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Dispersive barotropic equations for stratified mesoscale ocean dynamics

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Dispersive effects induced by weak hydrostatic imbalance in the presence of topography and stratification are incorporated into a new model of barotropic (vertically integrated) mesoscale ocean dynamics. This barotropic model is obtained by first expanding the solutions of three dimensional Euler–Boussinesq equations in a regular perturbation expansion in terms of the several small dimensionless parameters appropriate to mesoscale ocean dynamics. Vertically integrating from a fixed bottom topography to the free surface interface with the atmosphere and balancing orders in the expansion at fourth order in the small aspect ratio parameter then yields a system of reduced barotropic equations. These reduced barotropic equations are considerably more tractable than the starting equations and have appropriate limits to known dispersive wave equations such as the forced Kadomtsev–Petviashvili equation in one limit, and the rotating shallow water equations in another. This new model of barotropic ocean dynamics may be of use in developing numerical algorithms for global ocean circulation modeling.

1. Introduction

The fundamental equations describing ocean dynamics are the incompressible Navier–Stokes equations written in a rotating frame, with appropriate boundary conditions imposed at the free surface and fixed bottom boundaries. To complete the problem statement, one must add to these equations a thermodynamic equation of state for the density's dependence on temperature and salinity, as well as dynamical equations governing the advection and diffusion of these two thermodynamic quantities.

In three-dimensional numerical simulations of global ocean circulation, the code currently accepted as the standard in the oceanographic community is the finite difference model associated with the names of Bryan, Cox, Semtner, and Chervin (see ref. [9]). This model takes advantage of the smallness of the ratio of material to gravity-wave speeds, by distinguishing between the 2D barotropic (or vertically integrated) velocity field and the deviation from barotropic, called baroclinic, 3D velocity field. Since the fastest gravity wave speed, to a good approximation, only affects the barotropic velocity field, the equations determining this velocity field are solved taking, for instance, small time substeps in order to assure resolution of the gravity wave time scale, while the baroclinic velocity field, which evolves on a much longer time scale, is kept fixed and treated as a forcing. The baroclinic field is then computed using the full time step, updated and the process repeated. The effective speed of the computation is thus strongly influenced by the barotropic velocity field solver. Hence, a scheme that would provide a good approximation for the barotropic velocity field for sufficiently long times would be extremely valuable in cutting down the computation time. Our approach to developing such a scheme is to attempt to take advantage of *all* the small parameters in the problem. This paper reports preliminary results in this direction.

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Ocean dynamics is characterized by several dimensionless parameters that are small in magnitude. In particular, if one focuses on a resolution of about 50 km for the smallest resolved horizontal structures, then the aspect ratio between the largest vertical scale (the ocean's depth) and smallest interesting horizontal length scale is small (of order 0.1, or less). Other small dimensionless parameters are: the ratio of a typical fluid particle speed to the gravity wave speed (the ratio mentioned earlier as the basis for the finite difference model of Bryan et al.); the ratio between gravity wave amplitude and the ocean's depth; and the relative changes of temperature and salinity over the ocean's depth. In contrast to other treatments, the Rossby number measuring the ratio of the eddy turnover frequency to the Earth's rotation frequency is *not* taken to be small here, because of the applications envisioned for our model to relatively fine horizontal resolution.

Our analysis of this problem so far takes advantage of the scale separations implied by these small dimensionless parameters. To do this, we expand the ocean's dynamical variables in powers of these small parameters and seek a balance of scales in the vertically-integrated equations. In the work reported here, we study the nondissipative case and *retain* the small vertical acceleration terms in the Euler–Boussinesq equations that would otherwise be neglected in the so called "primitive equations" hydrostatic approximation (see, e.g., [2,9]). The balance of scales we have found results in a new closure scheme for the vertically-integrated dynamics of the ocean in the nondissipative case. The first advantage of this new closure scheme for the vertically-integrated equations is that it accounts for stratification, rotation and bottom topography in a two-dimensional set of equations, rather than the three-dimensional Euler–Boussinesq equations. The second advantage is that the boundary conditions are incorporated into the closure equations in their derivation, rather than being separately imposed. Both of these advantages allow the new closure model to be numerically simulated much more quickly and easily than the three-dimensional Euler–Boussinesq equations. Moreover, the new closure scheme can extend known equations in the classical Boussinesq dispersive water wave family to account for the effects of rotation, vorticity, stratification, and bottom topography.

In section 2 we nondimensionalize the Euler–Boussinesq equations by using the scaling appropriate to mesoscale ocean dynamics to identify small dimensionless parameters. We find a particular balance among these small parameters that eventually produces nontrivial dynamics from a regular perturbation expansion. The equations resulting from this expansion at fourth order in the small vertical-to-horizontal aspect ratio are integrated vertically and found to close as a dynamical system. Thus we obtain a set of closed barotropic equations that are *decoupled* from the baroclinic dynamics up to sixth order in the small aspect ratio. Finally we use the classic Marsigli example ([4], pp. 96–98), to provide an appropriate interpretation of the stratification variable summoned by our closure scheme. In section 3 we discuss the linearized equations, their dispersion relation and their energy balance. In section 4 we discuss the limits of the barotropic model to the forced Kadomtsev–Petviashvili ((f)KP) equation and the Korteweg-de Vries ((f)KdV) equation.

2. Derivation of the model equations

The dynamics of an inviscid, incompressible fluid in a three dimensional domain is described, in the Boussinesq approximation, by the motion equation

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\frac{1}{\rho_{\mathrm{ref}}}\boldsymbol{\nabla}_0 p + 2\Omega\, \boldsymbol{U} \times \hat{\boldsymbol{z}} - \frac{g}{\rho_{\mathrm{ref}}}(\rho_{\mathrm{ref}} + \rho)\hat{\boldsymbol{z}}, \tag{2.1}$$

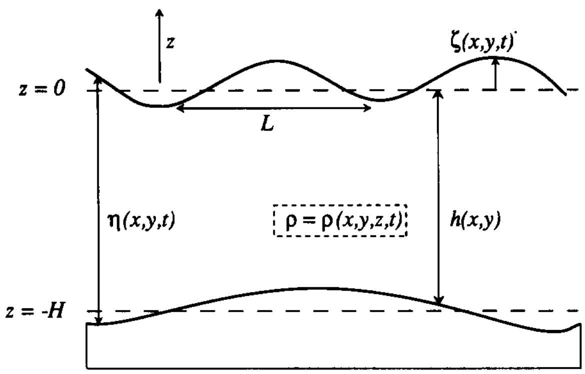

Fig. 1. The geometry of the fluid layer.

the incompressibility condition,

$$d\rho/ dt = 0, \tag{2.2}$$

and the conservation of mass (continuity),

$$\nabla_0 \cdot \boldsymbol{U} = 0. \tag{2.3}$$

In these equations $d/dt = \partial_t + \boldsymbol{U}\cdot\nabla_0$ is the material derivative, ∇_0 is the three dimensional gradient

$$\nabla_0 := (\partial_x, \partial_y, \partial_z) := (\nabla, \partial_z),$$

and $\boldsymbol{U}$ is the fluid velocity

$$\boldsymbol{U} := (u, v, w) := (\boldsymbol{u}, w).$$

Other notation is: density deviation ρ; pressure p; vertical unit vector $\hat{\boldsymbol{z}}$; and constant parameters, ρ_{ref}, the reference density; 2Ω the Coriolis parameter; and g, the gravitational acceleration.

