CH

Chaos from Nonlinear Mappings

1 Flows vs. Maps

A system of ordinary differential equations can often be written in the form [1, 2]

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_i) \\ \frac{dx_2}{dt} &= f_2(x_i) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_i), \end{aligned} \tag{1}$$

where t is the independent variable (normally time) and

$$x_i \equiv x_1, x_2, \ldots, x_n. \tag{2}$$

Equation (1) consists of n first-order differential equations. If the original set of equations is of higher-order, they can often be transformed to Eq. (1) by redefining variables and expanding their number. (It may be recalled that in classical mechanics the m second-order differential equations arising from a Lagrangian formulation are equivalent to the $n = 2m$ first-order equations from the Hamiltonian formulation.)

The system of equations in (1) is an example of a *flow*, which gives rise to a *continuous* evolution of "field lines" in the n-dimensional space (sometimes called the "phase space")

[2]. We say that the flow is *conservative* if the volume in phase space remains constant with time. If the volume decreases with time, there is a damping effect in the dynamics, and we refer to the resulting flow as *dissipative*. (A frictional force in a mechanical system gives rise to dissipation, as does a resistor in an electrical system.)

Also, it is sometimes convenient to write Eq. (1) in column vector form

$$\frac{d}{dt}X(t) = F(X(t)) \tag{3}$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}; \qquad F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}. \tag{4}$$

Chaotic effects can arise from Eq. (1) if two conditions are satisfied. First, at least one of the functions f_i must contain a *nonlinear* term (e.g., x_1^2, $x_1^2 x_2$, $x_1 x_2^3$). Second, there is a theorem [1] which states that chaos only occurs in Eq. (1) if $n \geq 3$. For example, the famous Lorenz equations are given by [1, 2]

$$\begin{aligned} \frac{dx_1}{dt} &= \sigma(x_2 - x_1) \\ \frac{dx_2}{dt} &= r x_1 - x_2 - x_1 x_3 \\ \frac{dx_3}{dt} &= x_1 x_2 - b x_3, \end{aligned} \tag{5}$$

where σ, r, and b are all constants. Note the nonlinear terms $x_1 x_3$ and $x_1 x_2$ in the second and third equations, respectively, of Eq. (5). For $\sigma = 10$, $b = 8/3$, and $r = 28$, one finds that the system of equations in (5) gives rise to the so-called Lorenz attractor [1, 2] (a "strange attractor"), which in 1963 was historically the first evidence of chaos in a dissipative system. (Shortly before 1900, Henri Poincaré discovered chaos in a conservative system.)

On the other hand, dynamical systems are often modelled as *maps*, e.g., [2, 3]

$$X_{N+1} = G(X_N), \quad N = 1, 2, \ldots \tag{6}$$

where we define the column vectors

$$X_N = \begin{pmatrix} (x_1)_N \\ (x_2)_N \\ \vdots \\ (x_n)_N \end{pmatrix}; \qquad G(X_N) = \begin{pmatrix} g_1(X_N) \\ g_2(X_N) \\ \vdots \\ g_n(X_N) \end{pmatrix}. \tag{7}$$

The symbol N labels the N^{th} *iteration* of the map. Comparing Eqs. (3), (4), (6), and (7), we see that the iteration index N can, in some cases, be considered a label for the time (taken in discrete "jumps").

Indeed, a map can be generated from a flow by taking:

$$X(t), X(t+\tau), X(t+2\tau), \ldots \quad \longrightarrow \quad X_0, X_1, X_2, \ldots \tag{8}$$

where τ is the "time delay" [2]. In the right hand part of Eq. (8) we indicate the iterations corresponding to the discrete flow times in the left hand side. In other examples, maps can arise from flows by taking various types of Poincaré sections, in which one or more variables are fixed [2]. However, in many cases maps have been constructed as pure dynamical systems, without recourse to an underlying system of flows. Also, just as with flows, one can determine whether the map is conservative or dissipative [2, 3].

As before, in order to obtain chaos for a map the equations (6) must contain at least one nonlinear term. However, *for a map chaos can arise even for a one-dimensional system* [3], which should be contrasted with the corresponding criterion for a flow. This is a very important point since, in order to study chaotic effects one need only consider one- or two-dimensional maps, whereas for a flow at least three dimensions is mandatory. Moreover, for a given dimensionality n, the map equations (6) are much easier to use than the flow equations (3). For a mapping, we study the dynamics by simply iterating, whereas for a flow we must solve a system of differential equations, which in general is nontrivial. Thus, for the remainder of this chapter we will focus on chaos in several simple one- and two-dimensional maps.

