Determining Efficiency Differences from Time-Integrated Data

R. Cahn (LBNL)

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Riccardo Faccini has pointed out that we have not so far allowed for differences in tagging and reco efficiencies on B^0 and $\overline{B}^0$. We propose to determine these differences using time-independent data, avoiding recoding tFit. These results can then be used in the CP analysis.

Here we investigate the effect of efficiency differences on the TagMix and CP analyses using maximum likelihood methods, both analytic and numerical.

We find that the various efficiences can be determined with sufficient accuracy simply by the method of counting the numbers of events with the various kinds of tags (and the events that are untagged). With these results, the coefficient of the $\cos\Delta mt$ term can be determined with no significant loss in precision. The coefficient of the $\sin\Delta mt$ term is in any event insensitive to these issues.

A simulation with 6600 reconstructed B^0 and a 0.7 probability of an event being tagged results in uncertainties in the differences of tagging efficiences on B^0 and $\overline{B}^0$ of order 1%. These contribute negligibly to the uncertainty in the coefficients of $\sin\Delta mt$ and $\cos\Delta mt$. Using counting methods to determine the efficiencies rather than a full likelihood fit increases the errors in the efficiency difference by only 6% of the 1%.

These results justify using counting methods to determine the efficiencies as inputs to the determination of the coefficient of $\cos\Delta mt$. The efficiency differences have no impact on the coefficient of $\sin\Delta mt$.

As expected, the errors on the coefficients of both $\sin dmt$ and $\cos\Delta mt$ are entirely dominated by the statistics of the CP sample. We demonstrate this by displaying how the errors in these coefficients behave as various parameters are freed in the fit.

We also investigate the effect of fitting double-gaussian data with a single gaussian.

1 TagMix Determination of Efficiency Differences

Let's let the reco efficiency on B^0 be r and that on $\overline{B}^0$ be $\overline{r}$. Similarly, let the tagging efficiencies on B^0 and $\overline{B}^0$ be t and $\overline{t}$.

It is simplest if we express the results in terms of small quantities:

$$\mu = \frac{t-\overline{t}}{t+\overline{t}}; \qquad \nu = \frac{r-\overline{r}}{r+\overline{r}} \tag{1}$$

We write the distributions as

$$\begin{aligned}
f_{B^0t,B^0r} &= r\left[t(1-w)(1-A)+\overline{t}\overline{w}(1+A)\right]\frac{\Gamma e^{-\Gamma|t|}}{4(r+\overline{r})}\\
f_{\overline{B}^0t,B^0r} &= r\left[\overline{t}(1-\bar{w})(1+A)+tw(1-A))\right]\frac{\Gamma e^{-\Gamma|t|}}{4(r+\overline{r})}\\
f_{B^0t,\overline{B}^0r} &= \overline{r}\left[t(1-w)(1+A)+\overline{t}\overline{w}(1-A))\right]\frac{\Gamma e^{-\Gamma|t|}}{4(r+\overline{r})}\\
f_{\overline{B}^0t,\overline{B}^0r} &= \overline{r}\left[\overline{t}(1-\overline{w})(1-A)+tw(1+A))\right]\frac{\Gamma e^{-\Gamma|t|}}{4(r+\overline{r})}
\end{aligned} \tag{2}$$

where here $A = \cos\Delta mt$. If we only count events in the different categories it is the time-integrated distributions that count. Integrating over $-\infty < t < \infty$ we obtain

$$\begin{aligned}
F_{B^0t,B^0r} &= \frac{r}{2(r+\overline{r})}\left[t(1-w)(1-A_{ave})+\overline{t}\overline{w}(1+A_{ave})\right]\\
F_{\overline{B}^0t,B^0r} &= \frac{r}{2(r+\overline{r})}\left[\overline{t}(1-\bar{w})(1+A_{ave}+)+tw(1-A_{ave}))\right]\\
F_{B^0t,\overline{B}^0r} &= \frac{\overline{r}}{2(r+\overline{r})}\left[t(1-w)(1+A_{ave})+\overline{t}\overline{w}(1-A_{ave}))\right]\\
F_{\overline{B}^0t,\overline{B}^0r} &= \frac{\overline{r}}{2(r+\overline{r})}\left[\overline{t}(1-\overline{w})(1-A_{ave})+tw(1+A_{ave}))\right]
\end{aligned} \tag{3}$$

where, with $x_d = \Delta m/\Gamma$

$$A_{ave} = \frac{1}{1+x_d^2} \approx \frac{2}{3} \tag{4}$$

We see that the normalization is such that

$$F_{B^0t,B^0r}+F_{\overline{B}^0t,B^0r}+F_{B^0t,\overline{B}^0r}+F_{\overline{B}^0t,\overline{B}^0r} = \frac{(rt+\overline{r}\overline{t})(1-A_{ave})+(r\overline{t}+\overline{r}t)(1+A_{ave})}{2(r+\overline{r})} \tag{5}$$

There are, however, more events to consider: the reconstructed B^0s and $\overline{B}^0$s that were not tagged. For these we can write

$$\begin{aligned} F_{no\ t,B^0r} &= \int_{-\infty}^{\infty} dtr[(1-t)(1-A)+(1-\overline{t})(1+A)]\frac{\Gamma e^{-\Gamma|t|}}{4(r+\overline{r})} \\ &= \frac{r}{2(r+\overline{r})}[(1-t)(1-A_{ave})+(1-\overline{t})(1+A_{ave})] \\ F_{no\ t,\overline{B}^0r} &= \int_{-\infty}^{\infty} dt\overline{r}[(1-t)(1+A)+(1-\overline{t})(1-A)]\frac{\Gamma e^{-\Gamma|t|}}{4(r+\overline{r})} \\ &= \frac{\overline{r}}{2(r+\overline{r})}[(1-t)(1+A_{ave})+(1-\overline{t})(1-A_{ave})] \end{aligned} \tag{6}$$