Equation (2.2) is consistent with nondiffusive advection equations for temperature and salinity. (Since we are dealing with ideal fluids, all diffusivity is being neglected.) The kinematic boundary conditions appropriate for an inviscid fluid are (see fig. 1)

$$w = d\zeta/dt \quad \text{at } z = \zeta(x,y,t), \qquad w = -\boldsymbol{u}\cdot\nabla h \quad \text{at } z = -h(x,y), \tag{2.4}$$

and $\boldsymbol{U}$ is tangential on any vertical lateral boundaries (free-slip). The dynamic boundary condition is (neglecting surface tension)

$$p = \tilde{p} \quad \text{at } z = \zeta(x,y,t). \tag{2.5}$$

We nondimensionalize eqs. (2.1)–(2.5) by introducing six units L, H, c, $\mathcal{U}$, ζ_0 and ρ_{ref} and three dimensionless small parameters α, β, ϵ as follows:

$$(x,y,z) = L(x^*,y^*,\epsilon z^*), \quad \epsilon := \frac{H}{L} \ll 1, \quad t = \frac{L}{c}t^*, \quad c = \sqrt{gH},$$
$$(u,v,w) = \beta c\ (u^*,v^*,\epsilon w^*), \quad \beta := \frac{\mathcal{U}}{c} \ll 1,$$
$$\rho = \rho_{\mathrm{ref}}\rho^*, \quad p = \rho_{\mathrm{ref}}c^2 p^*, \quad \zeta = \zeta_0\zeta^*, \quad \alpha := \frac{\zeta_0}{H} \ll 1. \tag{2.6}$$

By introducing these units we are preconditioning the solution to lie in a thin domain, and to have fluid velocities that are small compared to the gravity wave speed. Furthermore, the typical amplitude ζ_0 of the free surface above the equilibrium $z = 0$ is taken to be small compared to the depth H (that is $\alpha = \zeta_0/H \ll 1$). In these units the dimensionless rotation frequency becomes $\mathrm{Ro}^{-1} := 2\Omega L/\beta c = 2\Omega L/\mathcal{U}$, the inverse of the Rossby number (which is taken to be $\mathcal{O}(1)$ for mesoscale ocean dynamics).

In nondimensional form equations (2.1) and (2.2) become (after dropping $*$)

$$\beta\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla p + \mathrm{Ro}^{-1}\beta^2\boldsymbol{u}\times\hat{\boldsymbol{z}}, \quad \beta\epsilon\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{1}{\epsilon}\left(\frac{\partial p}{\partial z} + 1 + \rho\right), \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0, \tag{2.7}$$

where the material derivative is now $\mathrm{d}/\mathrm{d}t := \partial/\partial t + \beta\boldsymbol{u}\cdot\nabla + \beta w\partial/\partial z$. In addition to (2.7) we have the rescaled divergence-free condition with $\nabla_0 = (\partial_x, \partial_y, \epsilon^{-1}\partial_z)$,

$$u_x + v_y + w_z = 0 \tag{2.8}$$

and the boundary conditions

$$\beta w = \alpha\frac{\mathrm{d}\zeta}{\mathrm{d}t} \text{ at } z = \zeta(x,y,t), \quad w = -\boldsymbol{u}\cdot\nabla h \text{ at } z = -h(x,y). \tag{2.9}$$

The first equation in (2.9) implies $\alpha = \mathcal{O}(\beta)$. To keep the notation compact, in the following we will drop the parameter α in front of ζ, thinking of ζ and its derivatives as order $\mathcal{O}(\alpha)$ quantities. The system (2.7) is similar to system (2.1) in ref. [2], except that we ignore sound waves from the beginning, by imposing exact incompressibility. The oceanographic "primitive equations" [2,9] are obtained in dimensionless form by dropping the vertical acceleration term in (2.7), thereby strictly enforcing hydrostatic balance. This vertical acceleration term is *retained* here for the barotropic closure scheme we derive; it introduces additional dispersion arising from hydrostatic imbalance.

These notes focus on the effects of dispersion due to hydrostatic imbalance, topography and stratification on the barotropic (vertically integrated) dynamics. For this, we need the transport equations for vertically integrated quantities (see for example Wu [10]). Namely, the material derivative of a function $f = f(x,y,z,t)$ satisfies the equation

$$\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = \frac{\partial}{\partial t}[f], + \beta\nabla\cdot[f\boldsymbol{u}], \tag{2.10}$$

in which straight brackets denote vertical integration:

$$[f] := \int_{-h}^{\zeta} f(x,y,z,t)\,\mathrm{d}z =: (\zeta + h)\overline{f}. \tag{2.11}$$

So $[f]$ is the integral of f across the whole fluid layer $-h < z < \zeta$ and $\overline{f}$ is the vertical average of f.

In particular, setting $f = 1, \boldsymbol{u}, w$, and ρ, gives respectively,

$$
\begin{aligned}
&\partial_t \eta + \beta \nabla \cdot (\eta \overline{\boldsymbol{u}}) = 0, \quad \eta := \zeta + h, \\
&\beta \partial_t (\eta \overline{\boldsymbol{u}}) + \beta^2 \nabla \cdot (\eta \overline{\boldsymbol{u}\boldsymbol{u}}) = -\eta \overline{\nabla p} + \mathrm{Ro}^{-1} \beta^2 \eta \overline{\boldsymbol{u}} \times \hat{z}, \\
&\beta \epsilon \partial_t (\eta \overline{w}) + \beta^2 \epsilon \nabla \cdot (\eta \overline{w \boldsymbol{u}}) = -\frac{\eta}{\epsilon} (\overline{\partial_z p} + (1 + \overline{\rho})), \\
&\partial_t (\eta \overline{\rho}) + \beta \nabla \cdot (\eta \overline{\rho \boldsymbol{u}}) = 0.
\end{aligned}
\tag{2.12}
$$

These four equations represent conservation of volume, horizontal momentum, vertical momentum, and mass, respectively. Although exact, these relations cannot be used as equations of motion in general, since new unknowns in the form of higher order moments appear as a consequence of averaging. Our aim here is to obtain closure for these equations under suitable approximations for the dependent variables $\boldsymbol{u}, w, p$.