2 Simple Maps

One of the truly amazing aspects about chaotic behavior is that it can arise from very *simple systems*. We have already mentioned the strange attractor generated from the three-dimensional Lorenz equations (5). In this section we will study a variety of one- and two-dimensional maps from which one can obtain chaos.

Exercise 1 *The logistic map.*

This famous mapping from biology was first studied by R. May and describes the change in an animal population from one year to the next. The basic equation is the following simple one-dimensional map [5]

$$x_{n+1} = g(x_n) = a x_n (1 - x_n), \tag{9}$$

where x_n is the population for the n^{th} year and a is a constant. The first term in the above equation gives unrestricted Malthusian growth, while the second term, from Verhulst, models factors which limit the population (competition for food, predators, etc.). Note that x_n has been suitably scaled so that it always lies between 0 and 1. Write a script for following the iterations of the map for each value of a. You will also need scripts for displaying a table of values and for plotting x_n as a function of n. We want to study the "steady-state" behavior of this system for moderately large n values. That is, for each a value, examine what happens to the x_n value as n is increased. As n gets large, does x_n approach a single limit, or are there two (or more!) limiting values?

First, study the behavior for the following values of a:

$$a = 0.5, \qquad a = 1.50, \qquad a = 2.50. \tag{10}$$

You see, for the first case, that the population cannot sustain itself. Are the results sensitive to the choice of the initial x value? How many iterations are required to give reasonable convergence? What is the (subtle) difference between the second and third cases? The behavior so far is for a single limiting value as n increases indefinitely.

For determining the final, steady-state behavior, it is sometimes interesting to compare several initial values. However, for *verification* of the precise steady state, you should always use an initial x_1 of $1/2$. This is because $x = 1/2$ is a "critical value" of the map, or a point at which $g'(x) = 0$. An orbit arising from a critical value must lie in the "basin of attraction" [2] for an *attracting periodic point* [4] (provided that it exists).

Then, a definite change in behavior occurs for the following a values:

$$a = 3.22 \quad a = 3.50 \quad a = 3.55 \quad a = 3.5666. \tag{11}$$

(The very adventurous may wish to examine $a = 3.5692255$.) You are now looking at the famous *period-doubling cascade to chaos*. Do you see why the period of the population doubles for each new case? Each of the above a values is in the *center* of a period-multiplying region; how would you characterize each region? The *boundaries* of these regions are specified in Figure 1. Again, how many iterations are required, and is there "sensitivity to initial conditions"? Check the convergence if you perform the calculation closer to a boundary (where a "bifurcation" takes place); the behavior here should make it clear what occurs at the bifurcation points. (Note: close to the boundary, it is especially important to use $x_1 = 1/2$.)

Next, consider the case:

$$a = 3.65, \tag{12}$$

which is in the *chaotic* region. How does this case compare with the previous ones and what can you say about chaos here? Notice, however, that the points are distributed only within certain regions. Also, what can you say about sensitivity to initial conditions? Within the chaotic region, there are very short "windows" of regular behavior (periodicity) which open up. The largest of these, a period-tripling region, begins at $a = 3.828427$ and ends at $a \sim 3.855$. Thus, examine the cases:

$$a = 3.838 \qquad a = 3.845. \tag{13}$$

How would you characterize the behavior for these cases and how are they related? You might want to see if you can find the very small regular, period = 5 window for $a \sim 3.74$. Finally, consider the two cases:

$$a = 3.95 \qquad a = 4.0. \tag{14}$$

The Logistic Map

$$x_{n+1} = ax_n(1 - x_n)$$

There are major transitions at the following values of a:

1.0—below this value the population cannot survive

2.0—oscilllatory approach to the asymptotic value

3.0—"period" of the population doubles (beginning of the period-doubling)

3.449499—period quadruples

3.544090—period of 8

3.564407—period of 16

3.568759—period of 32

3.569692—period of 64

...

...

3.569946—period doubling region ends and chaos begins

$\sim$ 3.74—very small period = 5 window opens up

3.828427—small period tripling window opens up

$\sim$ 3.855—period tripling cascade ends and chaos resumes

$\sim$ 4.0—chaos reigns!!

