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Slender Vortex Filament with Slowly Varying Core Structure

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ABSTRACT We give a brief review of the asymptotic theory of slender vortex filaments with emphases on the choices of scalings characterizing the physical problems and the corresponding assumptions and/or restrictions introduced in the formation of the asymptotic theory of Callegari and Ting (1978) and its extension by Klein and Ting (1992). In particular, the slender filaments considered are assumed to be forming loops or tori. Because of this restriction, the theory is not applicable to the trailing vortex system of a rotorcraft. We describe the multiple length scales characterizing the vortex system, formulate the expansion scheme, derive the governing equations and then identify the assumptions or restrictions inherent in the multi-scale analysis and needed for the validity of the asymptotic theory of the trailing vortex system.

1. Introduction

It is well known that the velocity of a filament depends on the vorticity distribution in the core and the velocity is undefined when the filament is modeled by a vortex line of zero core radius [1]. Consider a slender filament, with centerline $\mathcal{C} : \mathbf{x} = \mathbf{X}(t, s)$, characterized by a parameter s. Here we choose s to be the arc length at $t = 0$. The velocity $\mathbf{Q}$ induced by a vortex line $\mathcal{C}$ is given by the Biot-Savart formula

$$\mathbf{Q}(t, \mathbf{x}) = [\Gamma/(4\pi)] \int_C [\mathbf{X}' - \mathbf{x}] \times d\mathbf{X}'/|\mathbf{X}' - \mathbf{x}|^3. \tag{1.1}$$

For a point $\mathbf{x}$ in the neighborhood $\mathcal{C}$, i. e., in the normal plane of point $\mathbf{X}(t, s)$ on $\mathcal{C}$, we write $\mathbf{x} = \mathbf{X} + r\hat{r}$, where $\hat{r} = \hat{n}\cos\phi + \hat{b}\sin\phi$, r and ϕ are the polar coordinates and $\hat{\tau}, \hat{n}$ and $\hat{b}$ are the unit tangent, normal and binormal vectors of $\mathcal{C}$. In approaching the vortex line $\mathcal{C}$, i. e., $r = |\mathbf{x} - \mathbf{X}| \to 0$, the formula gives three singular terms, [2] - [6],

$$\mathbf{Q} = [\Gamma/(2\pi r)]\hat{\theta} + [\Gamma\kappa/(4\pi)]\ln[S/r]\hat{b} + [\Gamma\kappa/(4\pi)]\hat{\theta}\cos\phi + \mathbf{Q}^f \ , \tag{1.2}$$

where $\mathbf{Q}^f$ denotes the remainder of the Biot-Savart integral, and $\hat{\theta} = \hat{\tau} \times \hat{r}$ denotes the unit circumferential vector. The first term with $1/r$ singularity corresponds to the circumferential velocity of a 2-D vortex point in the normal plane. The second term, the binormal velocity with $\ln r$ singularity, represents the curvature effect. The third term, a circumferential velocity depending on ϕ, does not have a limit as $r \to 0$. After subtracting these three singular terms from $\mathbf{Q}$, the remainder $\mathbf{Q}^f$ has a limit, known as the finite part of the integral. These singular behaviors are not valid for a real fluid because with viscosity the flow field has to have a continuous velocity gradient.

The B-S formula (1.1) for the vortex line $\mathcal{C}$ is valid in the "outer region", i. e., away from the filament. of a normal length scale ℓ. Near the filament, of a typical core size δ, the singular terms on the centerline $\mathcal{C}$ in (1.2) shall be matched with or removed by the *inner solution* for the core structure using the method of matched asymptotics. The asymptotic analyses with the small expansion parameter, slenderness ratio,

$$\epsilon = \delta \ / \ \ell \ , \tag{1.3}$$

were carried out in a sequence of papers [2, 3, 4, 5] from the 2-D cases in 1965 [2] to filaments in 3-D space with large swirling and axial flow in the core in 1978 by Callegari and Ting (C-T), [5]. The analyses were summarized by Ting and Klein in 1990. [6].

In the analyses of C-T for a slender filament we assume that:

(i) The filament forms a slender torus, of length $S(t) = O(\ell)$, so that all the physical entities in the inner region around the centerline $\mathcal{C}$ and the geometry of $\mathcal{C}$, are periodic function of its arc length $\bar{s}$ with period S, or its initial arc length s with period S_0, i. e.,

$$\mathbf{X}(t, s + S_0) = \mathbf{X}(t, s), \quad \text{and} \quad f(t, r, \theta, s + S_0) = f(t, r, \theta, s) \ , \tag{1.4}$$

where (r, θ, s) denote the curvilinear coordinates with respect to $\mathcal{C}$ and f denotes an entity near $\mathcal{C}$, i. e, the velocity components and the pressure.

