MECHANICAL PROPERTIES OF SEA ICE

Theoretical phase, July 1982 - April 1983

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Preamble

The November 1981 Technical Report (subsequently designated TR) gave a detailed account of the non-linear viscoelastic behaviour of ice, and the construction of differential operator relations with the minimal ingredients necessary to describe the observed qualitative response. It was shown that bi-axial test data is essential for the construction of a constitutive relation to describe the deviatoric (shear) response of an incompressible material. Until bi-axial data is available, simplifying assumptions to restrict the tensor structure of the relation must be made so that correlation with uni-axial data determines the remaining response coefficients. During this phase we have supposed that an appropriate differential relation is available, and have examined the formulation of various prototype boundary-value problems in plane stress and plane strain, and methods of solution.

Part I of this Report presents the construction of a small strain, small rotation approximation of the non-linear differential relation for a viscoelastic solid; that is, neglecting strain and rotation compared to unity, but retaining the essential non-linear character of the response coefficients. Elastic compressibility is included, but it is assumed that elastic dilatation and the instantaneous elastic shear strain are small compared to typical creep strains of order a few per cent, which introduces a small parameter ε measuring the relative magnitudes of elastic and creep strains. The parameter ε arises in a normalised dimensionless formulation of the constitutive relation, and must be monitored carefully in numerical approximations of the time-derivative balances. The initial modulus at constant strain-rate has magnitude ε^{-1} in the normalised variables. Plane stress and plane strain equations are spelled out, together with an implicit finite difference scheme for the time derivatives which reduces the system to a sequence of plane elasticity problems for a material with non-homogeneous moduli. A set of prototype boundary-value porblems are described. First the impact of a moving ice plate with a plane rigid wall which compresses the plate during deceleration, to be followed by full or partial rebound. A generalisation is the indentation of a smooth continuously curving structure. Next is the situation when an ice plate frozen to a rigid inclusion is set in motion, to investigate the contact stresses. Finally, a scheme to determine the in-plane stresses in a uniformly stressed region by embedding an elastic disk and measuring its boundary displacements.

Part II examines the uni-axial stress configuration described by the differential relation and compares it to non-linear elastic and linear viscoelastic models which each exhibit a crucial feature of the non-linear viscoelastic relation; namely the large initial modulus compared to subsequent stress-strain ratios. The uni-axial impact problem is analysed for each model. Explicit analytic solutions are obtained for linear elastic and linear viscoelastic materials, a solution requiring a simple numerical quadrature is given for the non-linear elastic model, but a finite difference (or other) procedure is required for the non-linear viscoelastic model. Being much simpler than the two-dimensional problems, it would serve as a numerical stability and accuracy test for the plane stress and strain program.

Part III derives linear viscoelastic solutions for the embedded rigid disk and elastic disk problems mentioned above, without calculations for explicit models. The Report closes with some Concluding Remarks highlighting the theoretical progress and indicating possible lines of further development.

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SMALL DEFORMATION OF A NONLINEAR VISCOELASTIC SOLID

BY

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INTRODUCTION

In many ice plate-structure interactions, the deformation of the ice, both strain and rotation, will remain relatively small prior to crushing or at least during an initial phase while peak stresses are reached. Strains of under one percent are associated with peak stresses in uniaxial stress configuration, and reach only a few percent after considerable relaxation. The small deformation theory implies small rotation as well, and while small strain finite rotation applications may also be of interest, the corresponding approximation does not yield the same degrees of simplification. Assuming small deformation, current particle position may be identified with its reference position for the application of field equations and boundary conditions, and a finite deformation strain tensor may be expressed in terms of a small strain from the undeformed configuration. However, the significant nonlinear response of ice even at small strain requires response coefficients in the adopted differential operator relation which are not constants, but vary with invariants of both the strain and stress. Appropriate quadratic invariants of strain are required to measure the amount of shear which is regarded as the physical basis of the nonlinear response.

A first order differential relation between stress, stress-rate, strain, and strain-rate is adopted for the deviatoric (shear) viscoelastic response, and an elastic compressibility relation is included. The deviatoric relation is the nonlinear viscoelastic solid model discussed in the November 1981 Technical Report (hereafter abbreviated to TR). This contains the minimal structure necessary to describe the known qualitative uniaxial stress response, but the response coefficients and their dependence on stress and strain must be considerably restricted to be determined by uniaxial data. The tensor (or directional) structure awaits confirmation or modification by two-dimensional data, or indirectly by comparison of observation and solution of boundary value problems.

The conventional ~~linear ratio~~ linearization of deformation yields an additive decomposition into strain and rotation, and the ~~deviation~~ deviatoric tensor relation involves the strain and strain-rate tensors, independent of the rotation tensor. However, inclusion of quadratic terms to measure the shear variation, and for consistency, the dilation required for a compressibility relation or incompressibility approximation, involves the square of the rotation. This will lead to unfamiliar equations more complicated than conventional systems in which rotation is absent. An alternative strain ~~deposition~~ definition is introduced with corresponding invariants to yield a system independent of the rotation. Within the linear approximation of the deformation geometry, both strain measures are equivalent, but model coefficient dependence on squares of the strain implies different response with the different measures. It is unlikely that small strain data could distinguish the two models, and the simpler form is adopted for application.

The constitutive equations for both plane stress and plane strain are derived with a view to investigating a series of two-dimensional contact problems; for example, indentation of an ice-plate by a rigid structure and movement of a rigid inclusion frozen into a plate (neglecting variations with depth), and contact with an inclined vertical structure (neglecting horizontal variation). For the very slow motion envisaged we have the conventional linear equilibrium equation (and linear strain-compatibility if required), and ~~line crossed~~ linearized boundary conditions, but nonlinear differential viscoelastic constitutive relations. Formulation in terms of dimensionless variables with normalized stress and strain measures, taking into account the small elastic strains (strain jumps when stress applied instantaneously) compared to the creep strains of order one percent, introduces a dimensionless parameter which will typically take values in the range .01→.1. The presence of a small parameter in one or more coefficients indicates that care must be taken with numerical schemes, but its influence is explicitly shown in the normalized system.

