}$$

With these definitions and Eq. (7.19) we return to Eq. (7.18) and integrate by parts the terms in the area integral involving variations of derivatives (or derivatives of variations). The contour integral thereby obtained is combined with the original one, and the resulting integral is rewritten in terms of the normal and tangential coordinates n and s. We thus find the final form for the virtual work δW_e:

$$\delta W_e = \iint_{S_0} \left\{ F_1 \delta\bar{v}_1 + F_2 \delta\bar{v}_2 + \left[F_3 + \frac{\partial m_1}{\partial \theta_1} + \frac{\partial m_2}{\partial \theta_2} + \frac{\partial}{\partial \theta_1}\left(m_3 \frac{\partial \bar{v}_3}{\partial \theta_1} \right) \right.\right.$$

$$\left.\left. + \frac{\partial}{\partial \theta_2}\left(m_3 \frac{\partial \bar{v}_3}{\partial \theta_2} \right) \right] \delta\bar{v}_3 \right\} d\theta_1 d\theta_2 + \int_{C_0} \left\{ F_n^* \delta\bar{v}_n + F_s^* \delta\bar{v}_s \right.$$

$$+ \left[F_3^* + \frac{\partial m_s^*}{\partial s} + \frac{\partial}{\partial s}\left(m_3^* \frac{\partial \bar{v}_3}{\partial s} \right) - m_n - m_3 \frac{\partial \bar{v}_3}{\partial n} \right] \delta\bar{v}_3$$

$$\left. - \left(m_n^* + m_3^* \frac{\partial \bar{v}_3}{\partial n} \right) \frac{\partial (\delta\bar{v}_3)}{\partial n} \right\} ds \qquad (7.22)$$

The three variations can now be combined. After grouping the coefficients of each of the variations, we have

$$\delta T - \delta W + \delta W_e = \iint_{S_0} \left\{ \left(-\rho_0 h \ddot{\bar{v}}_1 + \frac{\partial N_{11}}{\partial \theta_1} + \frac{\partial N_{12}}{\partial \theta_2} + F_1 \right) \delta\bar{v}_1 \right.$$

$$+ \left(-\rho_0 h \ddot{\bar{v}}_2 + \frac{\partial N_{22}}{\partial \theta_2} + \frac{\partial N_{12}}{\partial \theta_1} + F_2 \right) \delta\bar{v}_2 + \left[-\rho_0 h \ddot{\bar{v}}_3 + \frac{\rho_0 h^3}{12} \left(\frac{\partial^2 \ddot{\bar{v}}_3}{\partial \theta_1^2} + \frac{\partial^2 \ddot{\bar{v}}_3}{\partial \theta_2^2} \right) \right.$$

$$+ \frac{\partial}{\partial \theta_1}\left(N_{11} \frac{\partial \bar{v}_3}{\partial \theta_1} + N_{12} \frac{\partial \bar{v}_3}{\partial \theta_2} \right) + \frac{\partial}{\partial \theta_2}\left(N_{22} \frac{\partial \bar{v}_3}{\partial \theta_2} + N_{12} \frac{\partial \bar{v}_3}{\partial \theta_1} \right) + \frac{\partial^2 M_{11}}{\partial \theta_1^2}$$

$$\left.\left. + 2 \frac{\partial^2 M_{12}}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 M_{22}}{\partial \theta_2^2} + F_3 + \frac{\partial}{\partial \theta_1}\left(m_1 + m_3 \frac{\partial \bar{v}_3}{\partial \theta_1} \right) + \frac{\partial}{\partial \theta_2}\left(m_2 + m_3 \frac{\partial \bar{v}_3}{\partial \theta_2} \right) \right] \delta\bar{v}_3 \right\} d\theta_1 d\theta_2$$

$$+ \int_{C_0} \left\{ (-N_{nn} + F_n^*)\, \delta\bar{v}_n + (-N_{ns} + F_s^*)\, \delta\bar{v}_s + \left[-\frac{\rho_0 h^3}{12} \frac{\partial \ddot{\bar{v}}_3}{\partial n} - N_{nn} \frac{\partial \bar{v}_3}{\partial n} \right.\right.$$

$$\left. - N_{ns} \frac{\partial \bar{v}_3}{\partial s} - 2 \frac{\partial M_{ns}}{\partial s} - \frac{\partial M_{nn}}{\partial n} - m_n - m_3 \frac{\partial \bar{v}_3}{\partial n} + F_3^* + \frac{\partial m_s^*}{\partial s} + \frac{\partial}{\partial s}\left(m_3^* \frac{\partial \bar{v}_3}{\partial s} \right) \right] \delta\bar{v}_3$$

$$\left. + \left(M_{nn} - m_n^* - m_3^* \frac{\partial \bar{v}_3}{\partial n} \right) \delta\left(\frac{\partial \bar{v}_3}{\partial n} \right) \right\} ds \qquad (7.23)$$