Altogether then,

$$F_{no\ t,B^0r}+F_{no\ t,\overline{B}^0r}+F_{B^0t,B^0r}+F_{\overline{B}^0t,B^0r}+F_{B^0t,\overline{B}^0r}+F_{\overline{B}^0t,\overline{B}^0r}=1 \tag{7}$$

These relations remain true even if we convolute the time distributions with some resolution functions. Moreover, we can understand the time-dependent distributions also to be convoluted with some resolution function.

If we re-express the distributions in terms of

$$\begin{aligned} D &= 1-w-\overline{w}; \qquad \Delta w=(w-\overline{w})/2 \\ T &= (t+\overline{t})/2; \qquad \Delta T=(t-\overline{t})/2; \qquad \Delta T/T=\mu \\ R &= (r+\overline{r})/2; \qquad \Delta R=(r-\overline{r})/2; \qquad \Delta R/R=\nu \end{aligned} \tag{8}$$

In the present notation

$$\begin{aligned} F_{B^0t,B^0r} &= \frac{T}{4}(1+\nu)\left[1-DA_{ave}-2\Delta w+\mu D-\mu A_{ave}+2\Delta w\mu A_{ave}\right] \\ F_{\overline{B}^0t,B^0r} &= \frac{T}{4}(1+\nu)\left[1+DA_{ave}+2\Delta w-\mu D-\mu A_{ave}-2\Delta w\mu A_{ave}\right] \\ F_{B^0t,\overline{B}^0r} &= \frac{T}{4}(1-\nu)\left[1+DA_{ave}-2\Delta w+\mu D+\mu A_{ave}-2\Delta w\mu A_{ave}\right] \end{aligned}$$

$$
\begin{aligned}
F_{\overline{B}^0 t,\overline{B}^0 r} &= \frac{T}{4}(1-\nu)\left[1 - DA_{ave} + 2\Delta w - \mu D + \mu A_{ave} + 2\Delta w \mu A_{ave}\right] \\
F_{not,B^0 r} &= \frac{1+\nu}{2}\left[1 - T + \mu A_{ave} T\right] \\
F_{not,\overline{B}^0 r} &= \frac{1-\nu}{2}\left[1 - T - \mu A_{ave} T\right]
\end{aligned}
\tag{9}
$$

We form the combinations

$$
\begin{aligned}
F_{B^0 t} &= F_{B^0 t,B^0 r} + F_{B^0 t,\overline{B}^0 r} \\
&= \frac{T}{2}\left[1 - 2\Delta w + \mu D + \nu(-DA_{ave} - \mu A_{ave} + \Delta w \mu A_{ave})\right] \\
F_{\overline{B}^0 t} &= F_{B^0 t,B^0 r} + F_{B^0 t,\overline{B}^0 r} \\
&= \frac{T}{2}\left[1 + 2\Delta w - \mu D + \nu(DA_{ave} - \mu A_{ave} - \Delta w \mu A_{ave})\right] \\
F_{B^0 t} - F_{\overline{B}^0 t} &= T\left[\mu D - 2\Delta w - DA_{ave}\nu + 2\Delta w \mu \nu A_{ave}\right] \\
F_{any\ t,B^0 r} &= F_{B^0 t,B^0 r} + F_{\overline{B}^0 t,B^0 r} = \frac{T}{4}(1+\nu)(2 - 2\mu A_{ave}) \\
F_{any\ t,\overline{B}^0 r} &= F_{B^0 t,\overline{B}^0 r} + F_{\overline{B}^0 t,\overline{B}^0 r} = \frac{T}{4}(1-\nu)(2 + 2\mu A_{ave}) \\
F_{any\ t} &= F_{any\ t,B^0 r} + F_{any\ t,\overline{B}^0 r} = T(1 - \mu\nu A_{ave}) \\
F_{any\ t,B^0 r} - F_{any\ t,\overline{B}^0 r} &= T(\nu - \mu A_{ave}) \\
F_{B^0} &= F_{no\ t,B^0} + F_{any\ t,B^0 r} = \tfrac{1}{2}(1+\nu) \\
F_{\overline{B}^0} &= F_{no\ t,\overline{B}^0} + F_{any\ t,\overline{B}^0 r} = \tfrac{1}{2}(1-\nu)
\end{aligned}
\tag{10}
$$

From these we see that we can proceed by first determining ν:

$$
\nu = F_{B^0} - F_{\overline{B}^0} \tag{11}
$$

Then we form

$$
\begin{aligned}
T &= \frac{F_{any\ t} - \nu(F_{any\ t,B^0 r} - F_{any\ t,\overline{B}^0 r})}{1-\nu^2} \\
\mu &= \frac{\nu F_{any\ t} - (F_{any\ t,B^0 r} - F_{any\ t,\overline{B}^0 r})}{(1-\nu^2) T A_{ave}}
\end{aligned}
\tag{12}
$$

It is worth noting that we can in this way determine T, μ and ν without dealing with the wrong-tag probabilities: we need only the numbers of events in the tagged and untagged samples of reco-ed: B^0 and $\overline{B}^0$:

$$x = N_{any\ t,B^0}; y = N_{any\ t,\overline{B}^0}; z = N_{no\ t,B^0}; w = N_{no\ t,\overline{B}^0} \tag{13}$$

When $\mu = 0, \nu = 0$, $x = y = NT/2$, $z = w = N(1-T)/2$, where N is the total number of reco-ed events.