An implicit finite difference scheme for the time variation is introduced, in conjunction with an iteration procedure, at each time step to approximate the significant nonuniformity of the differential relation coefficients. It is shown that at each step of the iteration, current strain components can be related linearly to current "pseudo stress" components involving the current stresses and "residuals" constructed from previous stress and strain values. That is, there are strain-pseudo stress relations equivalent to linear elastic strain-stress relations, but with nonhomogeneous coefficients. Furthermore, the pseudo stresses satisfy the standard linear equilibrium equations with a nonhomogeneous body force determined from gradients of the residual stresses. The iteration and time step march therefore becomes a sequence of linear elastic equilibrium problems, each one for a material with (different) nonhomogeneous moduli and under nonhomogeneous body force. Boundary conditions of traction will involve the residual stresses. Thus, if the two-dimensional elliptic spatial problem for general (smooth) nonhomogeneous properties and body force can be solved accurately and quickly by finite element or finite difference methods, it is expected that the implicit time marching and iteration scheme will yield stable viscoelastic solutions.

VISCOELASTIC SOLID MODEL

Let $\underset{\sim}{\sigma}$ denote the Cauchy stress tensor and $\underset{\sim}{S}$ the deviatoric stress defined by

$$\underset{\sim}{S} = \underset{\sim}{\sigma} - \frac{1}{3}(\mathrm{tr}\underset{\sim}{\sigma})\underset{\sim}{1} \quad \text{or} \quad S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \quad , \tag{1}$$

where the components refer to rectangular Cartesian axes Ox_i $(i = 1,2,3)$. If $\underline{v}(\underline{x},t)$ is the spatial velocity field, where t denotes time, then the spatial velocity gradient has a symmetric-skew decomposition

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{2}$$

or

$$L_{ij} = D_{ij} + W_{ij} \quad , \tag{3}$$

where $\underset{\sim}{D}$ is the rate of strain and $\underset{\sim}{W}$ is the rate of rotation relative to the current configuration: σ, $\mathbf{S}$, and $\mathbf{D}$ are frame indifferent tensors, and a frame indifferent ~~deviation~~ deviatoric stress-rate is given by

$$\mathbf{S}^{(1)} = \dot{\mathbf{S}} + \mathbf{S}(\mathbf{D} + \mathbf{W}) + (\mathbf{D} - \mathbf{W})\mathbf{S} \quad , \tag{4}$$

where a ~~suspended~~ superposed $\cdot$ denotes material time derivative. If $\underline{X}$ denotes particle position in the reference configuration, then the deformation gradient tensor $\mathbf{F}$ is defined by

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad , \tag{5}$$

and the frame indifferent left Cauchy ~~Green~~ Green tensor is given by

$$\mathbf{B} = \mathbf{F}\,\mathbf{F}^T \quad . \tag{6}$$

It has been shown that a differential viscoelastic relation adequate to describe the known qualitative nonmonotonic strain-rate response at constant uniaxial stress and nonmonotonic stress response at constant strain-rate must contain stress and stress-rate and strain and strain-rate at least. The tensor (or directional) structure cannot be determined by uniaxial data, nor can the dependence of the response coefficients on the stress and deformation invariants (assuming isotropy in the reference configuration), or on rate invariants. We therefore adopt for the present a reduced form of the shear relation (6.12) in TR, presented in the alternative ~~nonrealization~~ normalization leading to the uniaxial relation (5.5) in TR:

$$\mathbf{S}^{(1)} - \frac{1}{3}\left(\mathrm{tr}\,\mathbf{S}^{(1)}\right)\mathbf{1} + \psi\mathbf{S} = \phi\left[\mathbf{D} - \frac{1}{3}\left(\mathrm{tr}\,\mathbf{D}\right)\mathbf{1}\right] + \omega\left[\mathbf{B} - \frac{1}{3}\left(\mathrm{tr}\,\mathbf{B}\right)\mathbf{1}\right] \quad . \tag{7}$$

The incompressibility assumption of TR is not made, so $\mathrm{tr}\,\mathbf{D} \neq 0$, and dependence on the tensor $\mathbf{D}^2$ is eliminated to obtain elastic jump relations as a smooth limit (see TR, page 42). Eliminating possible dependence on the tensor $\mathbf{B}^2$ has no similar justification, but could not be distinguished by uniaxial data. The response coefficients ψ, ϕ, and ω can depend on stress and deformation invariants, and on their rates, but for determination by uniaxial response only one stress and one deformation invariant can be included. We assume these should be measures of shear, so choose the deviatoric stress invariant

$$J = \frac{1}{2}\left(\mathrm{tr}\,\mathbf{S}^2\right) \quad , \tag{8}$$

but consider both principal deformation invariants

$$K_1 = \mathrm{tr}\mathbf{B},\; K_2 = \frac{1}{2}\left[K_1{}^2 - \mathrm{tr}\mathbf{B}^2\right] \quad , \tag{9}$$

to investigate the small deformation approximation and choose an appropriate measure. The third principal invariant

$$K_3 = \det \mathbf{B} = \left(\rho_o/\rho\right)^2 \tag{10}$$

measures the dilatation through the density change $\rho_o \rightarrow \rho$.

The introduction of elastic compressibility

$$- p = \frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma} = k\,\left(K_3^{1/2} - 1\right) \quad , \tag{11}$$

where k is a constant bulk modulus, anticipating the small elastic strain assumption, will modify the details of the correlation of uniaxial stress data with the response coefficients ψ, ϕ, ω, described in TR (this set is denoted by ψ^*, ϕ^*, ω^*). That correlation still determines magnitudes of the coefficients useful for our later normalization. The jump or elastic relation TR (4.21) with the correlation TR (5.15) shows that $\phi = O(G_o)$ where G_o is an elastic shear modulus, so that if σ_o is a stress magnitude, then $(\sigma_o/G_o) \ll 1$ is an elastic strain. We suppose also that $k = O(G_o)$, and is known. It has been shown that each of the terms of (7) is necessary for the uniaxial stress response at contrast stress and contrast strain-rate. If e is the axial strain, and the strain and strain-rate terms make similar contributions at contact stress $\sigma \equiv \sigma_o$, then TR (5.5) implies $\psi\sigma_o = O(G_o\dot{e}) = O(\omega e)$. It is conjectured that the strain at minimum strain-rate, e_o say, is of order one percent, highly independent of σ_o, so that $\varepsilon = \sigma_o/(G_o e_o) \ll 1$. If t_m is the time to minimum strain-rate, then $\dot{e} \sim e_o/t_m$ and so $\psi = O(1/\varepsilon t_m)$ and $\omega = O(G_o/t_m)$. Similarly, if t_M is the time to peak stress σ_o at contrast strain-rate, where the strain is also conjectured to be e_o, TR (5.5) implies $\psi = O(1/\varepsilon t_M)$ and $\omega = O(G_o/t_m)$, that $t_m = O(t_M)$. Here ε, t_m, and t_M depend on the stress level σ_o. As σ_o increases, the times t_m and t_M decrease, but ε increases, and $\varepsilon t_m = \sigma_o t_m$ will likely decrease due to the significant nonlinear dependency of t_m on σ_o. Thus a maximum magnitude of ψ should be attained when σ_o denotes a maximum stress level in the application, and similarly for ω.