For Hamilton's principle to be satisfied, we require that the area integral and the contour integral be separately zero. Furthermore, since the variations are independent, we require that the coefficient of each variation be zero. From the area integral we obtain thereby the Euler equations of the problem:

$$-\rho_0 h\ddot{\bar{v}}_1 + \frac{\partial N_{11}}{\partial \theta_1} + \frac{\partial N_{12}}{\partial \theta_2} + F_1 = 0$$

$$-\rho_0 h\ddot{\bar{v}}_2 + \frac{\partial N_{22}}{\partial \theta_2} + \frac{\partial N_{12}}{\partial \theta_1} + F_2 = 0$$

$$-\rho_0 h\ddot{\bar{v}}_3 + \frac{\rho_0 h^3}{12}\left(\frac{\partial^2 \ddot{\bar{v}}_3}{\partial \theta_1^2} + \frac{\partial^2 \ddot{\bar{v}}_3}{\partial \theta_2^2}\right) + \frac{\partial}{\partial \theta_1}\left(N_{11}\frac{\partial \bar{v}_3}{\partial \theta_1} + N_{12}\frac{\partial \bar{v}_3}{\partial \theta_2}\right) \tag{7.24}$$

$$+ \frac{\partial}{\partial \theta_2}\left(N_{22}\frac{\partial \bar{v}_3}{\partial \theta_2} + N_{12}\frac{\partial \bar{v}_3}{\partial \theta_1}\right) + \frac{\partial^2 M_{11}}{\partial \theta_1^2} + 2\frac{\partial^2 M_{12}}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 M_{22}}{\partial \theta_2^2}$$

$$+ F_3 + \frac{\partial}{\partial \theta_1}\left(m_1 + m_3\frac{\partial \bar{v}_3}{\partial \theta_1}\right) + \frac{\partial}{\partial \theta_2}\left(m_2 + m_3\frac{\partial \bar{v}_3}{\partial \theta_2}\right) = 0$$

From the contour integral we obtain first of all the relations between the applied loads and the integrated plate stresses:

$$N_{nn} = F_n^*$$

$$N_{ns} = F_s^*$$

$$M_{ns} = m_s^* + m_3^*\frac{\partial \bar{v}_3}{\partial s} \tag{7.25}$$

$$F_3^* = \frac{\rho_0 h^3}{12}\frac{\partial \ddot{\bar{v}}_3}{\partial n} + N_{nn}\frac{\partial \bar{v}_3}{\partial n} + N_{ns}\frac{\partial \bar{v}_3}{\partial s} + \frac{\partial M_{ns}}{\partial s} + \frac{\partial M_{nn}}{\partial n} + m_n + m_3\frac{\partial \bar{v}_3}{\partial n}$$

$$M_{nn} = m_n^* + m_3^*\frac{\partial \bar{v}_3}{\partial n}$$

We then find the quantities that must be specified at the edge by examining the contour integral in the expression for δW_e, Eq. (7.22). Any combination of loads or displacements, totaling four, must be specified; a load or a displacement from each of the four products in the integrand of Eq. (7.22) must be included. Thus we might specify $\bar{v}_n$, $\bar{v}_s$, $\bar{v}_3$, and $\frac{\partial \bar{v}_3}{\partial n}$, or F_n^*, F_s^*, $F_3^* + \frac{\partial m_s^*}{\partial s} + \frac{\partial}{\partial s}\left(m_3^* \frac{\partial \bar{v}_3}{\partial s}\right) - m_n - m_3 \frac{\partial \bar{v}_3}{\partial n}$, $m_n^* + m_3^* \frac{\partial \bar{v}_3}{\partial n}$, or any combination of four except ones containing, say, both F_n^* and $\bar{v}_n$ or the like.