To assess the uncertainties expected in T, μ, and ν we need to express these in terms of the observed numbers of events x, y, z, w:

$$\begin{aligned} \nu &= \frac{x-y+z-w}{x+y+z+w} \\ T &= \frac{2xy+yz+xw}{2(x+z)(y+w)} \\ \mu &= \frac{1}{A_{ave}} \frac{yz-xw}{2xy+xw+yz} \end{aligned} \tag{14}$$

We obtain a clearer representation by introducing $u = N_{B^0} = x+z, v = N_{\overline{B}^0} = y+w$.

$$\begin{aligned} \nu &= \frac{u-v}{u+v} \\ T &= \frac{1}{2}[\frac{x}{u} + \frac{y}{v}] \\ \mu &= \frac{1}{A} \frac{y/v - x/u}{y/v + x/y} \end{aligned} \tag{15}$$

If we evaluate the uncertainties expected in T, μ, and ν when the true values of μ and ν vanish, we find

$$\begin{aligned} \sigma(T) &= \sqrt{\frac{T(1-T)}{N}} \\ \sigma(\nu) &= \frac{1}{\sqrt{N}} \\ \sigma(\mu) &= \frac{1}{A_{ave}} \sqrt{\frac{1-T}{TN}} \end{aligned} \tag{16}$$

2 CP Analysis

These errors will propagate into the CP analysis. The distributions there are given by

$$
\begin{aligned}
f_{B^0 t,CP} &= [t(1-w)(1-A') + \bar{t}\bar{w}(1+A')]\,\frac{\Gamma e^{-\Gamma|t|}}{4} \\
&= T\{1 - 2\Delta w - DA' + \mu[D - A'(1-2\Delta w)]\}\frac{\Gamma e^{-\Gamma|t|}}{4} \\
f_{\overline{B}^0 t,CP} &= [\bar{t}(1-\bar{w})(1+A') + tw(1-A'))]\,\frac{\Gamma e^{-\Gamma|t|}}{4} \\
&= T\{1 + 2\Delta w + DA' - \mu[D + A'(1+2\Delta w)]\}\frac{\Gamma e^{-\Gamma|t|}}{4} \\
F_{B^0 t,CP} &= \tfrac{1}{2}\,[t(1-w)(1-A'_{ave}) + \bar{t}\bar{w}(1+A'_{ave})] \\
F_{\overline{B}^0 t,CP} &= \tfrac{1}{2}\,[\bar{t}(1-\overline{w})(1+A'_{ave}) + tw(1-A'_{ave}))] \\
F_{any\ t,CP} &= \tfrac{1}{2}\,[t(1-A'_{ave}) + \bar{t}(1+A'_{ave})] \\
&= T - \mu T A'_{ave} \\
F_{no\ t,CP} &= 1 - T + \mu T A'_{ave}
\end{aligned} \tag{17}
$$

where here $A' = a \sin \Delta mt + b \cos \Delta mt,\ A'_{ave} = b/(1+x_d^2)$. In the Standard Model, $a = -\sin 2\beta; b = 0$.

We proceed without worrying about the Δz resolution. The we can find relatively simple expressions for the matrix whose inverse is the covariance matrix. We compute a series of partial derivatives at $\mu = 0, b = 0$:

$$
\begin{aligned}
\frac{\partial f_{B^0 t,CP}}{\partial a} &= T\frac{\Gamma e^{-\Gamma|t|}}{4}(-D \sin \Delta mt) \\
\frac{\partial f_{\overline{B}^0 t,CP}}{\partial a} &= T\frac{\Gamma e^{-\Gamma|t|}}{4}(D \sin \Delta mt) \\
\frac{\partial f_{B^0 t,CP}}{\partial b} &= T\frac{\Gamma e^{-\Gamma|t|}}{4}(-D \cos \Delta mt) \\
\frac{\partial f_{\overline{B}^0 t,CP}}{\partial b} &= T\frac{\Gamma e^{-\Gamma|t|}}{4}(D \cos \Delta mt) \\
\frac{\partial f_{B^0 t,CP}}{\partial \mu} &= T\frac{\Gamma e^{-\Gamma|t|}}{4}(D - A')
\end{aligned}
$$

$$\frac{\partial f_{\overline{B}^0 t,CP}}{\partial \mu} = T\frac{\Gamma e^{-\Gamma|t|}}{4}(-D - A') \tag{18}$$

At this point

$$\begin{aligned}
f_{B^0 t,CP} &= T\frac{\Gamma e^{-\Gamma|t|}}{4}(1 - DA') \\
f_{\overline{B}^0 t,CP} &= T\frac{\Gamma e^{-\Gamma|t|}}{4}(1 + DA')
\end{aligned} \tag{19}$$