Hence, choose σ_o to be a maximum stress level and define dimensionless coefficients by

$$k = G_o K, \quad \phi = G_o \Phi, \quad \psi = \Psi/(\varepsilon t_m), \quad \omega = G_o \Omega/t_m, \tag{12}$$

where

$$\varepsilon = \sigma_o/(G_o e_o) \ll 1, \tag{13}$$

and K, Φ, Ψ, Ω, have magnitude of order unity or less. The natural time scale of the viscoelastic response is t_m (associated with the maximum stress level σ_o), so a dimensionless time T defined by

$$t = t_m T \tag{14}$$

is order unity when significant viscoelastic creep/relaxation has occurred.

Complete determination of ψ, ϕ, ω, by uni-axial stress response requires dependence of ϕ and ω on one strain invariant, dependence of ψ on one strain and one stress invariant, and the response to full unloading from each stress level σ, provided that the response is compatible with this reduced model. For the subsequent analysis and numerical scheme we suppose that ψ, ϕ, and ω depend on one shear invariant of stress and one shear invariant of strain, but dependence on further stress and strain invariants can easily be included. Dependence on rate invariants will affect the numerical scheme required for the time steps.

SMALL DEFORMATION APPROXIMATION

Let $\underline{u}$ ($\underline{X}$, t) denote displacement, then

$$\underline{x} = \underline{X} + \underline{u}, \quad \underline{v} = \partial \underline{u}/\partial t, \tag{15}$$

and

$$F_{ij} \equiv \delta_{ij} + \frac{\partial u_i}{\partial X_j} = \delta_{ij} + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \tag{16}$$

$$\text{or } \underset{\sim}{F} = \underset{\sim}{1} + \underset{\sim}{e} + \underset{\sim}{\omega}\ , \tag{17}$$

is an exact additive decomposition. Small deformation implies

$$\left|\frac{\partial u_i}{\partial X_j}\right| << 1 :- \|\underset{\sim}{e}\| << 1 \text{ and } \|\underset{\sim}{\omega}\| << 1\ , \tag{18}$$

when $\underset{\sim}{e}$ and $\underset{\sim}{\omega}$ measure strain (stretching and shear) and rotation respectively from the reference configuration. Let $e_o(<<1)$ denote a magnitude of $\underset{\sim}{e}$ or $\underset{\sim}{\omega}$, then

$$\underline{u}\ (\underline{X},t) = \underline{u}(\underline{X}_o,t) + 0\ (e_o|\underline{X} - \underline{X}_o|)\quad , \tag{19}$$

where $\underline{u}\ (\underline{X}_o,t)$ represents a rigid body displacement which can be eliminated by choice of coordinate origin. If the maximum body ~~stress~~ span is L, then

$$|\underline{x}/L - \underline{X}/L| = |\underline{u}/L| < O(e_o) << 1\quad , \tag{20}$$

so that on the length scale L:

$$\underline{x} = \underline{X}\quad . \tag{21}$$

That is, we identify reference and current-particle positions for application of field equations and boundary conditions, making the approximations

$$\left.\frac{\partial}{\partial x_j}\right|_t = \left.\frac{\partial}{\partial X_j}\right|_t \tag{22}$$

so that from (2), (3),

$$\underset{\sim}{D} = \dot{\underset{\sim}{e}}\ ,\quad \underline{W} = \dot{\underset{\sim}{\omega}}\quad , \tag{23}$$

and the equilibrium equations in the absence of body force become

$$\frac{\partial \sigma_{ij}}{\partial X_j} = 0 \quad . \tag{24}$$

For a long aspect ratio body with width $\ell \ll L$, the approximation (21) is required on the length scale ℓ which implies a much stronger restriction on $\underline{u}$ than (20). Small strain with finite rotation may be necessary for such bodies; for example, bending of thin plates. Now $\|\underset{\sim}{S}\| = O(\|\underset{\sim}{S}\|/t_m)$, $\|\underset{\sim}{S}\underset{\sim}{D}\| = O(\|\underset{\sim}{S}\| e_o/t_m)$, $\|\underset{\sim}{S}\underset{\sim}{W}\| = O(\|\underset{\sim}{S}\| e_o/t_m)$, and hence the linear approximation of (4) is

$$\underset{\sim}{S}^{(1)} = \dot{\underset{\sim}{S}}, \quad \mathrm{tr}\, \underset{\sim}{S}^{(1)} = 0 \quad . \tag{25}$$

To exhibit a shear influence in the invariants (9) of $(\underset{\sim}{B})$, it is necessary to retain the quadratic terms in the expansion (6):

$$\underset{\sim}{B} \equiv \underset{\sim}{1} + 2\underset{\sim}{e} + \underset{\sim}{e}^2 - \underset{\sim}{e}\underset{\sim}{\omega} + \underset{\sim}{\omega}\underset{\sim}{e} - \underset{\sim}{\omega}^2 \quad , \tag{26}$$

where to second order in e_o,

$$K_1 \equiv 3 + 2\,\mathrm{tr}\underset{\sim}{e} + \mathrm{tr}\underset{\sim}{e}^2 - \mathrm{tr}\underset{\sim}{\omega}^2 \quad , \tag{27}$$

$$K_2 = 3 + 4\,\mathrm{tr}\underset{\sim}{e} + 2\,(\mathrm{tr}\underset{\sim}{e})^2 - 2\,\mathrm{tr}\underset{\sim}{\omega}^2 \quad , \tag{28}$$

$$K_3^{1/2} = \det \underset{\sim}{F} = 1 + \mathrm{tr}\underset{\sim}{e} + \frac{1}{2}(\mathrm{tr}\underset{\sim}{e})^2 - \frac{1}{2}\mathrm{tr}\underset{\sim}{e}^2 - \frac{1}{2}\mathrm{tr}\underset{\sim}{\omega}^2 \quad . \tag{29}$$