To complete the formulation, we must determine the integrated plate stresses N_{ij} and M_{ij} in terms of the displacements $\bar{v}_i$. This is accomplished by using the stress-strain relations, Eqs. (5.2), and the strain-displacement relations, Eqs. (7.5). The stress-strain relations become

$$\begin{aligned}
s_{11} &= [E(1-\nu)/(1+\nu)(1-2\nu)]\epsilon_{11} + [E\nu/(1+\nu)(1-2\nu)]\epsilon_{22} \\
s_{22} &= [E(1-\nu)/(1+\nu)(1-2\nu)]\epsilon_{22} + [E\nu/(1+\nu)(1-2\nu)]\epsilon_{11} \\
s_{12} &= [E/(1+\nu)]\epsilon_{12} \\
s_{33} &= [E\nu/(1+\nu)(1-2\nu)](\epsilon_{11}+\epsilon_{22}) \\
s_{13} &= s_{23} = 0
\end{aligned} \tag{7.26}$$

Even though it is not zero, the stress s_{33} does not appear in the equations of equilibrium or the boundary conditions. This is a result of taking ϵ_{33} approximately zero, so that s_{33} does no work and therefore does not appear in the expression for the strain energy. Now we substitute for the strains from Eqs. (7.5) in Eqs. (7.26), integrate with respect to θ_3 to get the N_{ij}, and then multiply by θ_3 and integrate to get the M_{ij}. We obtain

$$N_{11} = \frac{Eh(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\frac{\partial \bar{v}_1}{\partial \theta_1} + \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \theta_1}\right)^2 + \frac{\nu}{1-\nu}\left[\frac{\partial \bar{v}_2}{\partial \theta_2} + \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \theta_2}\right)^2\right]\right\}$$

continued

$$N_{22} = \frac{Eh(1-\nu)}{(1+\nu)(1-2\nu)} \left\{ \frac{\partial \bar{v}_2}{\partial \theta_2} + \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \theta_2}\right)^2 + \frac{\nu}{1-\nu}\left[\frac{\partial \bar{v}_1}{\partial \theta_1} + \frac{1}{2}\left(\frac{\partial \bar{v}_3}{\partial \theta_1}\right)^2\right]\right\}$$

$$N_{12} = \frac{Eh}{2(1-\nu)}\left[\frac{\partial \bar{v}_1}{\partial \theta_2} + \frac{\partial \bar{v}_2}{\partial \theta_1} + \frac{\partial \bar{v}_3}{\partial \theta_1}\frac{\partial \bar{v}_3}{\partial \theta_2}\right] \tag{7.27}$$

$$M_{11} = -\frac{Eh^3(1-\nu)}{12(1+\nu)(1-2\nu)}\left(\frac{\partial^2 \bar{v}_3}{\partial \theta_1^2} + \frac{\nu}{1-\nu}\frac{\partial^2 \bar{v}_3}{\partial \theta_2^2}\right)$$

$$M_{22} = -\frac{Eh^3(1-\nu)}{12(1+\nu)(1-2\nu)}\left(\frac{\partial^2 \bar{v}_3}{\partial \theta_2^2} + \frac{\nu}{1-\nu}\frac{\partial^2 \bar{v}_3}{\partial \theta_1^2}\right)$$

$$M_{12} = -\frac{Eh^3(1-\nu)}{12(1+\nu)(1-2\nu)}\frac{\partial^2 \bar{v}_3}{\partial \theta_1 \partial \theta_2}$$

Using these relations in Eqs. (7.24) allows the problem to be posed solely in terms of displacements.

Eqs. (7.24) are similar in form to those of Herrmann[3], whereas Eqs. (7.27) are not. Herrmann used the stress-displacement relations of von Kármán plate theory, where the stress s_{33} is in effect taken as zero while the strain ϵ_{33} is not. In other respects, the remarks in Ref. 3 apply here — there are terms that represent the effect of rotatory inertia, and in the absence of inertia terms Eqs. (7.24) would also reduce to the corresponding ones for the von Kármán theory.

7.2 Method of Solution

In order to obtain solutions to these equations, we return to Hamilton's principle and use a generalization of the Ritz method for dynamic systems. We take the strain energy as given by Eq. (7.14) and rewrite it in terms of the displacements alone with the aid of Eqs. (7.27). We can say then that the kinetic energy and the strain energy depend on the displacements and their derivatives as follows:

$$T = T\left(\dot{\bar{v}}_i, \frac{\partial \dot{\bar{v}}_3}{\partial \theta_\alpha}\right)$$

continued

$$W = W\left(\frac{\partial \bar{v}_i}{\partial \theta_\alpha}, \frac{\partial^2 \bar{v}_3}{\partial \theta_\alpha \partial \theta_\beta}\right) \qquad (i = 1,2,3;\alpha,\beta = 1,2) \qquad (7.28)$$