Figure 1: Major bifurcation points and chaotic boundaries for the logistic map.

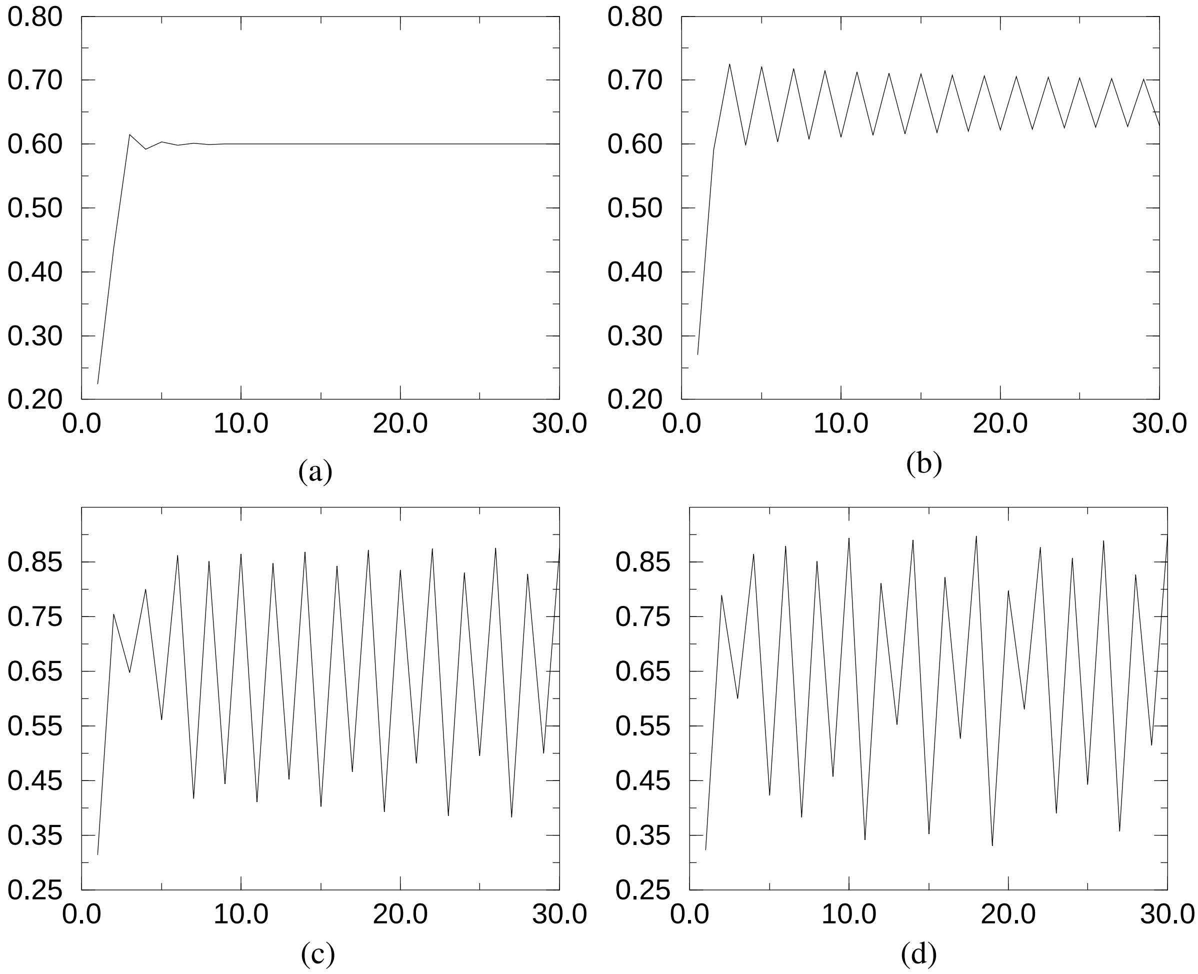

Figure 2: Population (x_n) vs. time (n) for the logistic map. The top-left picture is for $a < 3.0$, while the top-right and bottom-left pictures are in the period doubling and quadrupling regions, respectively. The bottom-right picture is for an a value which gives chaotic behavior.