We now introduce regular perturbation expansions with small parameters δ and γ whose magnitudes relative to α, β and ϵ are to be determined,

$$
\begin{aligned}
&\boldsymbol{u} = \boldsymbol{u}_0 + \delta \boldsymbol{u}_1 + \mathcal{O}(\delta^2), \quad w = w_0 + \delta w_1 + \mathcal{O}(\delta^2), \\
&p = p_0 + \gamma p_1 + \gamma^2 p_2 + \mathcal{O}(\gamma^3), \quad \rho = \gamma \rho_1 + \gamma^2 \rho_2 + \mathcal{O}(\gamma^3).
\end{aligned}
\tag{2.13}
$$

We substitute these expansions into the rescaled motion equations (2.7) and seek a particular balance in the equations resulting at order $\mathcal{O}(\gamma^2)$ that goes beyond geostrophic and quasi-geostrophic balance. In the rescaled horizontal motion equation (2.7), we require $\gamma \nabla p_1$ to be of the same order as $\beta \partial \boldsymbol{u}_0 / \partial t$, hence

$$\beta = \mathcal{O}(\gamma).$$

In this ordering, hydrostatic balance in the vertical motion equation will be broken at order $\mathcal{O}(\gamma^2)$, since $(\gamma^2/\epsilon)(\partial p_2/\partial z + \rho_2)$ is of the same order as $\beta \epsilon \partial w_0 / \partial t$, which implies

$$\gamma^2 = \mathcal{O}(\epsilon^2 \beta) \quad \text{and hence} \quad \beta = \mathcal{O}(\epsilon^2).$$

At the next order in γ, the term $\gamma^2 \nabla p_2$ in the rescaled horizontal motion equation balances with $\delta \beta \partial \boldsymbol{u}_1 / \partial t$, provided $\gamma^2 = \mathcal{O}(\delta \beta)$, which implies

$$\gamma = \mathcal{O}(\delta) = \mathcal{O}(\epsilon^2).$$

These considerations fix the order of the parameters α, β, γ and δ relative to ϵ. Namely $\mathcal{O}(\alpha) = \mathcal{O}(\beta) = \mathcal{O}(\gamma) = \mathcal{O}(\delta) = \mathcal{O}(\epsilon^2)$. (For convenience in what follows we shall suppress coefficients of order $\mathcal{O}(1)$ and simply identify $\gamma^2 = \beta \epsilon^2$ etc.) The order of the parameter Ro^{-1} is still free with respect to ϵ. Geostrophic balance is recovered in this ordering when $\mathrm{Ro} = \mathcal{O}(\beta^2)$. Quasi-geostrophy is recovered when $\mathrm{Ro} = \mathcal{O}(\beta)$ (see [8]). For mesoscale ocean dynamics $\mathrm{Ro} = \mathcal{O}(1)$; therefore the Coriolis force first comes into the mesoscale dynamics at order $\mathcal{O}(\beta^2)$, in the $\boldsymbol{u}_1$ equation.

Having found the scale relations that introduce nonhydrostatic dispersive effects at order $\mathcal{O}(\beta^2)$, we next use these relations to derive the equations of dispersive barotropic ocean dynamics. In the w equation at order $\mathcal{O}(\epsilon^{-1})$ we have hydrostatic equilibrium, i.e.,

$$\frac{\partial p_0}{\partial z} + 1 = 0, \tag{2.14}$$

and from the $\boldsymbol{u}$ equation in (2.7) at $\mathcal{O}(1)$ we have $\nabla p_0 = 0$. Hence $p_0 = -z +$ const.. Consider the w equation in (2.7) at $\mathcal{O}(\epsilon)$,

$$\frac{\partial p_1}{\partial z} + \rho_1 = 0 \quad \Rightarrow \quad p_1 = -\int^z \rho_1 \, \mathrm{d}z' + \psi(x,y,t), \tag{2.15}$$

and compare with the $\boldsymbol{u}$ equation at order $\mathcal{O}(\epsilon^2)$,

$$\nabla p_1 = -\frac{\partial \boldsymbol{u}_0}{\partial t}. \tag{2.16}$$

Taking cross derivatives in the two expressions above for p_1 gives

$$\frac{\partial^2 \boldsymbol{u}_0}{\partial t \partial z} = \nabla \rho_1. \tag{2.17}$$

Since advection of mass implies, at order $\mathcal{O}(\gamma)$,

$$\frac{\partial \rho_1}{\partial t} = 0, \tag{2.18}$$

the right hand side of (2.17) is independent of time. Hence, integration of (2.17) implies $\boldsymbol{u}_0$ is expressible as

$$\boldsymbol{u}_0 = t \int^z \nabla \rho_1 \, \mathrm{d}z' + \boldsymbol{u}_0'(x,y,t) + \tilde{\boldsymbol{u}}_0(x,y,z), \tag{2.19}$$

and the first term grows linearly in time unless

$$\nabla \rho_1 = 0. \tag{2.20}$$

That is, any vertical shear at order $\mathcal{O}(\beta)$ must be time independent; otherwise, secular terms will develop. Thus the term $\tilde{\boldsymbol{u}}_0(x,y,z)$ is the only z-dependence possible for $\boldsymbol{u}_0$ in the present balancing scheme. For definiteness, we will set $\tilde{\boldsymbol{u}}_0 = 0$ for the rest of this paper and drop the prime for $\boldsymbol{u}_0'(x,y,t)$. We will examine the effects of the time independent vertical shear term elsewhere, in [3]. Equations (2.18) and (2.20) imply $\rho_1 = \rho_1(z)$, and we will take for the equilibrium density stratification

$$\rho_1(z) = sz + \text{const.}, \tag{2.21}$$

where s is a negative constant. This equilibrium density stratification is typical over most of the ocean [7]. Since $\partial \rho_1 / \partial t = 0$ we must regard the dynamics of the fluid as taking place in a nonhomogeneous medium of fixed background density $\rho_{\text{ref}} + \gamma \rho_1(z)$.

Incompressibility implies

$$\frac{\partial w_0}{\partial z} = -\nabla \cdot \boldsymbol{u}_0 \quad \Rightarrow \quad w_0 = -z \nabla \cdot \boldsymbol{u}_0 + \phi(x,y,t). \tag{2.22}$$

We can evaluate the function ϕ from the boundary condition at $z = -h$,

$$-\boldsymbol{u}_0 \cdot \nabla h = w_0|_{-h} = h\nabla \cdot \boldsymbol{u}_0 + \phi,$$

hence (2.22) becomes

$$w_0 = -\nabla \cdot (z + h)\boldsymbol{u}_0, \tag{2.23}$$

which will be useful later. In the w equation at order $\mathcal{O}(\beta\epsilon^2)$ we have

$$\frac{\partial p_2}{\partial z} + \rho_2 = -\frac{\partial w_0}{\partial t}, \tag{2.24}$$

and so from (2.23)

$$p_2(x,y,z,t) = \frac{1}{2}z^2\partial_t\nabla \cdot \boldsymbol{u}_0 + z\partial_t\nabla \cdot (h\boldsymbol{u}_0) - \int^z \rho_2\,\mathrm{d}z' + \psi'(x,y,t). \tag{2.25}$$