To first order in e_o, the dilatation is measured by $\mathrm{tr}\underset{\sim}{e}$, and both K_1 and K_2 depend only on $\mathrm{tr}\underset{\sim}{e}$. Retaining second order terms necessarily involves $\mathrm{tr}\underset{\sim}{\omega}^2$ in K_1, K_2 and K_3. A pure shear measure is obtained through the tensor

$$\tilde{\mathbf{B}} = \mathbf{B} - \frac{1}{3} K_1 \mathbf{1} = 2\left(\mathbf{e} - \frac{1}{3}\,\mathrm{tr}\mathbf{e}\mathbf{1}\right) + \left(\mathbf{e}^2 - \frac{1}{3}\,\mathrm{tr}\mathbf{e}^2\mathbf{1}\right)$$

$$- \left(\boldsymbol{\omega}^2 - \frac{1}{3}\,\mathrm{tr}\boldsymbol{\omega}^2\mathbf{1}\right) - \mathbf{e}\boldsymbol{\omega} + \boldsymbol{\omega}\mathbf{e}, \tag{30}$$

which vanishes in pure dilatation. Since $\tilde{\mathbf{B}}^3$ is $O(e_o{}^3)$, the only second order shear measure is the quadratic approximation

$$\tilde{\mathbf{B}}^2 = 4\left[\mathbf{e}^2 - \frac{2}{3}\,\mathrm{tr}\mathbf{e}\;\mathbf{e} + \frac{1}{9}\,(\mathrm{tr}\mathbf{e})^2\mathbf{1}\right] \tag{31}$$

with invariant

$$\tilde{K} = \frac{1}{4}\,\mathrm{tr}\tilde{\mathbf{B}}^2 = \mathrm{tr}\mathbf{e}^2 - \frac{1}{3}\,(\mathrm{tr}\mathbf{e})^2 \quad , \tag{32}$$

which is independent of $\boldsymbol{\omega}$. However, an incompressibility approximation $K_3 = 1$ to second order would involve $\mathrm{tr}\boldsymbol{\omega}^2$.

Alternatively, we can define a first order strain measure by

$$\mathbf{e} \equiv \frac{1}{2}\,(\mathbf{B} - \mathbf{1}) \quad , \tag{33}$$

which is equivalent to the definitions (16) to first order. Now

$$K_1 \equiv 3 + 2\,\mathrm{tr}\mathbf{e}, \quad \tilde{\mathbf{B}} \equiv 2\left(\mathbf{e} - \frac{1}{3}\,\mathrm{tr}\mathbf{e}\;\mathbf{1}\right) \quad , \tag{34}$$

and $\tilde{\mathbf{B}}^2$ and $\tilde{K}$ are given exactly by (31) and (32), while to second order

O Jim, which is correct?

$$K_3^{1/2} = 1 + \operatorname{tr}\underset{\sim}{e} + \frac{1}{2}(\operatorname{tr}\underset{\sim}{e})^2 - \operatorname{tr}\underset{\sim}{e}^2 \quad , \tag{35}$$

so $\tilde{\underset{\sim}{B}}$, $\tilde{\underset{\sim}{K}}$, and K_3 are independent of $\underset{\sim}{\omega}$. Recall that $p/k = 0\,(\sigma_o/G_o) = 0\,(\varepsilon e_o)$ and $\varepsilon^2 < 0.01$ is expected, so that $\varepsilon^2 \lesssim e_o$. Hence (11) and (35) imply that $\operatorname{tr}\underset{\sim}{e} = 0\,(\varepsilon\, e_o)$ and $(\operatorname{tr}\underset{\sim}{e})^2 \lesssim e_o{}^3$, so to $0\,(e_o{}^2)$,

$$\tilde{K} = \operatorname{tr}\underset{\sim}{e}^2, \qquad K_3{}^{1/2} = 1 + \operatorname{tr}\underset{\sim}{e} - \operatorname{tr}\underset{\sim}{e}^2 \quad . \tag{36}$$

The arguments of ψ, ϕ, and ω will be J, defined by (8), and $\tilde{K}$, defined by (30) which is as simple as the approximation $(36)_1$; both require that $\operatorname{tr}\underset{\sim}{e}^2$ is calculated in addition to the strain tensor $\underset{\sim}{e}$. If incompressibility is prescribed, so $k \to \infty$, then $\operatorname{tr}\underset{\sim}{e} = \operatorname{tr}\underset{\sim}{e}^2$ to second order. The quadratic expansion $(36)_2$ is retained initially to examine the compressibility contributions in comparison with the shear terms, so (11) becomes

$$-p = k\,(\operatorname{tr}\underset{\sim}{e} - \operatorname{tr}\underset{\sim}{e}^2) \quad , \tag{37}$$

while the deviatoric relation (7) to first order in e_o is

$$\dot{\underset{\sim}{\sigma}} + \dot{p}\underset{\sim}{1} + \psi\,[\underset{\sim}{\sigma} + p\underset{\sim}{1}] = \phi\left[\dot{\underset{\sim}{e}} - \frac{1}{3}\operatorname{tr}\dot{e}\underset{\sim}{1}\right] + 2\omega\left[\underset{\sim}{e} - \frac{1}{3}\operatorname{tr}e\underset{\sim}{1}\right] \quad . \tag{38}$$

To complete the normalization (12)-(14), define dimensionless stress and strain by

$$\underset{\sim}{\sigma} = \sigma_o\underset{\sim}{\Sigma}, \quad p = \sigma_o P, \quad \underset{\sim}{e} = e_o\underset{\sim}{E} \quad . \tag{39}$$

Now (37) and (38) become

$$- P = K\left(\frac{\mathrm{tr}\underset{\sim}{E}}{\varepsilon} - \frac{e_o}{\varepsilon}\,\mathrm{tr}\underset{\sim}{E}^2\right) \quad , \tag{40}$$

$$\varepsilon \frac{d}{dT}\left(\underset{\sim}{\Sigma} + P\underset{\sim}{1}\right) + \Psi\left(\underset{\sim}{\Sigma} + P\underset{\sim}{1}\right) = \Phi \frac{d}{dT}\left(\underset{\sim}{E} - \frac{1}{3}\,\mathrm{tr}\underset{\sim}{E}\underset{\sim}{1}\right)$$

$$+ 2\Omega\left(\underset{\sim}{E} - \frac{1}{3}\,\mathrm{tr}\underset{\sim}{E}\underset{\sim}{1}\right) \quad . \tag{41}$$