From these we calculate

$$\begin{aligned}
-\frac{\partial^2 \langle \mathcal{L} \rangle}{\partial a^2} &= N_{CP}\int_{-\infty}^{\infty} dt \left[\frac{1}{f_{B^0 t,CP}}\frac{\partial f_{B^0 t,CP}}{\partial a}\frac{\partial f_{B^0 t,CP}}{\partial a} + \frac{1}{f_{\overline{B}^0 t,CP}}\frac{\partial f_{\overline{B}^0 t,CP}}{\partial a}\frac{\partial f_{\overline{B}^0 t,CP}}{\partial a}\right] \\
&= N_{CP}\int_{-\infty}^{\infty} dtT\frac{\Gamma e^{-\Gamma|t|}}{4}D^2\sin^2\Delta mt\left(\frac{1}{1-DA'} + \frac{1}{1+DA'}\right) \\
&= N_{CP}\int_{0}^{\infty} dtT\Gamma e^{-\Gamma|t|}\frac{D^2\sin^2\Delta mt}{1-D^2A'^2} \\
&= N_{CP}\int_{0}^{\infty} duTe^{-u}D^2\sin^2 xu[1 + D^2a^2\sin^2 xu + \ldots] \\
&= N_{CP}T\cdot 2x^2D^2\left[\frac{1}{1+4x^2} + D^2a^2\left(\frac{1}{1+4x^2} - \frac{1}{1+16x^2}\ldots\right)\right] \\
-\frac{\partial^2 \langle \mathcal{L} \rangle}{\partial b^2} &= N_{CP}\int_{0}^{\infty} dtT\Gamma e^{-\Gamma|t|}\frac{D^2\cos^2\Delta mt}{1-D^2A'^2} \\
&= N_{CP}T\cdot D^2\left[\frac{1+2x^2}{1+4x^2} + \frac{2x^2D^2a^2}{1+16x^2}\ldots)\right] \\
-\frac{\partial^2 \langle \mathcal{L} \rangle}{\partial a\partial \mu} &= N_{CP}\int_{-\infty}^{\infty} dt \left[\frac{1}{f_{B^0 t,CP}}\frac{\partial f_{B^0 t,CP}}{\partial a}\frac{\partial f_{B^0 t,CP}}{\partial \mu} + \frac{1}{f_{\overline{B}^0 t,CP}}\frac{\partial f_{\overline{B}^0 t,CP}}{\partial a}\frac{\partial f_{\overline{B}^0 t,CP}}{\partial \mu}\right] \\
&= N_{CP}\int_{-\infty}^{\infty} dtT\frac{\Gamma e^{-\Gamma|t|}}{4}\left[\frac{-D\sin\Delta mt(D-A')}{1-DA'} + \frac{D\sin\Delta mt(-D-A')}{1+DA'}\right] \\
&= N_{CP}\int_{-\infty}^{\infty} dtT\frac{\Gamma e^{-\Gamma|t|}}{4}\left[\frac{-2D^2\sin\Delta mt}{1-D^2A'^2} + \frac{2D^2A'^2\sin\Delta mt}{1-D^2A'^2}\right] \\
&= 0
\end{aligned} \tag{20}$$

where we have used $A' = a\sin\Delta mt$ when $b = 0$. Similarly

$$-\frac{\partial^2 \langle \mathcal{L} \rangle}{\partial b \partial \mu} = N_{CP} \int_{-\infty}^{\infty} dt T \frac{\Gamma e^{-\Gamma|t|}}{4} \left[\frac{-2D^2 \cos \Delta mt}{1 - D^2 A'^2} + \frac{2D^2 A'^2 \cos \Delta mt}{1 - D^2 A'^2} \right]$$
$$= -N_{CP} T \frac{D^2}{1+x^2} \left[1 - \frac{2x^2 a^2 (1-D^2)}{1+9x^2} + \ldots \right] \quad (21)$$

The element $\partial \langle \mathcal{L} \rangle / \partial^2 \mu$ is determined by the TagMix data. It is effectively

$$-\frac{\partial \langle \mathcal{L} \rangle}{\partial^2 \mu} = N_{TM} \frac{A_{ave}^2 T}{1-T} \quad (22)$$

The more detailed study in the Appendix shows that a full likelihood treatment increases this, i.e. decreases the error. A first order correction is the replacement $< A >^2 \rightarrow < A^2 >$, i.e.

$$\left[\int_{-\infty}^{\infty} dt \tfrac{1}{2} \Gamma e^{-\Gamma|t|} \cos \Delta mt \right]^2 \rightarrow \int_{-\infty}^{\infty} dt \tfrac{1}{2} \Gamma e^{-\Gamma|t|} \cos^2 \Delta mt$$
$$\left(\frac{1}{1+x^2} \right)^2 \rightarrow \frac{1+2x^2}{1+4x^2} \approx \frac{2}{3} \quad (23)$$

We can now write the covariance matrix in the variables b and μ, that is, we can determine how the uncertainty in μ induced in the counting method described above increases the error in the coefficient of the cos term in the CP measurement. Using just the first terms in the expansions above

$$-\begin{pmatrix} \frac{\partial^2 \langle \mathcal{L} \rangle}{\partial b^2} & \frac{\partial^2 \langle \mathcal{L} \rangle}{\partial b \partial \mu} \\ \frac{\partial^2 \langle \mathcal{L} \rangle}{\partial b \partial \mu} & \frac{\partial^2 \langle \mathcal{L} \rangle}{\partial \mu^2} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} N_{CP} T D^2 & -\frac{2}{3} N_{CP} T D^2 \\ -\frac{2}{3} N_{CP} T D^2 & \frac{2}{3} N_{TM} \frac{T}{1-T} \end{pmatrix} \quad (24)$$

The $\frac{2}{3}$ factors occur because we approximate $x_d = 1/\sqrt{2}$. Inverting this matrix and taking the square root of the $b-b$ element we find

$$\sigma(b) = \sqrt{\frac{3}{2TD^2 N_{CP}}} \left[1 + \tfrac{1}{2} \frac{N_{CP}}{N_{TM}} D^2 (1-T) + \ldots \right] \quad (25)$$

The first term is just the usual result for the uncertainty in b. The second term shows the effect of the uncertainty in μ, i.e. the difference in tagging efficiences for B^0 and $\overline{B}^0$. It is clear that this is not a problem since the second term is less than 0.01 times the first.