The last case is for the largest a value allowed; do you see why? How does the distribution of points differ now from the case $a = 3.65$ (near the *beginning* of the chaotic region)? By considering two initial points very close together (e.g., $x_1 = 0.34$ and $x_1 = 0.340002$, see if the "Butterfly effect" manifests itself. (The Butterfly effect is sensitivity to initial conditions.) If the two initial points are brought even closer together, how many iterations are required to see differences in the final patterns? Note: for $a = 4.0$, do *not* use $x_1 = 1/2$. (For $a = 4.0$, $1/2 \to 1 \to 0$, and 0 is an *unstable* fixed point.)

The results for all of these studies are summarized in Figure 1–3 [5]. Figure 2 shows typical plots of x_n vs. n for various values of a, while Figure 3 gives a plot of the final x value (or values!) for a wide range of the parameter a.

Exercise 2 *The Hénon attractor.*

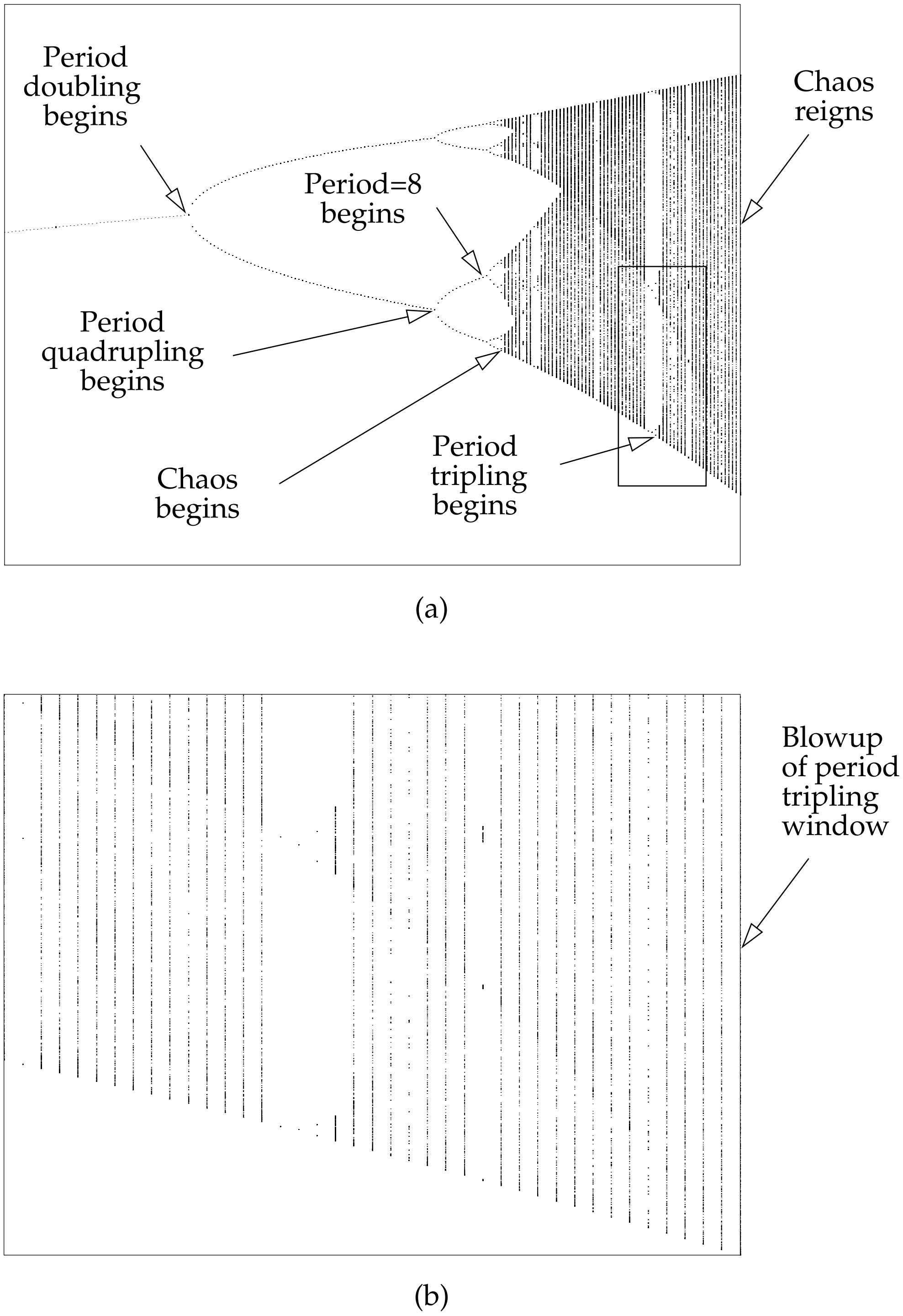

Figure 3: Bifurcation diagram for the logistic map.