Only partial differentiation with respect to time appears in (2.25), since our ordering scheme requires us to neglect in the material time derivative the advective transport term that would otherwise appear in (2.24). Next, we eliminate both ψ in (2.15) and ψ' in (2.25) by using the free boundary conditions for the pressure

$$\begin{aligned}\widetilde{p} := p|_{z=\zeta} &= [p_0 + \gamma p_1 + \gamma^2 p_2 + \mathcal{O}(\gamma^3)]|_{z=\zeta} \\ &= \text{const.} - \zeta + \gamma\left(-\int^\zeta \rho_1(z)\,\mathrm{d}z + \psi(x,y,t)\right) + \gamma^2 p_2(x,y,\zeta,t) + \mathcal{O}(\gamma^3).\end{aligned} \tag{2.26}$$

Hence the difference between the pressure at any z and the specified surface pressure, $\widetilde{p}$, is

$$\Delta p := p - \widetilde{p} = \zeta + \gamma\int_z^\zeta \rho_1(z')\,\mathrm{d}z' + \gamma^2\Delta p_2, \tag{2.27}$$

where

$$\Delta p_2 = \int_z^\zeta \rho_2(x,y,z',t)\,\mathrm{d}z' + \frac{1}{2}z^2\partial_t\nabla \cdot \boldsymbol{u}_0 + z\partial_t\nabla \cdot (h\boldsymbol{u}_0) + \mathcal{O}(\alpha). \tag{2.28}$$

We remark that $p_2|_\zeta$ is of the same order as ζ, and as the transport terms neglected in (2.24), namely $\mathcal{O}(\alpha)$. For the reader's convenience the scaling and matching considerations so far are summarized order by order in fig. 2.

We still need to evaluate $\overline{\nabla p}$, where

$$\nabla p = \nabla\widetilde{p} + [1 + \gamma\rho_1(0)]\nabla\zeta + \gamma^2\nabla(\Delta p_2) + \mathcal{O}(\gamma^3, \alpha^2\gamma). \tag{2.29}$$

Evaluating the average of (2.28) leads to

$$\overline{\nabla(\Delta p_2)} = \overline{(z+h)\nabla\rho_2} - \left(\tfrac{1}{2}h\nabla\partial_t(\nabla \cdot (h\boldsymbol{u}_0)) - \tfrac{1}{6}h^2\nabla\partial_t(\nabla \cdot \boldsymbol{u}_0)\right) + \mathcal{O}(\alpha). \tag{2.30}$$

"Flow Chart" of Derivation

$O(\varepsilon^{-1})$	*w-eq*	$\Rightarrow$	$p_0 = -z + \pi(x,y,t)$
$O(1)$	*u-eq*	$\Rightarrow$	$\nabla p_0 = 0$
$O(\varepsilon)$	*w-eq*	$\Rightarrow$	$p_1 = \int^z \rho_1\,(z)dz$
$O(\varepsilon^2)$	*u-eq*	$\Rightarrow$	$\nabla \rho_1 = 0$
$O(\varepsilon^2)$	*incompressibility and b.c. @ z= –h*	$\Rightarrow$	$\nabla\cdot(z+h)\boldsymbol{u}_0 = -w_0$
$O(\varepsilon^3)$	*w-eq*	$\Rightarrow$	$p_2 = p_2\,(\boldsymbol{u}_0,\rho_2)$

Fig. 2. "Flow-chart" of the derivation of eqs. (2.37), order by order in the asymptotic expansion.

Here the first term comes from taking the horizontal gradient inside the integral in (2.28), neglecting surface terms of order $\mathcal{O}(\alpha)$ then averaging and exchanging order of integration. The second set of terms in (2.30) again neglects surface terms and transport terms of order $\mathcal{O}(\alpha)$.

For closure in the barotropic equations we need a dynamical equation for the first term in (2.30). From equation (2.7) at order $\mathcal{O}(\beta^2)$, multiplied by $z+h$ and vertically averaged, we obtain after some algebra

$$
\begin{aligned}
\partial_t\overline{(z+h)\nabla p_2} + \mathcal{O}(\alpha) &= -s\overline{(z+h)\nabla w_0} \\
\text{by eq. (2.23)} \qquad &= s\frac{h}{\eta}\left(\frac{h}{2}\nabla(\nabla\cdot(h\boldsymbol{u}_0)) - \frac{h^2}{6}\nabla(\nabla\cdot\boldsymbol{u}_0)\right) \\
&= s\left(\frac{h}{2}\nabla(\nabla\cdot(h\boldsymbol{u}_0)) - \frac{h^2}{6}\nabla(\nabla\cdot\boldsymbol{u}_0)\right) + \mathcal{O}(\alpha).
\end{aligned}
\tag{2.31}
$$

We remark that a choice of the density stratification different from the linear law assumed in (2.21) would lead to the same structure for the right hand side of (2.31), i.e., the terms $h\nabla(\nabla\cdot(h\boldsymbol{u}_0))$ and $h^2\nabla(\nabla\cdot\boldsymbol{u}_0)$ would still be there, but with different constant coefficients depending on the particular form of $\rho_1(z)$.

Now we are in position to write a closed set of equations for the stratified barotropic fluid. Referring to the first and second equations in (2.12) gives

$$
\bar{\boldsymbol{u}}_t + \beta(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} = -\frac{1}{\beta}\overline{\nabla p} + \mathrm{Ro}^{-1}\beta\bar{\boldsymbol{u}}\times\hat{\boldsymbol{z}} - \frac{\beta}{\eta}\nabla\cdot\eta(\overline{\boldsymbol{u}\boldsymbol{u}} - \bar{\boldsymbol{u}}\,\bar{\boldsymbol{u}}).
\tag{2.32}
$$

Since $\boldsymbol{u} = \boldsymbol{u}_0 + \delta\boldsymbol{u}_1 + \mathcal{O}(\delta^2)$ and $\boldsymbol{u}_0$ has been taken to be independent of z, we have

$$
\overline{\boldsymbol{u}\boldsymbol{u}} - \bar{\boldsymbol{u}}\,\bar{\boldsymbol{u}} = \delta^2(\overline{\boldsymbol{u}_1\boldsymbol{u}_1} - \bar{\boldsymbol{u}}_1\bar{\boldsymbol{u}}_1) + \mathcal{O}(\delta^3),
\tag{2.33}
$$

and hence the last term in (2.32) is negligible. After substituting for $\overline{\nabla p}$ from (2.29) and (2.30) we find a closed set of dynamical equations for ζ, $\bar{\boldsymbol{u}}$ and A (since $\bar{\boldsymbol{u}} = \boldsymbol{u}_0 + \mathcal{O}(\delta)$, and reinstating the order parameter α for ζ):