Since $\mathrm{tr}\underset{\sim}{E} = O(\varepsilon)$ and $\mathrm{tr}\underset{\sim}{E}^2 = O(1)$, the terms in (40) are respectively order unity and order e_o/ε. Thus, eliminating P in (41) gives terms of order ε and order e_o from $\varepsilon dp/dT$, but terms of order unity and e_o/ε from ΨP, and with e_o/ε the term $\mathrm{tr}\underset{\sim}{E}^2$ must be retained. Alternatively, leaving P as $- 1/3$ to $\underset{\sim}{\Sigma}$, the same magnitude as $\underset{\sim}{\Sigma}$, and eliminating $\mathrm{tr}\underset{\sim}{E}$ in (41) by

$$\mathrm{tr}\underset{\sim}{E} = - \frac{\varepsilon P}{K} + e_o\,\mathrm{tr}\underset{\sim}{E}^2, \quad P = - \frac{1}{3}\,\mathrm{tr}\underset{\sim}{\Sigma} \quad , \tag{42}$$

allows $e_o \mathrm{tr}\underset{\sim}{E}^2$ to be neglected in comparison with $\underset{\sim}{E}$, and the first order viscoelastic constitutive relation becomes

$$\varepsilon \frac{d\underset{\sim}{\Sigma}}{dT} + \varepsilon\left(\frac{\Phi}{3K}\right)\frac{dP}{dT}\underset{\sim}{1} + \Psi\,\underset{\sim}{\Sigma} + \left(\Psi - \frac{2\varepsilon\Omega}{3K}\right)P\underset{\sim}{1} = \Phi \frac{d\underset{\sim}{E}}{dT} + 2\Omega\underset{\sim}{E} \quad . \tag{43}$$

The term $2\varepsilon\Omega/3K$ is retained in comparison with Ψ since $\varepsilon > e_o$ arises. In the normalized variables the elastic jump relations give

$$[\underset{\sim}{E}] \sim \varepsilon[\underset{\sim}{\Sigma}] \quad . \tag{44}$$

PLANE STRESS AND PLANE STRAIN

First consider plane stress in which

$$\Sigma_{33} = \Sigma_{32} = \Sigma_{31} = 0, \quad \mathrm{tr}\underset{\sim}{\Sigma} = \Sigma_{11} + \Sigma_{22}, \quad E_{32} = E_{31} = 0 \quad . \tag{45}$$

Let the superposed • denote differentiation with respect to T, and define

$$\Theta = 1 + \frac{2\Phi}{3K}, \quad \alpha = \Psi + \frac{4\varepsilon\Omega}{3K} \quad . \tag{46}$$

Now (43) gives a relation for E_{33}, not required in the plane field equation, together with th~~ose~~ree independent relations for E_{12}, E_{11}, and E_{22}, most conveniently expressed as

$$\Phi\dot{E}_{12} + 2\Omega E_{12} = \varepsilon\dot{\Sigma}_{12} + \Psi\Sigma_{12}, \tag{47}$$

$$\Phi\left(\dot{E}_{11} - \dot{E}_{22}\right) + 2\Omega\left(E_{11} - E_{22}\right) = \varepsilon\left(\dot{\Sigma}_{11} - \dot{\Sigma}_{22}\right) + \Psi\left(\Sigma_{11} - \Sigma_{22}\right) \quad , \tag{48}$$

$$\Phi\left(\dot{E}_{11} + \dot{E}_{22}\right) + 2\Omega\left(E_{11} + E_{22}\right) = \frac{1}{3}\varepsilon\Theta\left(\dot{\Sigma}_{11} + \dot{\Sigma}_{22}\right) + \frac{1}{3}\alpha\left(\Sigma_{11} + \Sigma_{22}\right) \quad . \tag{49}$$

The corresponding isotropic elastic relations are

$$E_{12} = \frac{\varepsilon}{2G}\Sigma_{12}, \quad E_{11} - E_{22} = \frac{\varepsilon}{2G}\left(\Sigma_{11} - \Sigma_{22}\right),$$

$$E_{11} + E_{22} = \frac{\varepsilon(K+4/3G)}{6KG}\left(\Sigma_{11} + \Sigma_{22}\right) \quad , \tag{50}$$

when the elastic shear modulus is $\mu = G_o G$, which follow from (47)-(49) if $\Omega = \Psi = 0$ and $\Phi = 2G$.

In plane strain,

$$E_{33} = E_{32} = E_{31} = 0, \quad \Sigma_{32} = \Sigma_{31} = 0, \quad \mathrm{tr}\,\underset{\sim}{E} = E_{11} + E_{22}, \tag{51}$$

and while Σ_{33} does not enter the plane field equations, it must be eliminated from $\mathrm{tr}\underset{\sim}{\Sigma}$ by applying the constraint $E_{33} = 0$. This is most simply achieved by replacing P in (42) by $-K\mathrm{tr}\underset{\sim}{E}/\varepsilon$, since only the first term in (40) was included in the derivation of (43) from (41). Hence (47) and (48) still apply, and (49) is replaced by

$$\beta\left(\dot{E}_{11} + \dot{E}_{22}\right) + \frac{2\gamma}{\varepsilon}\left(E_{11} + E_{22}\right) = \varepsilon\left(\dot{\Sigma}_{11} + \dot{\Sigma}_{22}\right) + \Psi\left(\Sigma_{11} + \Sigma_{22}\right), \tag{52}$$

where

$$\beta = 2K + \frac{1}{3}\Phi, \quad \gamma = K\Psi + \frac{1}{3}\varepsilon\Omega \ . \tag{53}$$

Recall that $E_{11} + E_{22} = \mathrm{tr}\underset{\sim}{E} = O(\varepsilon)$, so the L.H.S. of (52) is order unity; in the plane stress relation (48), $E_{11} + E_{22}$ is order unity. The corresponding elastic relation is

$$E_{11} + E_{22} = \frac{\varepsilon}{2(K + 1/3\ G)}\left(\Sigma_{11} + \Sigma_{22}\right) \ . \tag{54}$$

In both plane stress and plane strain, equilibrium in the absence of body force implies

$$\frac{\partial\Sigma_{11}}{\partial X_1} + \frac{\partial\Sigma_{12}}{\partial X_2} = 0, \quad \frac{\partial\Sigma_{12}}{\partial X_1} + \frac{\partial\Sigma_{22}}{\partial X_2} = 0 \ , \tag{55}$$

and strain compatibility with a continuous displacement field requires

$$\frac{\partial^2 E_{11}}{\partial X_2^{\,2}} + \frac{\partial^2 E_{22}}{\partial X_1^{\,2}} = 2\,\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \quad , \tag{56}$$

which are identical to the corresponding equations for linear elasticity.