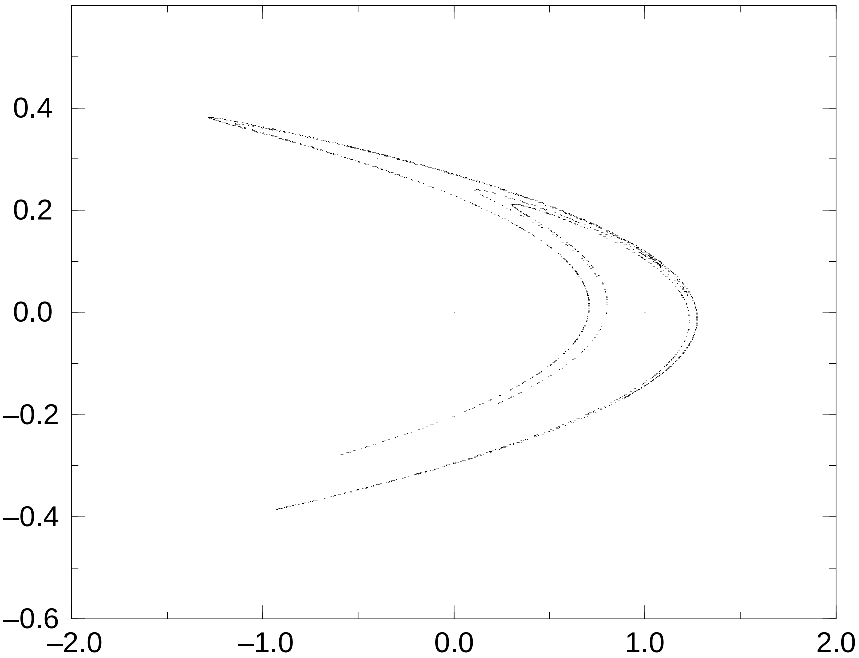

Figure 4: x–y plot of the Hénon attractor.

This two-dimensional map is given by the equations [1, 2]

$$\begin{aligned} x_{n+1} &= y_n + 1 - \alpha x_n^2 \\ y_{n+1} &= \beta x_n, \end{aligned} \tag{15}$$

where $\alpha = 1.4$ and $\beta = 0.3$. Note the nonlinear term in the first equation. Write scripts for iterating with these equations and for making an x–y plot of all of the points generated. For sensitivity to initial conditions (the Butterfly effect), consider the following two cases:

$$\textit{(i)}\ x_1 = 0 \qquad y_1 = 0 \tag{16}$$

and

$$\textit{(ii)}\ x_1 = 10^{-10} \qquad y_1 = -10^{-10}. \tag{17}$$

and compare the results after 10, 20, 30, 40, and 50 iterations. Next, perform a large number of iterations (400 or more) and plot the resulting object. You can see that the general form of the attractor does not depend on the initial x and y values, even though the Butterfly effect is very obvious. What does this exercise tell us about strange attractors? Examine the behavior of the system for other choices of the constants α and β. Are there values of α and β for which the attractor disappears?

In Figure 4 we present a picture of the Hénon attractor for a fairly large number of points (~ 2000). If one could increase the number of points by one or two orders of magnitude, the self-similarity inherent in the attractor would manifest itself, as we show in Figure 5.