$$\alpha\frac{\partial\zeta}{\partial t}+\beta\nabla\cdot(\alpha\zeta+h)\bar{\boldsymbol{u}}=0,$$
$$\beta\frac{\partial\bar{\boldsymbol{u}}}{\partial t}+\beta^2(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}}+\mathrm{Ro}^{-1}\beta^2\hat{\boldsymbol{z}}\times\bar{\boldsymbol{u}}=-\gamma\nabla\tilde{p}-\alpha(1+\gamma\tilde{\rho})\nabla\zeta-\boldsymbol{A}+\frac{\partial\boldsymbol{D}}{\partial t},$$
$$\frac{\partial\boldsymbol{A}}{\partial t}=s\boldsymbol{D}, \tag{2.34}$$

where we define

$$\boldsymbol{A}:=\gamma^2\overline{(z+h)\nabla\rho_2}, \tag{2.35}$$

$$\boldsymbol{D}:=\gamma\beta\left(\frac{h}{2}\nabla(\nabla\cdot(h\bar{\boldsymbol{u}}))-\frac{h^2}{6}\nabla(\nabla\cdot\bar{\boldsymbol{u}})\right), \tag{2.36}$$

and $\tilde{p}(x,y,t)$ and $\tilde{\rho}$ denote the specified surface pressure and the surface equilibrium density deviation, respectively. We remark that the first equation in (2.34) is exact, while the second and third equations have an error term of order $\mathcal{O}(\beta\epsilon^4,\ \gamma^3$ etc.). Restoring dimensions now gives our final equations

$$\frac{\partial\zeta}{\partial t}+\nabla\cdot(\zeta+h)\bar{\boldsymbol{u}}=0,$$
$$\frac{\partial\bar{\boldsymbol{u}}}{\partial t}+(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}}+2\Omega\hat{\boldsymbol{z}}\times\bar{\boldsymbol{u}}=-\frac{1}{\rho_{\mathrm{ref}}}\nabla\tilde{p}-g\left(1+\frac{\tilde{\rho}}{\rho_{\mathrm{ref}}}\right)\nabla\zeta-\frac{g}{\rho_{\mathrm{ref}}}\boldsymbol{A}+\frac{\partial\boldsymbol{D}}{\partial t},$$
$$\frac{\partial\boldsymbol{A}}{\partial t}=\sigma\boldsymbol{D}, \tag{2.37}$$

where now

$$\boldsymbol{A}:=\overline{(z+h)\nabla\rho}, \tag{2.38}$$

$$\boldsymbol{D}:=\left(\frac{h}{2}\nabla(\nabla\cdot(h\bar{\boldsymbol{u}}))-\frac{h^2}{6}\nabla(\nabla\cdot\bar{\boldsymbol{u}})\right), \tag{2.39}$$

and $\sigma:=s\gamma\rho_{\mathrm{ref}}/H=d\rho_1/\mathrm{d}z$ is a negative constant.

Equations (2.37) are dispersive shallow water equations that incorporate effects of weak deviations from hydrostatic balance, weak stratification and strong, $\mathcal{O}(1)$, topography. The key approximations in deriving equations (2.37) are: weak baroclinic horizontal velocity dependence, imposed after equation (2.20); and weak horizontal density gradients. The latter of these approximations is imposed in order to eliminate secular velocity growth. The effects of the mean vertical shear still allowed by (2.19) will be addressed elsewhere [3]. Equations (2.37) restrict to those of Wu [10], provided the stratification is absent ($\boldsymbol{A}=0$ and $\sigma=0$), and there is no rotation ($\Omega=0$). In Wu's derivation, the initial flow has no vorticity. Hence, there exists a velocity potential for $\boldsymbol{U}$ at all times, and the asymptotic expansion proceeds in terms of this potential. In this case, the mean vertical shear $\tilde{\boldsymbol{u}}_0(x,y,z)$ in (2.19) is absent.

We notice that the solutions $\bar{\boldsymbol{u}}$ of eqs. (2.37) give information about $\boldsymbol{u}_1$ and ρ_2. In fact, at order $\mathcal{O}(\delta\beta)$, we have

$$\frac{\partial\boldsymbol{u}_1}{\partial t}+(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0+\mathrm{Ro}^{-1}\hat{\boldsymbol{z}}\times\boldsymbol{u}_0=-\nabla p_2, \tag{2.40}$$

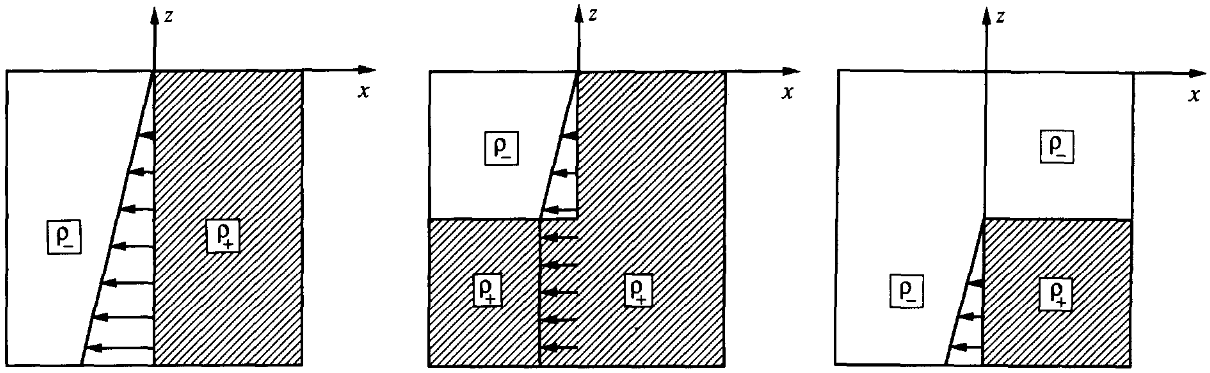

Fig. 3. Three different initial conditions for Marsigli's experiment. The vertically averaged forces acting on the partition at $x = 0$ are: (a) $-\gamma^2(gh/2/)\Delta\rho$; (b) $-\gamma^2(3gh/8/)\Delta\rho$; (c) $-\gamma^2(gh/8/)\Delta\rho$.

where p_2 is given by (2.28) in terms of $\boldsymbol{u}_0$ and powers of z, after ρ_2 has been determined from

$$\frac{\partial \rho_2}{\partial t} = -sw_0. \tag{2.41}$$

Hence, replacing $\boldsymbol{u}_0$ $(\equiv \bar{\boldsymbol{u}} + \mathcal{O}(\delta))$ by $\bar{\boldsymbol{u}}$ in (2.40) and (2.41), and neglecting terms of order $\mathcal{O}(\delta)$ gives a system of equations recovering $\boldsymbol{u}_1(x,y,z,t)$ and $\rho_2(x,y,z,t)$.