For the virtual work of the external forces, we rewrite Eq. (7.22) in a more compact form as

$$\delta W_e = \iint_{S_0} (F_1 \delta\bar{v}_1 + F_2 \delta\bar{v}_2 + \bar{F}_3 \delta\bar{v}_3) d\theta_1 d\theta_2$$

$$+ \int_{C_0} \left[F_n^* \delta\bar{v}_n + F_s^* \delta\bar{v}_s + \bar{F}_3^* \delta\bar{v}_3 - \bar{m}_n^* \delta\left(\frac{\partial \bar{v}_3}{\partial n}\right)\right] ds \qquad (7.29)$$

with the simplification in notation being easily discerned by comparison of this equation with Eq. (7.22). The displacements are then represented as follows:

$$\bar{v}_i(\theta_1, \theta_2, t) = \sum_{k=0}^{\infty} a_{ik}(t) \Psi_{ik}(\theta_1) \Phi_{ik}(\theta_2) \qquad (7.30)$$

The trial functions Ψ_{ik} and Φ_{ik} are chosen so that they form a complete set of orthogonal functions that satisfy (at least) the geometric boundary conditions. For example, one might choose as trial functions for a simply supported rectangular plate functions of the form

$$\Psi_{ik}(\theta_1) = \sin \frac{k\pi\theta_1}{\ell_1} \qquad (7.31)$$

where ℓ_1 is the dimension of the plate in the θ_1 direction; for a clamped rectangular plate one could use

$$\Psi_{ik}(\theta_1) = 1 - \cos \frac{2k\pi\theta_1}{\ell_1} \qquad (7.32)$$

The amplitude functions a_{ik} thus become the unknowns. Note that the displacements $\bar{v}_n$ and $\bar{v}_s$ are linear functions of $\bar{v}_1$ and $\bar{v}_2$, being given as sums of products of $\bar{v}_1$ and $\bar{v}_2$ with the proper direction cosines. We can therefore write the variations $\delta\bar{v}_n$ and $\delta\bar{v}_s$ in terms of δa_{1k} and δa_{2k}. The same situation obtains as well for $\partial\bar{v}_3/\partial n$, since it can be written in terms of $\partial\bar{v}_3/\partial\theta_1$ and $\partial\bar{v}_3/\partial\theta_2$. When the expressions given in Eqs. (7.30) are substituted for the displacements in the kinetic-energy and strain-energy expressions, the dependence on the new unknowns a_{ik} becomes simply

$$
\begin{aligned}
T &= T(\dot{a}_{ik}) \\
W &= W(a_{ik})
\end{aligned}
\qquad (7.33)
$$

For the virtual work δW_e, we find

$$
\delta W_e = \sum_{k=0}^{\infty} \left[(F_{1k}+F^*_{1k})\delta a_{1k} + (F_{2k}+F^*_{2k})\delta a_{2k} + (\bar{F}_{3k} + \bar{F}^*_{3k} - \bar{m}^*_{nk})\delta a_{3k} \right] \qquad (7.34)
$$

The unstarred coefficients of the variations in this expression come from the area integral in Eq. (7.29), and are given by expressions like

$$
F_{1k} = \iint_{S_0} F_1 \Psi_{1k} \Phi_{1k} d\theta_1 d\theta_2 \qquad (7.35)
$$

whereas the starred coefficients are derived from the contour integral in Eq. (7.29) and are given by expressions like

$$
F^*_{1k} = \int_{C_0} [F^*_n \cos(\theta_1, n) + F^*_s \cos(\theta_1, s)] \Psi_{1k} \Phi_{1k} ds \qquad (7.36)
$$

The direction cosines $\cos(\theta_1, n)$ and $\cos(\theta_1, s)$ are those of the undeformed plate, so $\cos(\theta_1, n)$, for example, is the cosine of the angle between the $\theta_1(X_1)$ axis and the pre-deformation edge normal.

We then rewrite Hamilton's principle as

$$\int_{t_1}^{t_2} \sum_{k=0}^{\infty} \left[\frac{\partial T}{\partial \dot{a}_{1k}} \delta\dot{a}_{1k} + \frac{\partial T}{\partial \dot{a}_{2k}} \delta\dot{a}_{2k} + \frac{\partial T}{\partial \dot{a}_{3k}} \delta\dot{a}_{3k} - \frac{\partial W}{\partial a_{1k}} \delta a_{1k} - \frac{\partial W}{\partial a_{2k}} \delta a_{2k}\right.$$

$$\left. - \frac{\partial W}{\partial a_{3k}} \delta a_{3k} + (F_{1k}+F^*_{1k})\delta a_{1k} + (F_{2k}+F^*_{2k})\delta a_{2k} + (\bar{F}_{3k}+\bar{F}^*_{3k}-\bar{m}^*_{nk})\delta a_{3k}\right] dt = 0 \tag{7.37}$$