3 Numerical Monte Carlo Calculations

Here we consider the effects of Δz resolution by allowing for a single Gaussian. As a result we encounter the integrals

$$\begin{aligned} C(\Gamma, \Delta m, \sigma, t) &= \int_{-\infty}^{\infty} dt \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma|t'|} e^{-(t-t')^2/2\sigma^2} \cos \Delta m t \\ S(\Gamma, \Delta m, \sigma, t) &= \int_{-\infty}^{\infty} dt \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma|t'|} e^{-(t-t')^2/2\sigma^2} \sin \Delta m t \end{aligned} \quad (26)$$

so we calculate

$$\begin{aligned} Z(\Gamma, \Delta m, \sigma, t) &= \int_{-\infty}^{\infty} dt \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma|t'|} e^{-(t-t')^2/2\sigma^2} e^{i\Delta m t} \\ &= C + iS \end{aligned} \quad (27)$$

Now

$$\begin{aligned} Z &= \int_{0}^{\infty} dt \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma t'} e^{-(t-t')^2/2\sigma^2} e^{i\Delta m t} + \int_{0}^{\infty} dt \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma t'} \frac{e^{-(t+t')^2/2\sigma^2}}{\sqrt{2\pi}} e^{-i\Delta m t'} \\ &= \int_{0}^{\infty} dt \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma(t'-t)} e^{-(t-t')^2/2\sigma^2} e^{i\Delta m(t-t')} e^{-(\Gamma - i\Delta m)t} + \\ &\qquad \int_{0}^{\infty} dt \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma(t'+t)} e^{-(t+t')^2/2\sigma^2} e^{-i\Delta m(t'+t)} e^{(\Gamma + i\Delta m)t} \\ &= \int_{-t}^{\infty} d\tau \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma\tau} e^{-\tau^2/2\sigma^2} e^{i\Delta m\tau} e^{-(\Gamma - i\Delta m)t} + \\ &\qquad \int_{t}^{\infty} d\tau \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\Gamma\tau} e^{-\tau^2/2\sigma^2} e^{-i\Delta m\tau} e^{(\Gamma + i\Delta m)t} \\ &= \int_{-t}^{\infty} d\tau \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\tau + \Gamma\sigma^2 - i\Delta m\sigma^2)^2/\sigma^2} e^{\frac{1}{2}\sigma^2(\Gamma - i\Delta m)^2} e^{-(\Gamma - i\Delta m)t} + \\ &\qquad \int_{t}^{\infty} d\tau \frac{\Gamma}{2\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\tau + \Gamma\sigma^2 + i\Delta m\sigma^2)^2/\sigma^2} e^{\frac{1}{2}\sigma^2(\Gamma + i\Delta m)^2} e^{(\Gamma + i\Delta m)t} \end{aligned} \quad (28)$$

Now using

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} du e^{-u^2} \quad (29)$$

and defining

$$H(z, \tau) = e^{z^2} e^{-2z\tau} \mathrm{erfc}(-\tau + z) \quad (30)$$

we have

$$Z(\Gamma, \Delta m, \sigma, t) = \frac{\Gamma}{4}\left[H(\frac{\sigma}{\sqrt{2}}(\Gamma - i\Delta m), \frac{t}{\sqrt{2}\sigma}) + H(\frac{\sigma}{\sqrt{2}}(\Gamma + i\Delta m), -\frac{t}{\sqrt{2}\sigma})\right] \tag{31}$$

We can confirm that in the limit $\sigma \to 0$ we reproduce the initial distribution using

$$\begin{aligned} \lim_{z\to\infty} \mathrm{erfc}(z) &= 0; \qquad \lim_{z\to-\infty} \mathrm{erfc}(z) = 2 \\ \lim_{\tau\to\infty} H(z,\tau) &= 2e^{z^2}e^{-2z\tau}; \qquad \lim_{\tau\to\infty} H(z,-\tau) = 0 \end{aligned} \tag{32}$$

A Monte Carlo was written to determine the asymptotic behavior of the analysis.This was done not by casting random events, but by evaluating numerically the Symbolically,

$$\langle \mathcal{L} \rangle = N \int dt f(t, \overline{\alpha}) \ln f(t, \alpha) \tag{33}$$

where $\overline{\alpha}$ are the true values of the parameters and α are their varied values. Minuit will return the true values (we hope!) and the covariance matrix, from which we find the asymptotic errors. Alternatively, we can use this approach to determine the error introduced by an error in an assumed value by fixing an element of α to a value that differs from its corresponding value in $\overline{\alpha}$.