In the absence of stratification ($\sigma = 0$) and without the dispersion caused by hydrostatic pressure imbalance ($\boldsymbol{D} = 0$), these equations reduce to the standard, topographically forced, rotating shallow water equations. Dropping overbars and surface forcing terms, these are

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\eta\boldsymbol{u} = 0, \quad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + 2\Omega\hat{\boldsymbol{z}}\times\boldsymbol{u} = -g\nabla\eta + g\nabla h. \tag{2.42}$$

It remains to give an interpretation of the acceleration due to stratification, $-(g/\rho_{\text{ref}})\boldsymbol{A}$. For this, we refer to ancient experiments of Marsigli, as described in [4].

In the seventeenth century, L.M. Marsigli considered convective adjustment due to horizontal density gradients as the mechanism for producing the undercurrent flowing through the Bosphorus Strait toward the Black Sea from the Mediterranean. To verify this mechanism, Marsigli set up laboratory experiments with a partition separating two compartments containing water of different salinities. When the partition was removed, water at lower depths would flow from the compartment of higher salinity (as in the Mediterranean) toward the compartment of lower salinity (as in the Black Sea), thus explaining the mechanism for the undercurrent. Of course, the explanation comes from differences in hydrostatic force for the two compartments, which is precisely the effect being modeled in eqs. (2.37) by the acceleration vector $-(g/\rho_{\text{ref}})\boldsymbol{A}$. To see this, we compute the hydrostatic forces acting on the partition for the set-up shown in fig. 3a. Here we assume for simplicity $\rho = \rho_{\text{ref}} + \gamma^2\rho_2$, i.e., no equilibrium stratification ($\rho_1 = 0$), and restrict density variations to ρ_2 only. Immediately after the removal of the partition, the difference in hydrostatic forces along the partition would yield an initial acceleration field for the fluid particles. Namely, a (small) force $\Delta\boldsymbol{f}$

$$\Delta\boldsymbol{f} = -g\gamma^2(\rho_+ - \rho_-)z\hat{\boldsymbol{x}} := -g\gamma^2\Delta\rho z\hat{\boldsymbol{x}}$$

is directed along the x-axis from the higher-density ($\gamma^2\rho_+$) compartment toward the lower-density ($\gamma^2\rho_-$) one. Vertically averaging this force gives initial acceleration of a column of water located at the partition, i.e., $-\frac{1}{2}\gamma^2(gh\Delta\rho/\rho_{\text{ref}})$. It is easy to see from the definition (2.38) that this vertical average is equal to the acceleration vector $-(g/\rho_{\text{ref}})\boldsymbol{A}$ for this density distribution. Similarly, calculating differences in hydrostatic forces for other simple initial density distributions (such as in figs. 3b,c) again yields the same value as $-(g/\rho_{\text{ref}})\boldsymbol{A}$ for the initial acceleration of the barotropic motion. The ensuing dynamics will involve oscillation and circulation generated by the gravity wave force and the Coriolis force. This dynamics will be discussed in more detail elsewhere.

3. Linearized dynamics

In this section we discuss the linearized equations and their dispersion relation. We also give a brief description of the energy balance for the linearized dynamics.

Linearizing the system (2.37) around $\boldsymbol{u} = \boldsymbol{A} = \zeta = 0$ gives

$$\begin{aligned}
\partial_t\zeta &= -\nabla\cdot h\boldsymbol{u},\\
\partial_t\boldsymbol{u} &= 2\Omega\boldsymbol{u}\times\widehat{\boldsymbol{z}} - g\left(1+\frac{\widetilde{\rho}}{\rho_{\text{ref}}}\right)\nabla\zeta - \frac{g}{\rho_{\text{ref}}}\boldsymbol{A} + \partial_t\boldsymbol{D},\\
\partial_t\boldsymbol{A} &= \sigma\boldsymbol{D},
\end{aligned}\tag{3.1}$$

where overbars for $\bar{\boldsymbol{u}}$ have been dropped for ease of notation. Taking h = const., which implies

$$\boldsymbol{D} = \frac{h^2}{3}\nabla(\nabla\cdot\boldsymbol{u}),$$

and eliminating ζ and $\boldsymbol{A}$, reduces the system (3.1) to a dispersive vector wave equation for $\boldsymbol{u}$,

$$\partial_t^2\boldsymbol{u} = 2\Omega\partial_t\boldsymbol{u}\times\widehat{\boldsymbol{z}} + g\left(1+\frac{\widetilde{\rho}}{\rho_{\text{ref}}}\right)h\nabla(\nabla\cdot\boldsymbol{u}) - \frac{g\sigma h^2}{3\rho_{\text{ref}}}\nabla(\nabla\cdot\boldsymbol{u}) + \frac{h^2}{3}\nabla(\nabla\cdot\partial_t^2\boldsymbol{u}).\tag{3.2}$$

Substituting the periodic travelling wave form of the solution

$$\boldsymbol{u}(x,y,t)\to\exp[i(k_x x + k_y y - \omega t)]\,\boldsymbol{u}(k_x,k_y,\omega)$$

yields the dispersion relation

$$\omega^2 = \frac{4\Omega^2 + |\boldsymbol{k}|^2\,[gh\,(1+\widetilde{\rho}/\rho_{\text{ref}}) - g\sigma h^2/3\rho_{\text{ref}}]}{1+h^2|\boldsymbol{k}|^2/3}\tag{3.3}$$

with $\boldsymbol{k} = (k_x,k_y)$.

In the absence of stratification and nonhydrostatic dispersion (the last two terms in (3.2)), this dispersion relation reduces to $\omega = \sqrt{4\Omega^2 + ghk^2}$ (with $k := |\boldsymbol{k}|$), the dispersion relation for Poincare waves [8]. The effect of stratification in the absence of dispersion arising from hydrostatic imbalance is to shift the wave phase speed to $\omega/k = \sqrt{4\Omega^2 + ghk^2(1+\widetilde{\rho}/\rho_{\text{ref}}) - g\sigma h^2/3\rho_{\text{ref}}}/k$ (which for the ocean is generally a fraction of a percent). The effect of nonhydrostatic dispersion is to regularize the frequency at high wave numbers, as shown in fig. 4. As the wave number ranges from zero to infinity, the squared frequency ranges from $\omega_{\min}^2 = 4\Omega^2$ to $\omega_{\max}^2 = 3g(1+\widetilde{\rho}/\rho_{\text{ref}})/h - g\sigma/\rho_{\text{ref}}$. Next, solving

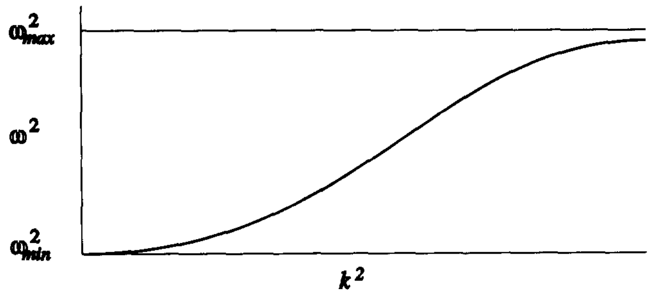

Fig. 4. The dispersion relation curve $\omega(k)$ for the linearized barotropic equations. $\omega^2_{\min} = 4\Omega^2$ and $\omega^2_{\max} = 3g(1+\tilde{p}/\rho_{\rm ref})/h - g\sigma/\rho_{\rm ref}$.