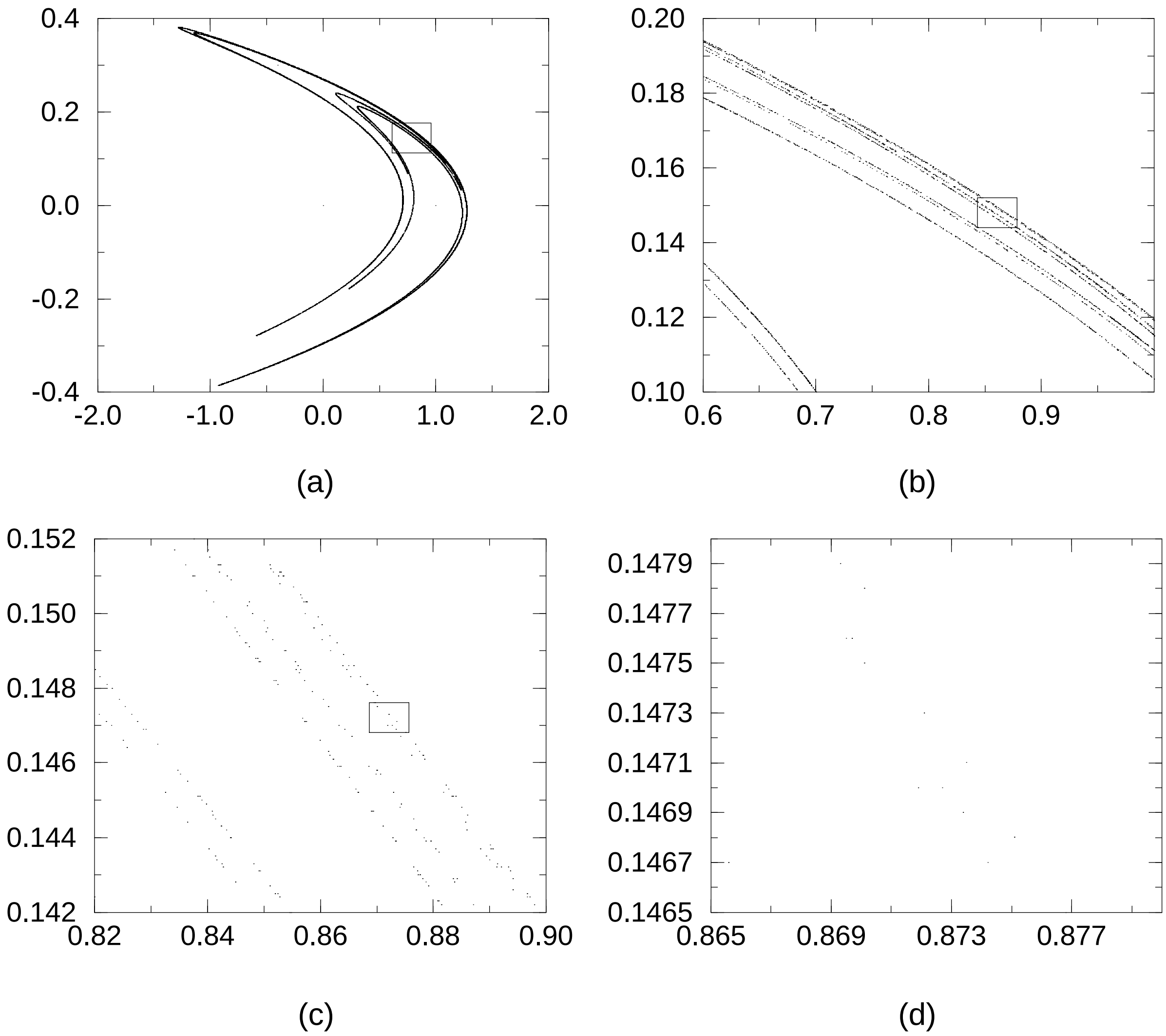

Figure 5: Pictures illustrating self similarity for the Hénon attractor [2].

Exercise 3 *Chaos esthétique.*

Consider the following two-dimensional mapping [6], by Gumolski and Mira, for modeling the dynamics of a particle accelerator:

$$\begin{aligned} x_{n+1} &= y_n + f(x_n) \\ y_{n+1} &= -bx_n + f(x_{n+1}), \end{aligned} \tag{18}$$

where a and b are constants and

$$f(x) = ax + \left[\frac{2(1-a)x^2}{1+x^2}\right]. \tag{19}$$

Note that the argument of f on the right-hand-side of the second equation above depends on x_{n+1}. Write a script for iterating with these equations. You will need to write a function program to calculate $f(x)$ and a script for displaying a graph of all iterations of the equations. In order to obtain striking pictures, you should try to perform at least 3000 iterations. It is important to *watch* the pattern of the points when they fill in on the graph as the iterations proceed.