It remains to construct the invariants J and $\tilde{K}$ in normalized variables for both plane stress and plane strain. Common ~~experiences~~ expressions for the stress and strain are

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ \Sigma_{12} & \Sigma_{22} & 0 \\ 0 & 0 & \mathrm{tr}\underset{\sim}{\Sigma} - \Sigma_{11} - \Sigma_{22} \end{pmatrix}, \quad \underset{\sim}{E} = \begin{pmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & \mathrm{tr}\underset{\sim}{E} - E_{11} - E_{22} \end{pmatrix}, \tag{57}$$

where in plane stress $\mathrm{tr}\underset{\sim}{\Sigma} = \Sigma_{11} + \Sigma_{22}$ and in plane strain $\mathrm{tr}\underset{\sim}{E} = E_{11} + E_{22}$. Hence

$$\hat{J} = \frac{1}{2}\,\mathrm{tr}\left(\underset{\sim}{\Sigma} - \frac{1}{3}\,\mathrm{tr}\,\underset{\sim}{\Sigma}\,\underset{\sim}{1}\right)^2$$

$$= \Sigma_{12}^2 + \Sigma_{11}^2 + \Sigma_{22}^2 + \Sigma_{11}\Sigma_{22} - \mathrm{tr}\underset{\sim}{\Sigma}\,(\Sigma_{11} + \Sigma_{22}) + \frac{1}{3}(\mathrm{tr}\underset{\sim}{\Sigma})^2 \quad , \tag{58}$$

$$\hat{I} = \frac{1}{2}\,\mathrm{tr}\left(\underset{\sim}{E} - \frac{1}{3}\,\mathrm{tr}\underset{\sim}{E}\,\underset{\sim}{1}\right)^2$$

$$= E_{12}^2 + E_{11}^2 + E_{22}^2 + E_{11}E_{22} - \mathrm{tr}\underset{\sim}{E}\,(E_{11} + E_{22}) + \frac{1}{3}(\mathrm{tr}\underset{\sim}{E})^2 \quad , \tag{59}$$

where $\hat{I}$ is a normalization of $\frac{1}{2}\tilde{K}$ given by the full expression (32). The dimensionless coefficients Φ, Ψ, Ω, will be given as functions of $\hat{J}$ and $\hat{I}$. By (40)

$$\mathrm{tr}\underset{\sim}{E} = e_o \, \mathrm{tr}\underset{\sim}{E}^2 + \frac{\varepsilon}{3K} \mathrm{tr}\underset{\sim}{\Sigma} \quad , \tag{60}$$

so $(\mathrm{tr}\underset{\sim}{E}) = O(\varepsilon)$, $(\mathrm{tr}\underset{\sim}{E})^2 = O(\varepsilon^2) \lesssim e_o$ as noted earlier.

Thus, in plane stress, the lead order approximations are

$$\hat{J} = \Sigma_{12}^2 + \frac{1}{3}\left(\Sigma_{11}^2 + \Sigma_{22}^2 - \Sigma_{11}\Sigma_{22}\right) \quad , \tag{61}$$

$$\hat{I} = E_{12}^2 + E_{11}^2 + E_{22}^2 + E_{11}E_{22} - \frac{\varepsilon}{3K}\left(\Sigma_{11} + \Sigma_{22}\right)\left(E_{11} + E_{22}\right) \quad , \tag{62}$$

where the order ε term has been retained in $\hat{I}$. In plane strain,

$$\hat{I} = E_{12}^2 + \frac{1}{3}\left(E_{11}^2 + E_{22}^2 - E_{11}E_{22}\right) \quad , \tag{63}$$

and $\hat{J}$ is given by (58) with

$$\mathrm{tr}\underset{\sim}{\Sigma} = \frac{3K}{\varepsilon}\left[E_{11} + E_{22} - e_o\left(E_{11}^2 + E_{22}^2 + 2E_{12}^2\right)\right] \quad . \tag{64}$$

IMPLICIT FINITE DIFFERENCE SCHEME FOR TIME STEPS

Time derivatives occur only in the constitutive relations (47), (48), and (49) or (52), and not in the equilibrium nor compatibility equations (55) and (56). An implicit point difference scheme for the time steps is generally more stable than an explicit scheme, and schemes with general weighting between time T_r and time T_{r+1} $(r=1,2,3,\ldots)$ are presented for the plane stress system (47)-(49), and the plane strain system (47), (48), (52). Let a subscript r denote quantities evaluated at time T_r and use the notation

$$W_r = W\left(\hat{J}_r, \hat{I}_r\right) \tag{65}$$

for all nonconstant coefficients, where $\hat{J}_r$, $\hat{I}_r$ denote the stress and strain ~~variants~~ invariants evaluated at time T_r. Define

$$\bar{W}_r = (1 - \lambda) W_r + \lambda W_{r+1}, \quad 0 < \lambda < 1, \tag{66}$$

as a weighted average of W between time T_r and time T_{r+1}.

An explicit scheme is given by $\lambda = 0$, when coefficients and spatial derivatives are all evaluated at time T_r, but are commonly unstable unless the time increments are extremely small compared to spatial increments.