(3.3) for k^2 gives $k^2 = (3/h^2)(\omega^2 - \omega^2_{\min})/(\omega^2_{\max} - \omega^2)$. That is, level surfaces of ω are circles in $\boldsymbol{k}$-space, so the group velocity $\mathrm{d}\omega/\mathrm{d}\boldsymbol{k}$ is along the wave vector $\boldsymbol{k}$.

Multiplying the linearized motion equation in (3.1) by $h\boldsymbol{u}$ and rearranging using the other linearized equations leads to the wave-energy balance relation,

$$\begin{aligned}&\tfrac{1}{2}\partial_t\left(h|\boldsymbol{u}|^2 + g\zeta^2 + \boldsymbol{A}\cdot S\boldsymbol{A} + \tfrac{1}{3}h^3(\nabla\cdot\boldsymbol{u})^2 + h^2(\boldsymbol{u}\cdot\nabla h)(\nabla\cdot\boldsymbol{u}) + h(\boldsymbol{u}\cdot\nabla h)^2\right)\\ &\quad = -\nabla\cdot\left\{\boldsymbol{u}\left[gh\zeta - \tfrac{1}{3}h^3\partial_t\nabla\cdot\boldsymbol{u} - \tfrac{1}{2}h^2\partial_t\boldsymbol{u}\cdot\nabla h\right]\right\} + \boldsymbol{A}\cdot(\sigma S\boldsymbol{D} - gh\boldsymbol{u}),\end{aligned} \tag{3.4}$$

where $h^{-1}Sh^{-1}$ is a symmetric operator and S satisfies $\sigma S\boldsymbol{D} - gh\boldsymbol{u} = 0$, and for notational convenience we have set $\rho_{\rm ref} = 1$ and $\tilde{p} = 0$. Thus, the linearized equations preserve the total energy

$$H := \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\left[h|\boldsymbol{u}|^2 + g\zeta^2 + \boldsymbol{A}\cdot S\boldsymbol{A} + h\left(\tfrac{1}{2}h\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla h\right)^2 + \tfrac{1}{12}h^3(\nabla\cdot\boldsymbol{u})^2\right] \tag{3.5}$$

provided the linearized velocity is tangential to the boundaries of the domain. The energy H consists of the sum of the horizontal kinetic energy, the gravity wave energy, the stratification energy and the vertical kinetic energy coming from topography and hydrostatic imbalance. Since the conserved quadratic form H is definite, it may be used as a norm to assure Liapunov stability for the linearized dynamics, since in this conserved norm the linearized motion is always bounded. (See, e.g., Holm et al. [6] for considerations of stability of fluid equilibria based on conservation laws.) The modifications to the energy H for the case of the nonlinear flow (2.37) and the associated Hamiltonian structure is discussed in [3].

4. Limits to classical equations in the Boussinesq family

In this section we discuss the limits of the barotropic model (2.37) to the (forced) Kadomtsev–Petviashvili equation and the (forced) Korteweg-de Vries equation. The starting point is eqs. (2.34), which we rewrite, setting $\alpha = \beta = \gamma = \delta = \epsilon^2$, for each of the components $\bar{u}, \bar{v}$ of $\bar{\boldsymbol{u}}$ (dropping bars from now on) as

$$\begin{aligned}&\frac{\partial\zeta}{\partial t} + \frac{\partial}{\partial x}\left[(1+\epsilon^2\zeta)u\right] + \frac{\partial}{\partial y}\left[(1+\epsilon^2\zeta)v\right] = 0,\\ &\frac{\partial u}{\partial t} + \epsilon^2\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \mathrm{Ro}^{-1}\epsilon^2 v = -\frac{\partial\tilde{p}}{\partial x} - (1+\epsilon^2\tilde{p})\frac{\partial\zeta}{\partial x} - A_1 + \frac{\partial D_1}{\partial t},\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + \epsilon^2 \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) - \mathrm{Ro}^{-1}\epsilon^2 u = -\frac{\partial \widetilde{p}}{\partial y} - (1 + \epsilon^2 \widetilde{p})\frac{\partial \zeta}{\partial y} - A_2 + \frac{\partial D_2}{\partial t}, \\
&\frac{\partial \boldsymbol{A}}{\partial t} = s\boldsymbol{D},
\end{aligned} \tag{4.1}
$$

with $\boldsymbol{D}$ given for constant h ($h = 1$, say) by

$$
D_1 = \frac{\epsilon^2}{3}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right), \quad D_2 = \frac{\epsilon^2}{3}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right).
$$

Time dependent topographical variation could also be included in this formalism, but we are neglecting it for now.

First, we eliminate $\boldsymbol{A}$ by taking a time derivative of the equations for u and v above, obtaining

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + \epsilon^2 \frac{\partial}{\partial t}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) + \mathrm{Ro}^{-1}\epsilon^2 \frac{\partial v}{\partial t} = -\frac{\partial^2 \widetilde{p}}{\partial x \partial t} - (1 + \epsilon^2 \widetilde{p})\frac{\partial^2 \zeta}{\partial t \partial x} + sD_1 + \frac{\partial^2 D_1}{\partial t^2} \\
&\frac{\partial^2 v}{\partial t^2} + \epsilon^2 \frac{\partial}{\partial t}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) - \mathrm{Ro}^{-1}\epsilon^2 \frac{\partial u}{\partial t} = -\frac{\partial^2 \widetilde{p}}{\partial y \partial t} - (1 + \epsilon^2 \widetilde{p})\frac{\partial^2 \zeta}{\partial t \partial y} + sD_2 + \frac{\partial^2 D_2}{\partial t^2},
\end{aligned} \tag{4.2}
$$

Next, we introduce the stretched and moving coordinates

$$
\xi = x + t(1 + \delta_F \epsilon^2), \quad \lambda = \epsilon y, \quad \tau = \epsilon^2 t. \tag{4.3}
$$

These coordinates describe the slow evolution and slow transverse modulation of solutions travelling in the x direction at nearly the critical speed. The derivatives with repect to the old variables become, in terms of the new variables,

$$
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} = \epsilon \frac{\partial}{\partial \lambda}, \quad \frac{\partial}{\partial t} = \epsilon^2 \frac{\partial}{\partial \tau} + (1 + \delta_F \epsilon^2)\frac{\partial}{\partial \xi}. \tag{4.4}
$$