(ii) There is only one typical length scale ℓ other than the typical core size δ. This implies that the radius of curvature, $1/\kappa$, the length of the filament, and the length scale of the background flow λ, are of the order ℓ,

$$1/\kappa = O(\ell), \quad S = O(\ell) \quad \text{and} \quad \lambda = O(\ell). \tag{1.5}$$

Here, the distance between two adjacent filaments is a length scale for the background flow to be $O(\lambda)$. These two typical length scales define the slenderness ratio,

$$\epsilon = \delta/\ell. \tag{1.6}$$

(iii) There is one typical velocity scale U for the background flow and the velocity of a filament centerline is of the order of U, i. e.,

$$\dot{\mathbf{X}}(t,s)\cdot\hat{n} = O(U) \quad \text{and} \quad \dot{\mathbf{X}}(t,s)\cdot\hat{b} = O(U) \quad \text{with} \quad \dot{\mathbf{X}}\cdot\hat{\tau} = 0, \tag{1.7}$$

where $\hat{\tau}$, $\hat{n}$ and $\hat{b}$ denote the unit tangential, principal normal and binormal vectors of $\mathcal{C}$.

(iv) The core structure does not vary along the centerline of the filament, i. e., independent of the axial variable s.

Since a real fluid is viscous, our problem is characterized by the physical parameter, the background Reynolds number, $R_e = U\ell/\nu$, where ν denotes the kinematic viscosity.† We consider $R_e \gg 1$ so that the viscous effect in the background flow is of higher order while in the core structure the viscous effect is important when the core size is very small initially and/or the effect is accumulated over a long time. We choose the "distinguished limit",

$$1/\sqrt{R_e} = O(\epsilon) \ \text{ or } \ 1/\sqrt{R_e} \leq K\epsilon\ , \tag{1.8}$$

so that the equations for the evolution of the core structure retain the nonlinear convection (inviscid stretching) and viscous diffusion terms. The constant K represents a typical ratio of viscous to inviscid effect. The inviscid limit is $K \to 0$.

In 1992, Klein and Ting, [7], extended C-T theory to allow for axial core structure variation, thus removing assumption (iv) while keeping (i), (ii) and (iii). They arrived at a system of integro-differential equations much more complex than that of C-T.

Because of assumptions (i) and (ii), the analyses of C-T and K-T are not applicable to the vortex system trailing rotor blades, for which the filaments do not form tori with length of the order of the radius of curvature. The filaments have at least three distinct length scales in addition to the core size δ. They are the typical distance between two filaments, d, the radius of curvature ℓ, and the length scale L for the axial variation of core structure and size. It is clear that the scale d is of the order of the pitch of a filament divided by the number of blades. The slender filaments reach a typical length L when they merge, i. e., when the core size becomes comparable to the distance between two filaments. We have

$$L \gg \ell, \ d \gg \delta \quad \text{while } d \ll \ell \text{ or } d = O(\ell), \tag{1.9}$$

We consider $d = O(\ell)$ and hence has only one more small length ratio

$$\ell/L = \varsigma, \tag{1.10}$$

in addition to the slenderness ratio, ϵ in (1.6).

To explain why or where the existing theory is not applicable to the rotorcraft vortex system, we review in §2. the formulation of C-T theory [5] based on assumptions (i) to (iv). We show the expansion schemes and the two systems of equations defining the motion of the filament centerline and the evolution of the core

† When the vortical core is turbulent, ν shall be replaced by an eddy viscosity ν_d, resulting in a relatively smaller R_e.

structures respectively and then the coupling of these two systems and the necessity of assumption (i) and (ii) for the closure of the second system.

In §3, we introduce the expansion schemes for filaments having three length scales δ, ℓ and L, removing assumption (i), modifying (ii) and (iii), while keeping (iv). We describe the multi-scale analysis yielding two new systems of governing equations and then discuss the conditions on the filament centerlines needed for the multi-scale analysis.

2. Filament in the form of a slender torus of length $O(\ell)$.

We review the C-T theory [5] for slender filaments submerged in an incompressible irrotational flow. The background flow in absence of the filaments is a potential flow, fulfilling the Navier Stokes (N-S) Equations. We consider the strength of a filament, the circulation Γ, to be $O(U\ell)$. Let ℓ and U be the unit length and velocity, i. e., $\ell = 1$ and $U = 1$, and hence $\ell/U = 1$ is the unit time. With the core size $\delta = O(\epsilon)$, we have the order of magnitude of the vorticity Ω and that of the velocity $\mathbf{v}$ in the vortical core:

$$|\Omega| = O(\epsilon^{-2}) \qquad \text{and} \quad |\mathbf{v}| = O(\epsilon^{-1}) \ . \tag{2.1}$$

To construct the inner solution, i. e., the core structure near a point $\mathbf{X}$ on the centerline, we introduce the intrinsic coordinates, r, ϕ, s and then replace the polar angle ϕ in the normal plane by the angle

$$\theta = \phi - \theta_0(t,s) \ , \qquad \text{with} \quad \partial\theta_0/\partial s = -\sigma T, \quad \sigma = |\mathbf{X}_s|, \tag{2.2}$$

so that the new coordinates, r, θ, s, are orthogonal with

$$d\mathbf{x} = \hat{r}\ dr + \hat{\theta}\ rd\theta + \hat{\tau}\ h_3 ds \quad \text{where} \quad h_3 = \sigma\lambda, \quad \lambda = 1 - r\kappa\cos\phi \ . \tag{2.3}$$

The rotation $\theta_0(t,s)$ from ϕ to θ accounts for the torsion $T(t,s)$ of $\mathcal{C}$. See [4].