As an example we consider the following case:

$$\begin{aligned} N_{TM} &= 6600 \\ N_{CP} &= 470 \\ D &= 0.55 \\ T &= 0.7 \\ \nu &= 0.0 \\ \mu &= 0.0 \\ \delta w &= 0.0 \\ \text{coef. of } \sin\Delta m t &= 0.7 \\ \text{coef. of } \cos\Delta m t &= 0.0 \end{aligned} \tag{34}$$

```
 EXT PARAMETER                                   STEP         FIRST
 NO.   NAME        VALUE          ERROR          SIZE      DERIVATIVE

 15   mu         -0.16413E-11   0.11157E-01   0.54335E-05  -0.14507E-06
 16   nu          0.44630E-11   0.12314E-01   0.60167E-05  -0.92169E-06
 17   dw           0.0000       constant
 18   ave_tag     0.69980       0.54530E-02   0.26644E-05   0.10022E-03
 19   sin_coef    0.69992       0.16297       0.79553E-04  -0.63742E-02
 20   cos_coef    0.19184E-12   0.12225       0.59521E-04   0.20518E-08

EXTERNAL ERROR MATRIX.    NDIM=  50    NPAR=  5    ERR DEF=  1.00
 0.124E-03 0.402E-10-0.468E-09 0.355E-09-0.111E-03
 0.402E-10 0.152E-03-0.131E-10-0.112E-10-0.430E-10
-0.468E-09-0.131E-10 0.297E-04-0.141E-10 0.416E-09
 0.355E-09-0.112E-10-0.141E-10 0.266E-01-0.127E-06
-0.111E-03-0.430E-10 0.416E-09-0.127E-06 0.149E-01

PARAMETER  CORRELATION COEFFICIENTS
      NO.  GLOBAL    15     16     18     19     20
      15  0.08173  1.000 0.000 0.000 0.000-0.082
      16  0.00000  0.000 1.000 0.000 0.000 0.000
      18  0.00001  0.000 0.000 1.000 0.000 0.000
      19  0.00001  0.000 0.000 0.000 1.000 0.000
      20  0.08173 -0.082 0.000 0.000 0.000 1.000
```

Table 1: Portion of the Minuit output with μ, ν, T, a and b as variables.

At first we use an unrealistically good Δz resolution, 10 μm.

The result is shown in Table 3 We see that $\sin 2\beta$ is not correlated with any of the other variables determined here. The uncertainty in $\sin 2\beta$ is in excellent agreement with

$$\begin{aligned}
\sigma(a)^{-2} &= N_{CP}T \cdot 2x^2 D^2 [\frac{1}{1+4x^2} + D^2 a^2 (\frac{1}{1+4x^2} - \frac{1}{1+16x^2} \ldots)] \\
&= 106 \times (0.319 + 0.032) = 372 \\
\sigma(a) &= 0.164
\end{aligned} \tag{35}$$

If we invert the covariance matrix in μ and b we find

$$10^{-4} \begin{pmatrix} 1.24 & -1.11 \\ -1.11 & 149 \end{pmatrix} \to 10^4 \begin{pmatrix} 0.81 & 0.0061 \\ 0.0061 & 0.0068 \end{pmatrix} \tag{36}$$

For comparison, we find

$$\frac{\partial^2 \mathcal{L}}{\partial b^2} = N_{CP}T \cdot D^2 [\frac{1+2x^2}{1+4x^2} + \frac{2x^2 D^2 a^2}{1+16x^2} \ldots)] = 67.2 \tag{37}$$

$$-\frac{\partial^2 \langle \mathcal{L} \rangle}{\partial b \partial \mu} = -N_{CP}T \frac{D^2}{1+x^2} \left[1 - \frac{2x^2 a^2 (1-D^2)}{1+9x^2} + \ldots \right] = -61 \tag{38}$$

In Appendix 5 we find

$$\frac{\partial^2 \mathcal{L}}{\partial \mu^2} \approx 8000. \tag{39}$$

Altogether we have very good agreement between the numerical Monte Carlo and our analytic calculation.

The general behavior of the coefficients of $\sin \Delta m t$ and $\cos \Delta m t$ can be seen by varying D, μ, ν and δw. The results are shown in Table 3

We see that $\sin s\beta$ (variable 19) is correlated only with D. The size of that correlation is precisely what we'd naively. Since $\sin 2\beta$ is effectively $1/D$ times an observed asymmetry, we predict

$$\delta \sin 2\beta = -\frac{\sin 2\beta}{D} \delta D \tag{40}$$

```
 EXT PARAMETER                                  STEP         FIRST
 NO.   NAME        VALUE          ERROR          SIZE      DERIVATIVE

  4  dilu_sig     0.55000       0.15641E-01   0.75860E-05  -0.30145E-03
 15  mu           0.44672E-12   0.11280E-01   0.54335E-05   0.19718E-06
 16  nu           0.55546E-11   0.12314E-01   0.60167E-05   0.99919E-06
 17  dw          -0.42519E-12   0.13073E-01   0.62058E-05   0.32918E-06
 18  ave_tag      0.69980       0.54530E-02   0.26644E-05  -0.26748E-02
 19  sin_coef     0.70000       0.16417       0.79631E-04   0.40348E-04
 20  cos_coef    -0.98184E-12   0.12423       0.59533E-04  -0.13987E-08

EXTERNAL ERROR MATRIX.    NDIM=  50    NPAR=  7    ERR DEF=  1.00
 0.245E-03
-0.140E-09 0.127E-03
 0.335E-10 0.584E-10 0.152E-03
-0.211E-08 0.217E-04-0.244E-10 0.171E-03
 0.116E-10-0.463E-09 0.525E-10-0.992E-10 0.297E-04
-0.311E-03 0.117E-10 0.583E-11-0.150E-08-0.470E-10 0.270E-01
-0.302E-07-0.749E-04-0.102E-09 0.289E-03 0.221E-09-0.956E-07 0.154E-01

PARAMETER  CORRELATION COEFFICIENTS
      NO.  GLOBAL      4     15     16     17     18     19     20
       4  0.12128  1.000 0.000 0.000 0.000 0.000-0.121 0.000
      15  0.16779  0.000 1.000 0.000 0.147 0.000 0.000-0.053
      16  0.00000  0.000 0.000 1.000 0.000 0.000 0.000 0.000
      17  0.23697  0.000 0.147 0.000 1.000 0.000 0.000 0.178
      18  0.00001  0.000 0.000 0.000 0.000 1.000 0.000 0.000
      19  0.12128 -0.121 0.000 0.000 0.000 0.000 1.000 0.000
      20  0.19506  0.000-0.053 0.000 0.178 0.000 0.000 1.000
```