Terms involving the kinetic energy are integrated by parts, with variations at times t_1 and t_2 assumed zero. This gives

$$\int_{t_1}^{t_2} \sum_{k=0}^{\infty} \left\{\left[- \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{a}_{1k}}\right) - \frac{\partial W}{\partial a_{1k}} + F_{1k}+F^*_{1k}\right]\delta a_{1k}\right.$$

$$+ \left[- \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{a}_{2k}}\right) - \frac{\partial W}{\partial a_{2k}} + F_{2k}+F^*_{2k}\right]\delta a_{2k}$$

$$\left. + \left[- \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{a}_{3k}}\right) - \frac{\partial W}{\partial a_{3k}} + \bar{F}_{3k}+\bar{F}^*_{3k}-\bar{m}^*_{nk}\right]\delta a_{3k} \right\} dt = 0 \tag{7.38}$$

Since the variations are independent, we require that each coefficient be separately equal to zero, thereby arriving at the sets of equations

$$\begin{aligned} - \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{a}_{1k}}\right) - \frac{\partial W}{\partial a_{1k}} + F_{1k} + F^*_{1k} &= 0 \\ - \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{a}_{2k}}\right) - \frac{\partial W}{\partial a_{2k}} + F_{2k} + F^*_{2k} &= 0 \\ - \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{a}_{3k}}\right) - \frac{\partial W}{\partial a_{3k}} + \bar{F}_{3k} + \bar{F}^*_{3k} - \bar{m}^*_{nk} &= 0 \end{aligned} \tag{7.39}$$

for each value of k. They are in general coupled, quasi-linear, second-order, ordinary differential equations. The problem is thus reduced to an initial-value problem, with the results being the plate motion as a function of time.

7.3 Extension to Higher Approximations

We recall that in applying the Love-Kirchhoff hypothesis we assumed that the strains ϵ_{23} and ϵ_{13} were approximately zero throughout the plate. It is therefore said that the effects of transverse shear deformations are neglected. If now a more accurate approximation to the plate problem is desired, the transverse shear deformations must be taken into account. Therefore the Love-Kirchhoff assumption is dropped. Series expansions for the displacements as in Eqs. (7.1) are assumed, and for the first approximation the series are truncated at two terms. Now, however, the linear terms in θ_3 are no longer given as functions of the middle-surface terms. The number of unknowns is doubled — from three to six — and a variational formulation as illustrated in Section 7.1 will give six plate equations and the proper boundary conditions to go with them. This formulation has been discussed by Habip[4]. In principle, it can be extended indefinitely simply by taking more and more terms in the series expansions for the displacements, although as a practical matter there seems to be very little justification for taking transverse shear deformations into account, at least for isotropic plates of thicknesses of interest in this study.

VIII. CONCLUDING REMARKS

Further research activity has been divided into three areas of investigation, as outlined below:

(1) The first case chosen for detailed analysis is the one considered by Dowell[5], who applied the Galerkin method to the von Kármán plate equations with first-order piston-theory aerodynamic loads. It is expected that a comparison of his results with those obtained from variational considerations (as outlined in Section VII) will help to clarify the relationships between the two approaches. Of particular interest is the effect of the boundary conditions on these relationships. This case will also provide a convenient means for assessing the influence of additional nonlinearities, such as aerodynamic nonlinearities, on stability and the behavior of the limit cycle. As this comparison nears completion, a new case will be chosen for detailed analysis. The theoretical model will simulate as closely as possible an experimental model, with the aim being to eliminate or explain any differences between experimental and theoretical results. Results from a number of experiments[6,7,8] are presently under consideration.

(2) Some less complex but very interesting problems, such as a beam with a follower force at one end and a Timoshenko beam, are being treated with the variational approach and the Galerkin method. Since there are for the most part exact solutions to these problems, it is expected that some enlightening and very general results will be obtained to be used as guidelines in the application of the variational approach and the Galerkin method to panel-flutter problems in general.

(3) Although the plate equations presented in this report are expected to be entirely adequate for the problems that will be considered, some further thought will be given to their development. There are very general theories for plates and shells, such as that of Koiter[9], but none of them appear to be suitable for problems of interest in this study. On the other hand, the possibility of using the plate thickness as a small parameter in an expansion scheme has been explored by a number of authors (Eringen[10], for example), and it appears that such a scheme would be most useful in providing a rational analytic means of assessing the relative

import of such effects as rotatory inertia and transverse shear deformations along with the geometric nonlinearities that arise when finite rotations are considered.

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