For the inner region, i. e., $r = O(\epsilon)$, we introduce the stretched radial variable

$$\bar{r} = r/\epsilon \quad \text{and assume} \quad \mathbf{X}_t = O(1) \quad \text{with} \quad \mathbf{X}_t \cdot \hat{\tau} = 0. \tag{2.4}$$

With the assumption (iii), we introduce the velocity $\mathbf{V}(t,\bar{r},\theta,s)$ relative to $\mathcal{C}$ and write

$$\mathbf{v} = \mathbf{X}_t + \mathbf{V} \quad \text{with,} \quad \mathbf{V} = u\hat{r} + v\hat{\theta} + w\hat{\tau} = O(\epsilon^{-1}) \quad \text{and} \quad u = v = 0 \quad \text{at } \bar{r} = 0. \tag{2.5}$$

The continuity and momentum equations in the moving curvilinear coordinates are

$$(r\lambda u)_r + (\lambda v)_\theta + (w_s + \mathbf{X}_{st}\cdot\hat{\tau})(r/\sigma) = 0, \tag{2.6}$$

$$d\mathbf{V}/dt + [\frac{w - r\hat{r}_t\cdot\hat{\tau}}{\sigma\lambda}]\mathbf{X}_t + \mathbf{X}_{tt} = -\nabla p + \nu\Delta\mathbf{V} + \frac{\nu}{h_3}[\frac{1}{h_3}\mathbf{X}_s]_s \tag{2.7}$$

To characterize a slender vortex filament, or to be consistent with (2.1), we need a large circumferential velocity, $v = O(\epsilon^{-1})$, and allow for a large axial flow w, but the radial component u remains $O(1)$. Note that the inner solutions are periodic in θ with period 2π and in s with period S_0 on account of (i). Using this physical model, we arrive at the expansion scheme,

$$f(t,\bar{r},\theta,s,\epsilon) = \epsilon^{-m}\{f^{(0)}(t,\bar{r},\theta,s) + \epsilon f^{(1)}(t,\bar{r},\theta,s) + O(\epsilon^2)\} \ . \tag{2.8}$$

We consider $\ln(1/\epsilon)$ to be $O(1)$ with respect to ϵ, therefore, $f^{(j)}, j = 0, \cdots$, could be functions of $\ln(1/\epsilon)$. If f stands for $\mathbf{v}$, we have $m = 1$, with $u^{(0)} \equiv 0$. If f stands for p, we obtain $m = 2$ from the radial momentum equation. We substitute the power series (2.8) into the N-S equations, equate the coefficients of like powers of ϵ, and obtain systematically the sets of leading and higher order equations.

With $u^{(0)} \equiv 0$, we obtain from the N-S equations, (2.6) and (2.7), the leading order continuity equation and axial and radial momentum equations,

$$v_\theta^{(0)} = 0, \quad w_\theta^{(0)} = 0 \quad \text{and} \quad p_{\bar{r}}^{(0)} = [v^{(0)}]^2/\bar{r} \,. \tag{2.9}$$

Then the circumferential momentum equation becomes an identity. The solution, independent of θ, represents a quasi steady two-dimensional axi-symmetric flow in the normal plane of $\mathcal{C}$.

Note that the dependence of the core structure, $v^{(0)}$ and $w^{(0)}$, on $t, \bar{r}$, and s is yet to be defined later by the compatibility conditions of the second order equations [5, 6, 7].

The next order (the first order) system of equations is

$$v_\theta^{(1)} + (\bar{r}u^{(1)})_{\bar{r}} + \bar{r}w^{(0)}/\sigma + \bar{r}v^{(0)}\kappa\sin\phi = 0, \tag{2.10}$$

$$u^{(1)}w_{\bar{r}}^{(0)} + w_\theta^{(1)}v^{(0)}/\bar{r} + w^{(0)}v^{(0)}\kappa\sin\phi = -(p_s^{(0)} + w^{(0)}w_s^{(0)})/\sigma, \tag{2.11}$$

$$v^{(0)}v_\theta^{(1)} + u^{(1)}(\bar{r}v^{(0)})_{\bar{r}} - (w^{(0)})^2\bar{r}\kappa\sin\phi = -p_\theta^{(1)} - \bar{r}w^{(0)}v_s^{(0)}/\sigma, \tag{2.12}$$

$$v^{(0)}u_\theta^{(1)} - 2v^{(0)}v^{(1)} + (w^{(0)})^2\bar{r}\kappa\cos\phi = -\bar{r}p_{\bar{r}}^{(1)}. \tag{2.13}$$

They are linear equations for the first order solutions, $u^{(1)}, v^{(1)}$ etc, with inhomogeneous terms nonlinear in the leading order solutions. We can decompose the first order solutions and their governing equations to their axi-symmetric and asymmetric parts in θ. For example,

$$f(\theta) = \langle f \rangle + f_a(\theta) \quad \text{with} \quad \langle f \rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta f(\theta) \quad \text{and} \quad \langle f_a(\theta) \rangle = 0 \,. \tag{2.14}$$

Here $\langle f \rangle$ denotes the θ-average of f and f_a denotes the asymmetric part of f. Likewise, we can do the decompositions for the higher order solutions and their governing equations. Because of the distinguished limit (1.8) and our assumption of only one time scale ℓ/U, the viscous terms and the time derivatives do not appear in the first order equations (but will appear in the second order equations).