Table 2: Portion of the Minuit output with D, μ, ν, T, a and b as variables.

a	b	σ	D	T	μ	ν	δw
0.190	0.127						
0.190	0.127	7.56					
0.190	0.127	7.56	0.017				
0.190	0.127	7.56	0.017	0.0055	0.012	0.012	0.013

Table 3: Errors in various parameters with 6600 TagMix events and 470 CP events. The errors in a (the coefficient of $\sin \Delta mt$) and b (the coefficient of $\cos \Delta mt$) are seen not to be affecting by opening up the other parameters since the statistics of the TagMix sample is to large compared to that of the CP sample.

Now for a one-sigma shift in D, 0.0156, we get a shift of $-(0.7/0.55)(0.0156) = -0.020$ which is $-0.121\sigma(\sin 2\beta)$.

As for the coefficient of $\cos \Delta mt$, we find correlations with μ, as discussed above, and with δw, the difference in the wrong-tag probabilities.We see that these correlations are relatively weak.

4 Errors in the coefficients of $\sin \Delta mt$ and $\cos \Delta mt$

The errors in a (the coefficient of $\sin \Delta mt$) and b (the coefficient of $\cos \Delta mt$) for the parameter set described above with $\sigma = 10\mu$m are 0.164 and 0.124 respectively. In Table 4 we display the errors obtained with $\sigma = 150\mu$m, showing at the same time the errors in various parameters as they are freed in the fit.

The uncertainty in a is quite sensitive to the Δz resolution. This shown in Table 4.

5 Multiple Tagging Categories

If there is more than one tagging category we need to find μ for each of them. Because the expression for μ depends both on the events with a given tag and events without that tag, there are correlations between the various μ_i.

σ_1	σ_2	f_1	a	b
$144. \pm 8.$		1.0	0.686 ± 0.188	0.0 ± 0.127
$100. \pm 18.$	300.	0.80 ± 0.06	0.70 ± 0.192	0.0 ± 0.128

Table 4: The result of fitting data that have a double gaussian with (1) a single gaussian and (2) with a double gaussian, fixing the width of the wider gaussian. The result shows that the single gaussian fit gives a bias in the coefficient of $\sin \Delta mt$, but one that is small (0.014) compared to the statistical error (0.188).

Let us calculate the correlations, since this is a simple problem in statistcs. We begin with the expression

$$\mu = \frac{1}{A} \frac{y_i/v - x_i/u}{y_i/v + x_i/u} \tag{41}$$

where $x_i = N_{i,B^0}, y_i = N_{i,B^0}$, $u = N_{B^0} = x + z, v = N_{\overline{B}^0} = y + w$.

Now x_i and u are correlated, as are y_i and v. Let us indicate correlations by $<< a, b >>=< ab > - < a >< b >$. Then

$$\begin{aligned} << x_i, u >> &= < x_i > \\ << x_i, v >> &= 0 \\ << y_i, u >> &= 0 \\ << y_i, v >> &= < y_i > \end{aligned} \tag{42}$$

However,

$$<< x_i, y_j >>= 0 \tag{43}$$

If we evaluate the fluctuations when $< \mu_i >= 0$, we have simply

$$\delta\mu_i = \frac{1}{A_{ave}} \frac{1}{2T_i} \left[\frac{1}{v}\delta y_i - \frac{y_i}{v^2}\delta v - \frac{1}{u}\delta x_i + \frac{x_i}{u^2}\delta u \right] \tag{44}$$

We then find

$$<< \delta\mu_i, \delta\mu_i >>= \frac{1}{(2T_i A_{ave})^2} \frac{4T_i(1-T_i)}{N} = \frac{1-T_i}{A_{ave} N T_i} \tag{45}$$

and

$$<< \delta\mu_i, \delta\mu_j >>= -\frac{1}{A_{ave}^2 N} \tag{46}$$

Now let us see how the presence of a second tagging category would affect our estimate of the uncertainty in the coefficient of the cosine term. In the case of a single tagging category we had

$$-\frac{\partial^2 \langle \mathcal{L} \rangle}{\partial \mu^2} = A_{ave}\frac{T}{1-T} \tag{47}$$

The corresponding quantity is evidently obtained from the inverse of the matrix

$$\frac{1}{A_{ave}^2 N}\begin{pmatrix} \frac{1-T_1}{T_1} & -1 \\ -1 & \frac{1-T_2}{T_2} \end{pmatrix} \tag{48}$$

whose inverse is

$$\frac{A_{ave}^2 N}{1-T_1-T_2}\begin{pmatrix} T_1(1-T_2) & T_1 T_2 \\ T_1 T_2 & T_2(1-T_1) \end{pmatrix} \tag{49}$$