This mapping nicely illustrates the difference between conservative and dissipative dynamics. There are many interesting cases to consider, e.g.,

$$\begin{aligned} (i)\ a &= 0.31 \qquad b = 1.0 \\ x_1 &= 12.0 \qquad y_1 = 0, \end{aligned}$$

and you can see from the way that the points fill in that the mapping is conservative. (Areas are conserved and there is no basin of attraction.) Figure 6 shows the results of this mapping for 10,000 points. We emphasize that this is a *single* orbit. A case showing dissipation is given by

$$\begin{aligned} (ii)\ a &= 0.7 \qquad b = 0.9998 \\ x_1 &= 15.0 \qquad y_1 = 0, \end{aligned}$$

and, as the iterations proceed, the orbit spirals inward, collecting around the eight oval-shaped objects surrounding the center. Figure 7 shows the mapping of this chaotic orbit for 10,000 points. Check sensitivity to initial conditions. Also, consider the orbits resulting from other combinations of a and b.

Exercise 4 *The Standard Map.*

This famous two-dimensional map, widely used to model accelerator dynamics, is given by

$$\begin{aligned} q_{n+1} &= q_n + p_{n+1} \qquad (\mathrm{mod}\ q = 1) \\ p_{n+1} &= p_n + \left(\frac{k}{2\pi}\right)\sin(2\pi q_n) \qquad (\mathrm{mod}\ p = 1). \end{aligned} \tag{20}$$

Notice in Eq. (20) that the restrictions (to mod $q = 1$ and mod $p = 1$) keep the values of q and p between 0 and 1. Write scripts to iterate this map and to plot it. For each k-value, you should generate a plot using several starting points, each of which lies in a different part of the unit $(p - q)$ square. For small values of k there is no chaos and for values of k above ~ 4 chaos reigns. Study the behavior of this map for different k values. The k region between .9 and 1.1 is thought to be especially important. In particular, the onset of widespread chaotic behavior is believed to occur at $k \sim .9716$ [7, 8]. In Figure 8 we show a plot of the standard map for $k = 1$, obtained purely from relatively low-order periodic orbits [8].

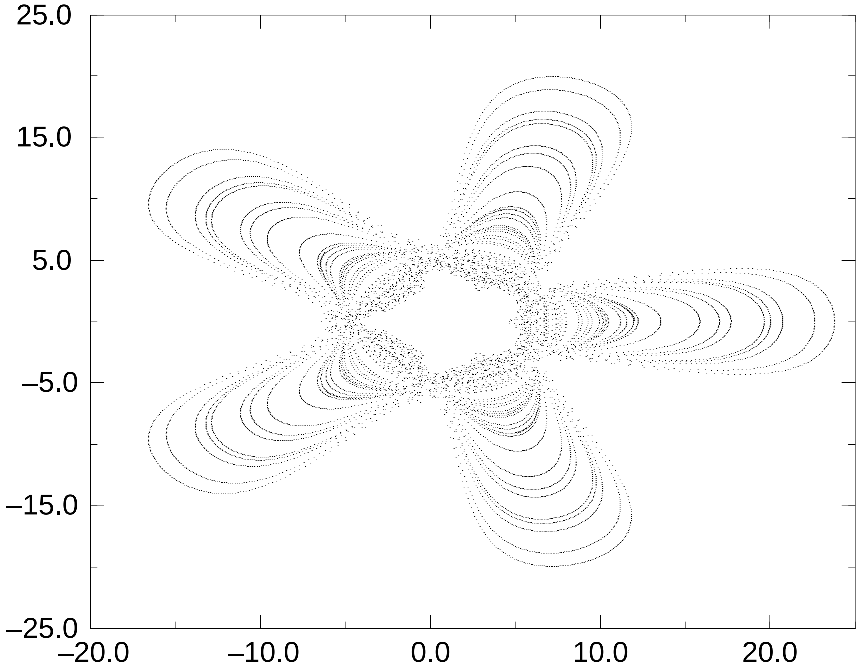

Figure 6: Conservative mapping [6] for Eqs. (18) and (19) ($a = 0.31$, $b = 1.0$). We plot 10,000 iterations.