The θ-averages of those first order equations yield a system of quasi-steady inviscid equations for the axi-symmetric parts of the solutions. Consequently, the core structure for $t \geq 0$ has to fulfill two be consistency conditions identified with two classical relationships for steady inviscid flows. See [7]. Early on, C-T [5] found a special case fulfilling both relationships. It is:

$$v_s^{(0)} = 0, \quad w_s^{(0)} = 0 \quad \text{and} \quad p_s^{(0)} = 0 \quad \text{if any one of them holds.}$$

The C-T theory is based on the restriction that

$$v_s^{(0)} = 0, \quad \text{hence} \quad w_s^{(0)} = 0 \quad \text{and} \quad p_s^{(0)} = 0. \tag{2.15}$$

Thus the leading order solutions, $v^{(0)}, w^{(0)}$ and $p^{(0)}$ are functions of t and $\bar{r}$ only.

By matching the asymmetric inner solution of the core structure as $\bar{r} \to \infty$ with the outer solution for $r \to 0$, *the second and the third singular terms* in the Biot-Savart integral (1.2) are removed and the velocity of the centerline, $\mathbf{X}_t$, is defined by

$$\mathbf{X}_t(t,s) = \hat{n}[\mathbf{Q}_0 \cdot \hat{n}] + \hat{b}[\mathbf{Q}_0 \cdot \hat{b}] + \hat{b}[\Gamma\kappa/(4\pi)][\ln(1/\epsilon) + C_v + C_w] \,, \tag{2.16}$$

where $\mathbf{Q}_0(t, \mathbf{X}) = \nabla\Phi + \mathbf{Q}^f$ denotes the background velocity without the filament plus the finite part of $\mathbf{Q}$ in (1.2). $C_v(t,s)$ and $C_w(t,s)$ denote respectively the contributions of the circumferential and axial velocity in the core to $\mathbf{X}_t$. They are,

$$C_v = \lim_{\bar{r}\to\infty}\left(\frac{4\pi^2}{\Gamma^2}\int_0^{\bar{r}} \bar{r}'v^2\,d\bar{r}' - \ln\bar{r}\right) + \frac{1}{2}\,, \quad C_w = \frac{-8\pi^2}{\Gamma^2}\int_0^\infty \bar{r}'w^2\,d\bar{r}' \,. \tag{2.17}$$

With $\ln(1/\epsilon)$ considered $O(1)$, we treat the terms in (2.15) as $O(1)$ while terms $O(\epsilon)$ have been omitted. Their omission implies that the superscripts (0) for $\mathbf{X}$ and its geometrical entities, $\sigma, \kappa, T, \hat{r}$ etc have been suppressed, i. e., we are considering $\mathbf{X}(t, s, \epsilon = 0)$. The terms omitted can come from the higher order core structure and from the effect of nonzero core size not accounted for in the B-S formula (1.1).

The asymptotic theory of C-T requires that the initial core structure should fulfill the consistency conditions (2.15), and the initial velocity of the filament centerline, if assigned, has to agree with (2.16). These restrictions on the initial data are the results of having only one time scale. In case the initial data violate any one of those restrictions, we need multi-time analysis. See [2], [4], [6].

The second order equations involve the first and second order unknowns, $v^{(1)}, v^{(2)}$, etc. To remove those unknowns, we first remove the second order unknowns, $u^{(2)}$ and $v^{(2)}$. by using the periodicity of the solution in the variable θ with period 2π. Carrying out the θ-average of the second order continuity equation and the axial and circumferential components of the second order momentum equations, we arrive at the following equations:

$$\begin{aligned} &\frac{1}{\bar{r}}(\bar{r} < u^{(2)} >)_{\bar{r}} + \frac{1}{\sigma^{(0)}}[< w^{(1)} >_s + \dot{\sigma}^{(0)}], \quad \text{or} \\ &< u^{(2)} > = -\frac{1}{\bar{r}\sigma^{(0)}} \int_0^{\bar{r}} < w^{(1)} >_s \bar{r}' d\bar{r}' - \frac{\bar{r}\dot{\sigma}^{(0)}}{2\sigma^{(0)}}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} w_t^{(0)} - \frac{K^2}{\bar{r}}(\bar{r} w_{\bar{r}}^{(0)})_{\bar{r}} = &- \frac{1}{\sigma^{(0)}}\left(w^{(0)} < w^{(1)} >_s + < p^{(1)} >_s\right) \\ &- \frac{w^{(0)}\dot{\sigma}^{(0)}}{\sigma^{(0)}} - w_{\bar{r}}^{(0)} < u^{(2)} > \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} v_t^{(0)} - K^2\left[\frac{1}{\bar{r}}(\bar{r} v_{\bar{r}}^{(0)})_{\bar{r}} - \frac{1}{\bar{r}^2} v^{(0)}\right] = &- \frac{w^{(0)}}{\sigma^{(0)}} < v^{(1)} >_s \\ &- (\bar{r} v^{(0)})_{\bar{r}} < u^{(2)} > / \bar{r} \ . \end{aligned} \tag{2.20}$$

Here $K^2 = \nu/(U\ell\epsilon^2) = O(1)$ on account of (1.8).