Let δ be the time increment $T_{r+1} - T_r$. Different increments δ may be chosen at different times T_r, which could be advantageous for the different time scales of variation which can arise. Using a forward time difference and a weighted average defined by (66), the relations (47) to (49) are approximated by

$$\bar{\Phi}_r[(E_{12})_{r+1} - (E_{12})_r] + 2\delta\bar{\Omega}_r(\overline{E_{12}})_r = \varepsilon[(\Sigma_{12})_{r+1} - (\Sigma_{12})_r] + \delta\bar{\Psi}_r(\overline{\Sigma_{12}})_r, \tag{67}$$

$$\bar{\Phi}_r[(E_{11} - E_{22})_{r+1} - (E_{11} - E_{22})_r] + 2\delta\bar{\Omega}_r(\overline{E_{11} - E_{22}})_r$$

$$= \varepsilon[(\Sigma_{11} - \Sigma_{22})_{r+1} - (\Sigma_{11} - \Sigma_{22})_r] + \delta\bar{\Psi}_r(\overline{\Sigma_{11} - \Sigma_{22}})_r, \tag{68}$$

$$\bar{\Phi}_r[(E_{11} + E_{22})_{r+1} - (E_{11} + E_{22})_r] + 2\delta\bar{\Omega}_r(\overline{E_{11} + E_{22}})_r$$

$$= \frac{1}{3}\varepsilon\bar{\Theta}_r[(\Sigma_{11} + \Sigma_{22})_{r+1} - (\Sigma_{11} + \Sigma_{22})_r] + \frac{1}{3}\delta\bar{\alpha}_r(\overline{\Sigma_{11} + \Sigma_{22}})_r, \tag{69}$$

and the plane strain relation (52) is approximated by

$$\bar{\beta}_r[(E_{11}+E_{22})_{r+1}-(E_{11}+E_{22})_r]+\frac{2\delta\bar{\gamma}_r}{\varepsilon}\overline{(E_{11}+E_{22})_r}$$

$$=\varepsilon\,[(\Sigma_{11}+\Sigma_{22})_{r+1}-(\Sigma_{11}+\Sigma_{22})_r]+\delta\bar{\Psi}_r\overline{(\Sigma_{11}+\Sigma_{22})}\;. \tag{70}$$

Now a weighted average of W involves both W_r and W_{r+1}. The scheme starts with known initial conditions - all variables W_o - and proceeds in steps, calculating each W_{r+1} given that each W_r has been determined in the previous step. It is therefore necessary to iterate within each step by taking

$$\bar{W}_r^{(o)}=W_r,\;\bar{W}_r^{(n+1)}=(1-\lambda)\,W_r+\lambda W_{r+1}^{(n)},\quad n=0,1,2\ldots \tag{71}$$

for all the coeficients $\bar{W}_r$ in (67)-(70), but not for the stress and strain components. Using (71) for coefficients, so they are prescribed at the start of each iteration, means that (67), (68), (69), and (70) are three linear relations connecting $(E_{11})_{r+1}$, $(E_{22})_{r+1}$, $(E_{12})_{r+1}$, and $(\Sigma_{11})_{r+1}$, $(\Sigma_{22})_{r+1}$, $(\Sigma_{12})_{r+1}$, which involve the known $(E_{ij})_r$ and $(\Sigma_{ij})_r$. The iteration in continued until some solution norm or parameter measuring a significant feature changes by less than a specified tolerance.

The deviatoric relations (67) and (68) have a common form

$$(E_{12})_{r+1}=\bar{J}_r[(\Sigma_{12})_{r+1}+\bar{A}_r(E_{12})_r-\bar{B}_r(\Sigma_{12})_r]\;, \tag{72}$$

$$(E_{11}-E_{22})_{r+1}=\bar{J}_r[(\Sigma_{11}-\Sigma_{22})_{r+1}+\bar{A}_r(E_{11}-E_{22})_r-\bar{B}_r(\Sigma_{11}-\Sigma_{22})_r], \tag{73}$$

where

$$\bar{J}_r=\frac{\varepsilon+\delta\lambda\bar{\Psi}_r}{\bar{\Phi}_r+2\delta\lambda\bar{\Omega}_r}\;,\quad \bar{A}_r=\frac{\bar{\Phi}_r-2\delta(1-\lambda)\bar{\Omega}_r}{\varepsilon+\delta\lambda\bar{\Psi}_r}\;,\quad \bar{B}_r=\frac{\varepsilon-\delta(1-\lambda)\bar{\Psi}_r}{\varepsilon+\delta\lambda\bar{\Psi}_r}\;, \tag{74}$$

The relations (69) and (70) have the form

$$(E_{11} + E_{22})_{r+1} = \bar{\chi}_r[(\Sigma_{11} + \Sigma_{22})_{r+1} + \bar{C}_r(E_{11} + E_{22})_r - \bar{D}_r(\Sigma_{11} + \Sigma_{22})_r] \ , \quad (75)$$

where for plane stress (69),

$$\bar{\chi}_r = \frac{\frac{1}{3}(\varepsilon\bar{\Theta}_r + \delta\lambda\bar{\alpha}_r)}{\bar{\Phi}_r + 2\delta\lambda\bar{\Omega}_r} \ , \quad \bar{C}_r = \frac{3[\bar{\Phi}_r - 2\delta(1-\lambda)\bar{\Omega}_r]}{\varepsilon\bar{\Theta}_r + \delta\lambda\bar{\alpha}_r} \ , \quad \bar{D}_r = \frac{\varepsilon\bar{\Theta}_r - \delta(1-\lambda)\bar{\alpha}_r}{\varepsilon\bar{\Theta}_r + \delta\lambda\bar{\alpha}_r} \ , \quad (76)$$

and for plane strain (70),

$$\bar{\chi}_r = \frac{\varepsilon + \delta\lambda\bar{\Psi}_r}{\bar{\beta}_r + (2\delta\lambda\bar{\gamma}_r)/\varepsilon} \ , \quad \bar{C}_r = \frac{\bar{\beta}_r - [2\delta(1-\lambda)\bar{\gamma}_r]/\varepsilon}{\varepsilon + \delta\lambda\bar{\Psi}_r} \ , \quad \bar{D}_r = \frac{\varepsilon - \delta(1-\lambda)\bar{\Psi}_r}{\varepsilon + \delta\lambda\bar{\Psi}_r} \ . \quad (77)$$