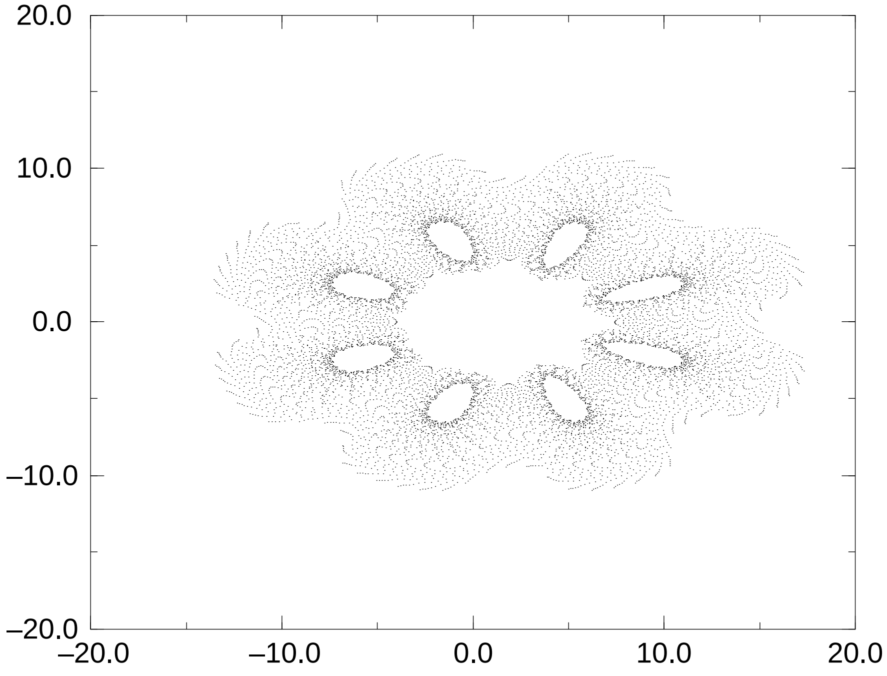

Figure 7: Dissipative mapping [6] for Eqs. (18) and (19) ($a = 0.7$, $b = 15.0$). We plot 10,000 iterations.

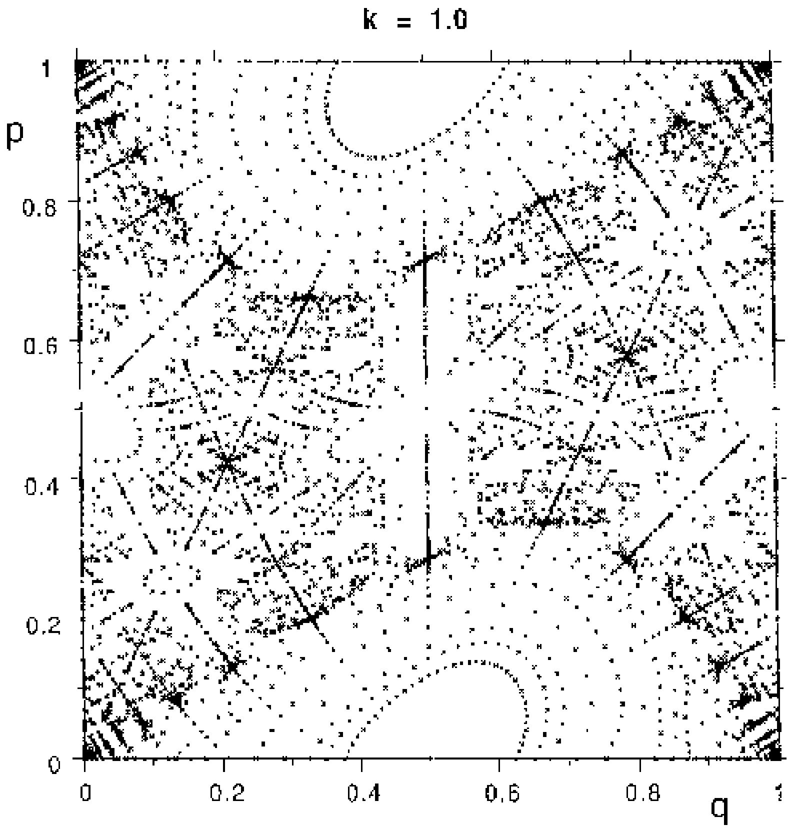

Figure 8: The standard map for $k = 1.0$, obtained entirely from periodic orbits (whose periods are ≤ 20). Notice the regular regions at the top and bottom (for $q = 0.5$, $p = 0$ and 1.0), at $p = q = .5$, $p = .5$ and $q = 0$ and 1.0, and at various other smaller regions. The "jumble" of points in other regions is indicative of chaotic behavior.

References

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