To remove the first order unknowns in (2.18), (2.19) and (2.20), which are the s-derivatives of their θ-averages, we make use of assumption (i), the periodicity of s with period S_0 expressed by (1.4). We use (2.18) to remove $< u^{(2)} >$ and then integrate (2.19) and (2.20) with respect to $\tilde{s}$ over the entile length $S(t)$ or s over S_0 with $d\tilde{s} = \sigma^{(0)} ds$. We arrive at the compatibility conditions of the second order equations, which in turn are the evolution equations of the core structure, $v^{(0)}(t, \bar{r})$ and $w^{(0)}(t, \bar{r})$. They are:

$$w_t^{(0)} = (K^2/\bar{r})[\bar{r} w_{\bar{r}}^{(0)}]_{\bar{r}} + (\bar{r}^3/2)(\dot{S}/S)[w^{(0)}/\bar{r}^2]_{\bar{r}} \ , \tag{2.21}$$

$$v_t^{(0)} = K^2\left[\frac{1}{\bar{r}}(\bar{r} v_{\bar{r}}^{(0)})_{\bar{r}} - \frac{1}{\bar{r}^2} v^{(0)}\right] + (\bar{r} v^{(0)})_{\bar{r}} \frac{\dot{S}}{2S} \ . \tag{2.22}$$

With the axial vorticity $\zeta^{(0)}$ related to $v^{(0)}$ by $\bar{r}\zeta^{(0)} = (\bar{r} v^{(0)})_{\bar{r}}$, (2.22) is then replaced by

$$\zeta_t^{(0)} = (K^2/\bar{r})[\bar{r}\zeta_{\bar{r}}^{(0)}]_{\bar{r}} + (\dot{S}/S)(\bar{r}^2\zeta^{(0)})_{\bar{r}}/(2\bar{r}) \ . \tag{2.23}$$

We use the axial vorticity as a primary variable instead of the circumferential velocity $v^{(0)}$, because $\zeta^{(0)}$ decays exponentially in $\bar{r}$, while $v^{(0)}$ becomes $\Gamma/(2\pi\bar{r})$ as $\bar{r} \to \infty$. Thus the inner solution vo matches or removes the leading singular term of the B-S integral (1.2). Recall that the second and third singular terms were removed by the asymmetric part of the first order solutions, therefore, we have shown that *the leading and first order inner solutions have matched or removed all three singular terms of the outer solution* (1.2).

Equations (2.21) and (2.23) can be considered as the evolution equations for $w^{(0)}(t, s)$ and $\zeta^{(0)}(t, s)$ respectively, but with coefficients depending on the filament length $S(t)$, they are coupled with the equation of motion (2.17) of the centerline, $\mathbf{X}(t, s)$, and hence with each other. The radial integrals of the core structure, C_v and C_w in (2.17) in turn couple the equation of motion with the two evolution equations. of the core structure. Together, (2.17), (2.21) and (2.23), form a close system of equations for the dynamics of filaments fulfilling assumption (i), i. e., in the forms of slender torii. Solutions of this system for a given set of initial data were presented in [5] and [6]. In the next section, we shall explain how to derive a close system of equation for slender filaments without assumption (i).

3. Filament trailing a rotor blade – Filament with multiple length scales in addition to the small core size δ.

The vortex system trailing a helicopter rotor is extremely complex. It can be treated as a steady axisymmetric flow only in hover flight. In a forward flight, the flow is unsteady and asymmetric. Mathematical modeling of the flow field with rotor tip speed nearly transsonic and large forward velocity is beyond reach in the near future. Here we shall describe how to modify the theory of C-T for flows with slender vortex filament [5, 6] to simulate the vortex system trailing a rotor. Note that the theory of C-T allows for unsteady asymmetric flow but incompressible. The last restriction implies that the flow field is at low Mach number.

As indicated in §1, we need to consider the length of a filament much larger than the reference radius of curvature ℓ of the filament and the filament structure varying along its centerline in a length scale, $L \gg \ell$, introduced in (1.10). The assumption (i) for the C-T theory shall be removed and be replaced by that for a two length scale analysis. See for example [8] and [9].

3.1. Multiple length scales and the expansion scheme

We replace the axial variable s by two scaled variables

$$\bar{s} = s/\ell \quad \text{and} \quad \xi = s/L\,. \tag{3.1}$$

A function $f(s)$ is expressed as a function of the two independent scaled variables, $F(\bar{s},\xi)$, with the assumption of the existence of the average of F in $\bar{s}$ over a large interval λ, but $\lambda \ll 1/\varsigma$, i. e.,

$$\begin{aligned} &f(s) = F(\bar{s},\xi), \quad \text{when } \xi = \bar{s}\varsigma, \text{ and} \\ &\lim_{\lambda\to\infty}\{[\int_{\xi/\varsigma-\lambda/2}^{\xi/\varsigma+\lambda/2}\sigma(\bar{s}',\xi)F(\bar{s}',\xi)d\bar{s}']/[\int_{\xi/\varsigma-\lambda/2}^{\xi/\varsigma+\lambda/2}\sigma(\bar{s}',\xi)d\bar{s}']\} = \widehat{F}(\xi)\,. \end{aligned} \tag{3.2}$$