Thus the 3×3 matrix of interest is

$$\begin{pmatrix} \frac{2}{3}N_{CP}(T_1 D_1^2 + T_2 D_2^2) & -\frac{2}{3}N_{CP}T_1 D_1^2 & -\frac{2}{3}N_{CP}T_2 D_2^2 \\ -\frac{2}{3}N_{CP}T_1 D_1^2 & \frac{A_{ave}^2 N}{1-T_1-T_2}T_1(1-T_2) & \frac{A_{ave}^2 N}{1-T_1-T_2}T_1 T_2 \\ -\frac{2}{3}N_{CP}T_2 D_2^2 & \frac{A_{ave}^2 N}{1-T_1-T_2}T_1 T_2 & \frac{A_{ave}^2 N}{1-T_1-T_1}T_2(1-T_1) \end{pmatrix} \tag{50}$$

What we need is the first diagonal element of the inverse of this matrix. Some algebra yields

$$\sigma(b)^{-2} = \tfrac{2}{3}N_{CP}(T_1 D_1^2 + T_2 D_2^2) \times \left\{ 1 - \tfrac{2}{3}\frac{N_{CP}}{N_{TM}}\frac{1}{A_{ave}^2}\left[\frac{T_1 D_1^4 + T_2 D_2^4}{T_1 D_1^2 + T_2 D_2^2} - (T_1 D_1^2 + T_2 D_2^2)\right]\right\} \tag{51}$$

A Appendix: Time-integrated vs. time-dependent analysis for μ.

If we use the integrated quantities, the correspondence with the partial derivatives of the likelihood function is

$$
\begin{aligned}
-\frac{\partial \langle \mathcal{L} \rangle}{\partial \mu^2} &= \frac{1}{F_{B^0 t, B^0 r}} \left(\frac{\partial F_{B^0 t, B^0 r}}{\partial \mu} \right)^2 + \frac{1}{F_{\overline{B}^0 t, B^0 r}} \left(\frac{\partial F_{\overline{B}^0 t, B^0 r}}{\partial \mu} \right)^2 + \frac{1}{F_{B^0 t, \overline{B}^0 r}} \left(\frac{\partial F_{B^0 t, \overline{B}^0 r}}{\partial \mu} \right)^2 + \\
& \frac{1}{F_{\overline{B}^0 t, \overline{B}^0 r}} \left(\frac{\partial F_{\overline{B}^0 t, \overline{B}^0 r}}{\partial \mu} \right)^2 + \frac{1}{F_{no\ t, B^0 r}} \left(\frac{\partial F_{no\ t, B^0 r}}{\partial \mu} \right)^2 + \frac{1}{F_{no\ t, \overline{B}^0 r}} \left(\frac{\partial F_{no\ t, \overline{B}^0 r}}{\partial \mu} \right)^2 \\
&= \frac{T}{4} \left[\frac{2(D-A)^2}{1-DA} + \frac{2(D+A)^2}{1+DA} \right] + \frac{1}{\frac{1}{2}(1-T)} 2 (\tfrac{1}{2} A_{ave} T)^2 \\
&= \frac{A_{ave}^2 T^2}{1-T} + T A_{ave}^2 + T D^2 \frac{(1-A_{ave}^2)^2}{1-D^2 A_{ave}^2}
\end{aligned}
\tag{52}
$$

Using $T = 0.7, A_{ave} = 0.652, D = 0.55$ we find for the first term 0.694, for the second 0.298 and for the third 0.080. The total is 1.07. In a complete maximum likelihood analysis the second and third contributions would increase:

$$
\begin{aligned}
T A_{ave}^2 &\rightarrow \int_0^\infty dt \Gamma e^{-\Gamma t} T A^2 \\
&= T \frac{1+2x^2}{1+4x^2} \\
T D^2 \frac{(1-A_{ave}^2)^2}{1-D^2 A_{ave}^2} &\rightarrow \int_0^\infty dt \Gamma e^{-\Gamma t} T D^2 \frac{(1-A^2)^2}{1-D^2 A^2}
\end{aligned}
\tag{53}
$$

We have a good bound on the second integral since

$$
\begin{aligned}
\int_0^\infty du e^{-u} \sin^4 xu < \int_0^\infty du e^{-u} \frac{\sin^4 xu}{1-D^2 \cos^2 xu} < \int_0^\infty du e^{-u} \sin^2 xu \\
\frac{24x^2}{(1+4x^2)(1+16x^2)} < \int_0^\infty du e^{-u} \frac{\sin^4 xu}{1-D^2 \cos^2 xu} < \frac{2x^2}{1+4x^2}
\end{aligned}
\tag{54}
$$

We have the numerical values

$$
\frac{A_{ave} T^2}{1-T} = 0.694
$$

$$
\begin{aligned}
T\frac{1+2x^2}{1+4x^2} &= 0.461 \\
TD^2\int_0^\infty due^{-u}\frac{\sin^4 xu}{1-D^2\cos^2 xu} &= 0.21(0.229-0.340) \\
&= (0.048-0.071) \qquad (55)
\end{aligned}
$$

Using the central value for the third we get a sum in full maximum likelihood of 1.21. Multiplying by 6600 we get 802.

What is really going on here is that the simple analysis is a binned likelihood fit, with just six bins, corresponding to B^0 tag, $\overline{B}^0$ tag, and no tag for both B^0 reco and $\overline{B}^0$ reco. When we include the time dependence we are going to an unbinned likelihood, which improves (slightly!) the precision. In fact the improvement in the second partial derivative is only from 1.07 top 1.21, which amounts to only a reduction of 6% in the uncertainty.