Notice that in the new ξ, λ coordinate system, stretched in the y direction, the parameter Ro^{-1} acquires a factor $1/\epsilon$ for the u equation and a factor ϵ for the v-equation in (4.2), respectively. We then expand the dependent variables

$$
\zeta = \zeta^{(0)} + \epsilon^2 \zeta^{(2)} + \cdots, \quad u = u^{(0)} + \epsilon^2 u^{(2)} + \cdots, \quad v = \epsilon v^{(1)} + \epsilon^3 v^{(3)} + \cdots, \tag{4.5}
$$

substitute in (4.2) and equate equal powers of ϵ. At order $\mathcal{O}(1)$ we obtain

$$
\zeta^{(0)}{}_{\xi} + u^{(0)}{}_{\xi} = 0 \tag{4.6}
$$

from the ζ-equation and

$$
\zeta^{(0)}{}_{\xi\xi} + u^{(0)}{}_{\xi\xi} = -\widetilde{p}_{\xi\xi}
$$

from the u-equation. Clearly these two equations are incompatible unless $\widetilde{p}_{\xi\xi} = 0$. That is, any nontrivial forcing from surface pressure has to be of higher order in ϵ. We will take it to be $\widetilde{p} = \mathcal{O}(\epsilon^2)$. If we assume that all the dependent variables vanish for $\xi \to \infty$, eq. (4.6) implies

$$\zeta^{(0)} = -u^{(0)}. \tag{4.7}$$

From the v-equation in (4.1) at order $\mathcal{O}(\epsilon)$ we obtain

$$v^{(1)}{}_{\xi} = -\zeta^{(0)}{}_{\lambda} + \mathrm{Ro}^{-1}u^{(0)}. \tag{4.8}$$

At order $\mathcal{O}(\epsilon^2)$ we obtain

$$\zeta^{(2)}{}_{\xi} + u^{(2)}{}_{\xi} + \zeta^{(0)}{}_{\tau} + \delta_F\zeta^{(0)}{}_{\xi} + (\zeta^{(0)}u^{(0)})_{\xi} + v^{(1)}{}_{\lambda} = 0 \tag{4.9}$$

from the ζ-equation in (4.1) and

$$u^{(2)}{}_{\xi\xi} + \zeta^{(2)}{}_{\xi\xi} + 2u^{(0)}{}_{\xi\tau} + (u^{(0)}u^{(0)}{}_{\xi})_{\xi} + \mathrm{Ro}^{-1}v^{(1)}{}_{\xi} + \tilde{p}\zeta^{(0)}{}_{\xi\xi} + \zeta^{(0)}{}_{\xi\tau} - \tfrac{1}{3}su^{(0)}{}_{\xi\xi} - \tfrac{1}{3}u^{(0)}{}_{\xi\xi\xi\xi}$$
$$= -\tfrac{1}{2}\tilde{p}_{\xi\xi}$$

from the u-equation in (4.2). Taking the derivative with respect to ξ of eq. (4.9), subtracting it from eq. (4.10) and taking into account (4.7) and (4.8) gives

$$[\zeta^{(0)}{}_{\tau} + \tfrac{1}{2}(\delta_F + \tilde{p} - \tfrac{1}{3}s)\zeta^{(0)}{}_{\xi} - \tfrac{3}{2}\zeta^{(0)}\zeta^{(0)}{}_{\xi} - \tfrac{1}{6}\zeta^{(0)}{}_{\xi\xi\xi}]_{\xi} - \tfrac{1}{2}\zeta^{(0)}{}_{\lambda\lambda} - \tfrac{1}{2}\mathrm{Ro}^{-2}\zeta^{(0)}$$
$$= -\tfrac{1}{2}\tilde{p}_{\xi\xi}, \tag{4.10}$$

which is the forced Kadomtsev–Petviashvili (fKP) equation for first term in the free surface expansion, with additional terms coming from the stratification (in Boussinesq approximation) and rotation through the parameters s ($\tilde{p}$) and Ro^{-2}, respectively. In particular, the rotation term proportional to $\mathrm{Ro}^{-2}\zeta^{(0)}$, in the homogeneous part of the KP equation, is in agreement with the one derived by Grimshaw [5]. In the absence of y (or λ) dependence (and rotation) this equation further reduces to the forced Korteweg-de Vries (fKdV) equation.

5. Conclusion

Our purpose in this paper has been to report preliminary results concerning the dispersive effects of hydrostatic imbalance and stratification in the barotropic approximation, by developing an asymptotic series for mesoscale ocean dynamics that closes at order $\mathcal{O}(\epsilon^4)$ for solutions with weak vertical shear and weak horizontal density gradients. The resulting closure scheme is the set of dispersive shallow water equations (2.37). These equations describe the barotropic (vertically integrated) motion, in the absence of dissipation and without any coupling to the baroclinic motion. The effect of the dispersive terms in (2.37) is to limit the range of frequencies available to the barotropic motion. (The barotropic motion has the highest frequencies present in the problem of mesoscale ocean dynamics.) Equations (2.37) also extend the rotating shallow water description to include hydrostatic imbalance and stratification, while they extend the classical Boussinesq family of equations such as KP to include stratification, rotation and vorticity. The relationships among these families of equations are sketched in fig. 5. In later work, we will address the effects of higher order ($\mathcal{O}(\epsilon^6)$) nonlinear transport, baroclinic coupling and dissipation.

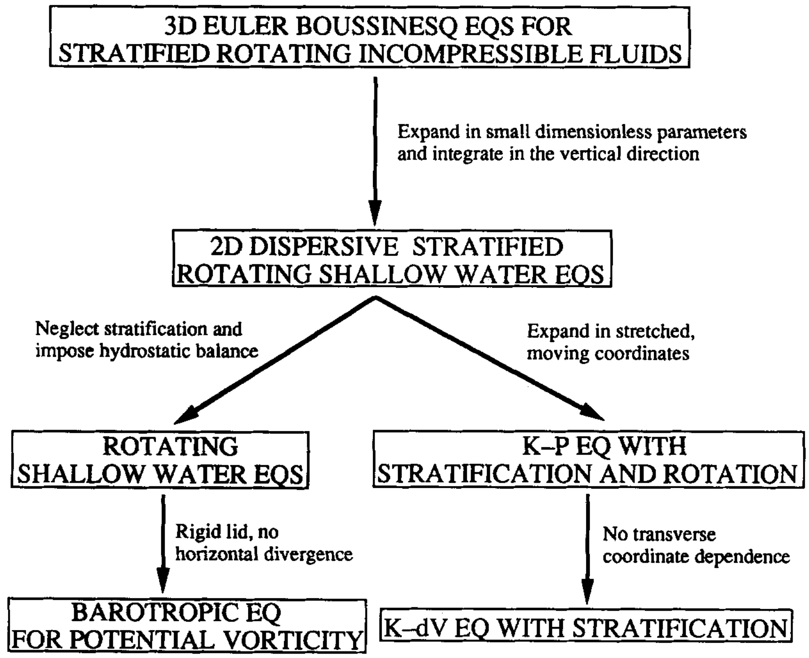

Fig. 5. Relationships among the various barotropic equations.

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