With $\sigma = d\tilde{s}/ds$, where $\tilde{s}$ denotes the arc length of C at instant t, the average of f is the line integral along a segment of C divided by its length, which is much longer than ℓ but much shorter than L. Note that the independent variables $t, \bar{r}$ and θ in f, F and $\widehat{F}$ have been suppressed. From (3.1) and (3.2), we have

$$\ell\,\partial_s f = \partial_{\bar{s}}F + \varsigma\partial_\xi F \quad \text{and} \quad \widehat{\sigma^{-1}\partial_s F} = \widehat{\partial_{\bar{s}}F} = 0\,. \tag{3.3}$$

For a filament f stands for the core structure variables, v, w, ζ, and the centerline position vector $\mathbf{X}$. Instead of introducing new symbols for the corresponding functions of two scaled axial variables, we consider v, w, ζ and $\mathbf{X}$ as functions of the two axial variables. This is equivalent to replace F by f and $\bar{s}$ by s noting $\ell = 1$.

The expansion scheme (2.8) for the inner solution in one axial variable is replaced by:

$$u(t,\bar{r},\theta,s,\xi,\epsilon) = u^{(1)}(t,\bar{r},\theta,s,\xi) + \epsilon u^{(2)} + \cdots, \tag{3.4}$$
$$v(t,\bar{r},\theta,s,\xi,\epsilon) = \epsilon^{-1}v^{(0)}(t,\bar{r},\xi) + v^{(1)}(t,\bar{r},\theta,s,\xi) + \epsilon v^{(2)} + \cdots, \tag{3.5}$$
$$w(t,\bar{r},\theta,s,\xi,\epsilon) = \epsilon^{-1}w^{(0)}(t,\bar{r},\xi) + w^{(1)}(t,\bar{r},\theta,s,\xi) + \epsilon w^{(2)} + \cdots \tag{3.6}$$
$$p(t,\bar{r},\theta,s,\xi,\epsilon) = \epsilon^{-2}p^{(0)}(t,\bar{r},\xi) + \epsilon^{-1}p^{(1)}(t,\bar{r},\theta,s,\xi) + p^{(2)} + \cdots \tag{3.7}$$
$$\zeta(t,\bar{r},\theta,s,\xi,\epsilon) = \epsilon^{-2}\zeta^{(0)}(t,\bar{r},\xi) + \epsilon^{-1}\zeta^{(1)}(t,\bar{r},\theta,s,\xi) + \zeta^{(2)} + \cdots \tag{3.8}$$

and

$$\dot{\mathbf{X}}(t,s,\xi) = \dot{\mathbf{X}}^{(0)}(t,s,\xi) + \epsilon\dot{\mathbf{X}}^{(1)} + \cdots. \tag{3.9}$$
$$\dot{\sigma}(t,s,\xi) = \dot{\mathbf{X}}_s\cdot\tau = \dot{\sigma}^{(0)}(t,s,\xi)\,. \tag{3.10}$$

For the core structure, the leading order solutions have only the slow axial variation in ξ, but both axial variables, s and ξ, can appear in the higher order solutions. For the velocity of the centerline, the leading term $\mathbf{X}^{(}0)$ has to depend on both s and ξ because its radius of curvature is $O(\ell)$. From (3.2), we know that a $\xi-$derivative term will not appear in the system of equations where s-derivative terms first appear.

Thus the leading order and first order systems of equations, (2.9) to (2.17), in §**2** remain valid with the long axial variable ξ appearing as a parameter. The removal of assumption (i), requires special attention to the evaluation of the finite part $\mathbf{Q}_f$ of the B-S integral (1) at point $\mathbf{X}(s)$ for $s > 0$ with t treated as a parameter. We need to assume that

(v) Far downstream, a trailing vortex centerline should be of the shape, e. g., a circular helix, which renders the B-S integral finite as its upper limit tends to infinity.

(vi) Let $\mathbf{X}(0)$ denote the starting point of the tip vortex, trailing from a rotor blade near its tip at distance b from the axis of the rotor. Then we need to model the vortex line for $s < 0$ in order to evaluate the B-S integral. A simple model is having the vortex line bounded or rotating with the blade for $s < 0$ until the root of the blade at distance r_0 from the rotor axis, where $s = -(b - r_0)$ and continue for $s < -(b - r_0)$ along a free "root" vortex line, trailing from the root to downstream as $s \to -\infty$.

This "root" vortex line has to be present so that the circulation vanishes along any a contour around the trailing vortex system. This mathematical model could be realized in a experimental setup for a single rotor blade. In a real problem, the root vortex lines would interact with the body. In the mathematical model, the radius of curvature of the "root" vortex is $O(r_0)$. For a small root radius $r_0 \ll b = O(\ell)$, the contribution of the small radius of curvature of the root vortex to the B-S integral at a point near the tip vortex can be ignored. Thus we can model the "root" vortex by a curve with radius of curvature $o(\ell)$. In a hover flight, the root vortex line can be approximate by a straight line, along the rotor axis. In this special case, the vortex line for $s \leq 0$ is composed of a finite line segment and a semi-infinite one, and explicit formula for the B-S integral for $s \leq 0$ is available. See for example [10].

3.2. System of governing equations for "long" filaments

To study the evolution of the core structure in the long length scale L, we introduce the distinguished limit,

$$\varsigma = O(\epsilon)\ , \quad \text{i. e., the constant } \ G = \varsigma/\epsilon = O(1)\ . \tag{3.11}$$

Then a $Gv_\xi^{0)}$ term shall appear in the second order equations together with a $v_s^{(1)}$ term. The same applies to w and p. With the addition of the ξ-derivative terms to (2.18), (2.19) and (2.20) in §**2**, they become respectively,

$$\begin{aligned} &\frac{1}{\bar r}(\bar r < u^{(2)} >)_{\bar r} + \frac{1}{\sigma^{(0)}}[< w^{(1)} >_s + G w_\xi^{(0)} + \dot\sigma^{(0)}], \quad \text{or} \\ &< u^{(2)} > = -\frac{1}{\bar r \sigma^{(0)}}\int_0^{\bar r} [< w^{(1)} >_s + G w_\xi^{(0)}]\bar r' d\bar r' - \frac{\bar r \dot\sigma^{(0)}}{2\sigma^{(0)}}, \end{aligned} \tag{3.12}$$

$$w_t^{(0)} - \frac{K^2}{\bar r}(\bar r w_{\bar r}^{(0)})_{\bar r} = \frac{w^{(0)}}{\bar r}(\bar r < u^{(2)} >_{\bar r} - \frac{1}{\sigma^{(0)}}[< p^{(1)} >_s + G p_\xi^{(0)}] - w_{\bar r}^{(0)} < u^{(2)} > \tag{3.13}$$

and

$$v_t^{(0)} - K^2\left[\frac{1}{\bar r}(\bar r v_{\bar r}^{(0)})_{\bar r} - \frac{1}{\bar r^2}v^{(0)}\right] = -\frac{w^{(0)}}{\sigma^{(0)}}[< v^{(1)} >_s + G v_\xi^{(0)}] - (\bar r v^{(0)})_{\bar r} < u^{(2)} > /\bar r. \tag{3.14}$$

Note that from (2.9) we have

$$p_\xi^{(0)}(t, \bar r, \xi) = -2\int_{\bar r}^{\infty} v^{(0)} v_\xi^{(0)} d\bar r'/\bar r'\ . \tag{3.15}$$

To remove the first order unknowns $< w^{(1)} >_s$ and $< vv >_s$ in (3.12), (3.13) and (3.14), which are the s-derivatives of their θ-averages, we carry out their s-average in the sense of (3.2) while making use of (3.3), we arrive at the equations for $\widehat{< u^{(2)} >}$, $w^{(0)}$ and $v^{(0)}$ which are functions of $t, \bar r$ and ξ. They are:

$$\frac{1}{\bar r}(\bar r \widehat{< u_{\bar r}^{(2)} >})_{\bar r} + G w_\xi^{(0)} + \frac{\dot S^{(0)}}{S^{(0)}} \quad \text{or} \quad \widehat{< u^{(2)} >} = -\frac{G}{\bar r}\int_0^{\bar r} w_\xi^{(0)} \bar r' d\bar r' - \bar r \dot\Xi, \tag{3.16}$$

$$w_t^{(0)} + G w^{(0)} w_\xi^{(0)} + \widehat{< u^{(2)} >} w_{\bar r}^{(0)} = (K^2/\bar r)[\bar r w_{\bar r}^{(0)}]_{\bar r} - G p_\xi^{(0)} - \bar r w^{(0)}\dot\Xi\ , \tag{3.17}$$

and

$$v_t^{(0)} + Gw^{(0)}v_\xi^{(0)} + \widehat{<u^{(2)}>}(\bar{r}v^{(0)})_{\bar{r}}/\bar{r} = K^2\left[\frac{1}{\bar{r}}(\bar{r}v_{\bar{r}}^{(0)})_{\bar{r}} - \frac{1}{\bar{r}^2}v^{(0)}\right] , \tag{3.18}$$

where

$$\dot{\Xi}(t,\xi) = \widehat{\dot{\sigma}^{(0)}/\sigma^{(0)}} = \lim_{\lambda\to\infty}\{[\int_{\xi/\varsigma-\lambda/2}^{\xi/\varsigma+\lambda/2}\sigma_t^{(0)}(t,\bar{s},\xi)d\bar{s}']/[\int_{\xi/\varsigma-\lambda/2}^{\xi/\varsigma+\lambda/2}\sigma^{(0)}(t,\bar{s},\xi)d\bar{s}]\} \tag{3.19}$$

denotes the average of the rate of change of the arc length $\mathcal{C}$ in the sence of (3.2) with $F(t,\bar{s},\xi) = \sigma_t^{(0)}(t,\bar{s},\xi)/\sigma^{(0)}$. Using (3.16) and (3.15) to eliminate $\widehat{<u^{(2)}>}$ and $p_\xi^{(0)}$ respectively, (3.17) and (3.18) become integro-differential equations for $v^{(0)}$ and $w^{(0)}$, in $t \geq 0$, $\bar{r} \geq 0$ and $\xi \geq 0$. The unknown $\Xi(t,\xi)$ provides the coupling with the velocity $\dot{\mathbf{X}}^{(0)}(t,s)$ of $\mathcal{C}$, given by (2.16) in variables, $t \geq 0$, $s \geq 0$, with the variable ξ in $C_v(t.\xi)$ and $C_w(t,\xi)$ identified as $s\varsigma$.

Now we need to specify the initial data for the filament centerline $\mathbf{X}(0,s)$ and the core structure $v^{(0)}(0,\bar{r},\xi)$ and $w^{(0)}(0,\bar{r},\xi)$. Note that the initial centerline of a filament simulating a rotor blade has to fulfill the assumption (v) far downstream. The centerline for $s \leq 0$, has to fulfill asumption (vi) for $t \geq 0$, for example, $\mathbf{X}(t,s) = r(\hat{\imath}\cos\omega t + \hat{\jmath}\sin\omega t)$ for $r = -s \in [r_0, b]$. In addition, the initial centerline has to fulfill the basic assumption that the average define by (3.19) exists for $\lambda \gg 1$ but $\ll 1/\varsigma$. We then have to solve numerically the system of equations and find an upper estimate of time $t_c > 0$, below which the basic assumption of two-scale analysis (3.2) remains valid.

3.3. Modeling initial filament centerlines

Finally, we shall present a few admissible initial centerlines in the form of helices. See for example [11]. Let $\hat{k}$ represent the axis of a helix and $\alpha \in [0,\pi/2]$ denote the constant angle between the axis and a tangent line of the helix. Here we have $\hat{k}$ pointing downward. Then its principal normal vector lies in the xy plane and $\kappa/T = \tan\alpha$. We can express the unit principal normal vector $\hat{n}$ and its orthogonal unit vector $\hat{m}$ in the xy plane in term of the angle ϕ as

$$\hat{n} = -\sin\phi\hat{\imath} + \cos\phi\,\hat{\jmath}\,, \quad \hat{m} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}\,, \tag{3.01}$$

and then the unit tangent and binormal vectors of the helix,

$$\hat{\tau} = \cos\alpha\,\hat{k} + \sin\alpha\,\hat{m}\,, \quad \hat{b} = \sin\alpha\,\hat{k} + \cos\alpha\,\hat{m}\,. \tag{3.21}$$

With $\hat{m}' = \hat{n}$, we have

$$\hat{\tau}'(\phi) = \sin\alpha\hat{n} = s'(\phi)\kappa\hat{n} \quad \text{hence} \quad \sigma = s' = \sin\alpha R(\phi). \tag{3.22}$$

where $R = 1/\kappa$. Given $R(\phi)$, we have the torsion $T(\phi) = \cot\alpha/R(\phi)$, the arc length,

$$s(\phi) = \sin\alpha\int_0^\phi R(\phi')d\phi'\,, \tag{3.23}$$

and the centerline

$$\mathbf{X}(\phi) - \mathbf{X}(0) = \sin\alpha\int_0^\phi d\phi' R(\phi')[\cos\alpha\hat{k} + \sin\alpha(\cos\phi'\hat{\imath} + \sin\phi'\hat{\jmath})]\,. \tag{3.24}$$

From (3.24) we see that $\mathbf{X}(\phi)$ is a linear function $R(\phi)$. We can get different helices by linear combinations of different $R(\phi)$'s.

§3.3.1. Circular Helices. Let $R(\phi) = R_1 =$ constant. We could identify R_0 as b. The centerline for $s = (b\sin\alpha)\phi \geq 0$ is

$$\mathbf{X}_1(\phi) = R_1\sin\alpha[\hat{k}\phi + \sin\alpha(\hat{\imath}\cos\phi + \hat{\jmath}\sin\phi)]. \tag{3.25}$$

§3.3.2. **Tapering Circular Helices.** Let $R_2(\phi) = R_1 + c\exp(-q\phi)$, where c and q are positive constants. We have $R_2(0) = R_1 + c$ and $R_2(\infty) = R_1$ and for $\phi > 0$,

$$\begin{aligned}\mathbf{X}_2(\phi) =&\mathbf{X}_1(\phi) + c\hat{k}\sin\alpha[1 - e^{-q\phi}]/q + \\ &[c\cos\alpha/(1+q^2)]\{\hat{\imath}[q + e^{-q\phi}(\sin\phi - q\cos\phi)] + \hat{\jmath}[q - e^{-q\phi}[\cos\phi + q\sin\phi]\}\ .\end{aligned} \tag{3.26}$$

§3.3.3. **Helices in forward Motion without Tapering.** Consider helix in forward motion with velocity $U\hat{\imath}$, being circular when $U = 0$. Let $R_3(\phi) = R_1 + g\cos\phi$. We have for $\phi > 0$,

$$\mathbf{X}_3(\phi) = \mathbf{X}_1(\phi) + g\{\hat{k}\sin\alpha\sin\phi + \cos\alpha[\hat{\imath}(2\phi + \sin 2\phi) + \hat{\jmath}(1 - \cos 2\phi)]/4. \tag{3.27}$$

If ϕ advances by 2π with period $P = 2\pi/\omega$ then $\mathbf{X}\cdot\hat{\imath}$ advances by $2\pi g = UP$. Thus we set $g = U/\omega$. It is clear that we can create a tapering helix in forward motion and other types of helices.

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