Define

$$S_r = \bar{C}_r(E_{11} + E_{22})_r - \bar{D}_r(\Sigma_{11} + \Sigma_{22})_r \ ,$$

$$Q_r = \bar{A}_r(E_{11} - E_{22})_r - \bar{B}_r(\Sigma_{11} - \Sigma_{22})_r \ , \quad (78)$$

$$(R_{11})_r = \frac{1}{2}(S_r + Q_r) \ , \quad (R_{22})_r = \frac{1}{2}(S_r - Q_r) \ ,$$

$$(R_{12})_r = \bar{A}_r(E_{12})_r - \bar{B}_r(\Sigma_{12})_r \ , \quad (79)$$

$$(\hat{\Sigma}_{ij})_{r+1} = (\Sigma_{ij})_{r+1} + (R_{ij})_r \ , \quad i,j = 1,2 \ , \quad (80)$$

where R_{ij} and $\hat{\Sigma}_{ij}$ are conveniently termed residual and pseudo stresses respectively. Now (72), (73), and (75) can be simply written

$$(E_{12})_{r+1} = \bar{J}_r(\hat{\Sigma}_{12})_{r+1}, \quad (E_{11} - E_{22})_{r+1} = \bar{J}_r(\hat{\Sigma}_{11} - \hat{\Sigma}_{22})_{r+1},$$

$$(E_{11} + E_{22})_{r+1} = \bar{\chi}_r(\hat{\Sigma}_{11} + \hat{\Sigma}_{22})_{r+1}, \tag{81}$$

and hence we have explicit strain-pseudo stress relations

$$(E_{11})_{r+1} = \frac{1}{2}(\bar{J}_r + \bar{\chi}_r)(\bar{\Sigma}_{11})_{r+1} - \frac{1}{2}(\bar{J}_r - \bar{\chi}_r)(\hat{\Sigma}_{22})_{r+1}, \tag{82}$$

$$(E_{22})_{r+1} = \frac{1}{2}(\bar{J}_r + \bar{\chi}_r)\cancel{(\bar{J}_r + \bar{\chi}_r)}(\hat{\Sigma}_{22})_{r+1} - \frac{1}{2}(\bar{J}_r - \bar{\chi}_r)(\hat{\Sigma}_{11})_{r+1}, \tag{83}$$

$$(E_{12})_{r+1} = \bar{J}_r(\hat{\Sigma}_{12})_{r+1} \tag{84}$$

at time t_{r+1}. The compatibility equation (56) holds at each time T_{r+1} and the equilibrium equation (55) can be ~~experienced~~ expressed in terms of the pseudo stresses at each time T_{r+1} as

$$\frac{\partial(\hat{\Sigma}_{11})_{r+1}}{\partial X_1} + \frac{\partial(\hat{\Sigma}_{12})_{r+1}}{\partial X_2} + b_1 = 0, \quad \frac{\partial(\hat{\Sigma}_{12})_{r+1}}{\partial X_1} + \frac{\partial(\hat{\Sigma}_{22})_{r+1}}{\partial X_2} + b_2 = 0, \tag{85}$$

where the pseudo body force components b_1 and b_2 are defined by

$$b_1 = -\frac{\partial(R_{11})_r}{\partial X_1} - \frac{\partial(R_{12})_r}{\partial X_2}, \quad b_2 = -\frac{\partial(R_{12})_r}{\partial X_1} - \frac{\partial(R_{22})_r}{\partial X_2}, \tag{86}$$

given by the solution at time T_r and values at the preceding iteration.

The system (82)-(85) with (56) is simply a plane linear isotropic elastic system with nonhomogeneous body force (b_1, b_2), and nonhomogeneous moduli, since $\bar{J}_r$, $\bar{\chi}_r$, $(R_{ij})_r$, at each iteration within the time step $T_r \to T_{r+1}$ depend on the nonuniform stress and strain fields just determined. Traction boundary conditions can be expressed in terms of the pseudo stresses. A complete spatial problem must be solved at each iteration. Note that if stress jumps are applied initially, the initial solution corresponding to $r = 0$ is the solution at $T = 0+$ determined by the elastic jump relations. The choice of δ at each step depends on variations occurring on the T scale; that is, on the real time scale t_m.

FEATURES OF THE TIME VARIATION

The typical response is described by the shear relations (47) and (48) which, deleting the component subscripts, have the form

$$\Phi\dot{E} + 2\Omega E = \varepsilon\dot{\Sigma} + \Psi\Sigma, \tag{87}$$

where the time unit is t_m and ε is a small parameter. If $\Sigma(T)$ is prescribed, with order unity variation on the t_m scale, then (87) determines a response $E(T)$ with $\dot{E}(T)$ order unity (assuming that the coefficients Φ, Ω, and Ψ are smooth). However, if $E(T)$ is prescribed with strain-rate $\dot{E}(T)$ order unity, for example, $\dot{E}(T) = r =$ constant, $E = rT$, then the time interval on which $\varepsilon\Sigma(T)$ enters the balance requires $\dot{\Sigma} = O(1/\varepsilon)$; that is, Σ varies on the time scale εt_m. This is precisely the observed constant strain-rate response in which initially $\dot{\sigma}/\dot{e} = G_o$ or $\dot{\Sigma}/\dot{E} = 1/\varepsilon$. For illustration, consider the linear viscoelastic case $\Phi = 2\Omega = \Psi \equiv 1$ where (87) defines a standard linear solid. Then, for

$$\Sigma(T) = H(T), \quad E(0+) = \varepsilon \quad , \tag{88}$$

there is a smooth bounded response on the time scale t_m:

$$T > 0: \quad E(T) = 1 - (1-\varepsilon)e^{-T}, \quad \dot{E}(T) = (1-\varepsilon)e^{-T} \quad . \tag{89}$$

Alternatively, for

$$\dot{E}(T) = H(T), \quad \Sigma(0+) = 0, \tag{90}$$

the stress response is

$$T > 0: \quad \Sigma(T) = T + (1-\varepsilon)\left(1 - e^{-T/\varepsilon}\right) \quad , \tag{91}$$

which exhibits rapid variation over an initial time period $T = \varepsilon$ ($t = \varepsilon t_m$).

We can assess the finite difference approximation (72), (74) applied to this linear viscoelastic solid. Here

$$\bar{J}_r = \frac{\varepsilon + \delta\lambda}{1 + \delta\lambda}, \quad \bar{A}_r = \frac{1 - \delta(1-\lambda)}{\varepsilon + \delta\lambda}, \quad \bar{B}_r = \frac{\varepsilon - \delta(1-\lambda)}{\varepsilon + \delta\lambda} \quad , \tag{92}$$

which are constant at each time step unless δ is changed, and no iteration within the time step is required. Now (72) becomes

$$E_{r+1} - \left(1 - \frac{\delta}{1 + \delta\lambda}\right) E_r = \frac{\varepsilon + \delta\lambda}{1 + \delta\lambda}\left\{\Sigma_{r+1} - \left(1 - \frac{\delta}{\varepsilon + \delta\lambda}\right)\Sigma_r\right\} . \tag{93}$$

When $\Sigma \equiv 1$ in $T > 0$,

$$\frac{E_{r+1} - E_r}{\delta} = \frac{1}{1 + \delta\lambda}\left(1 - E_r\right) = O(1) \quad , \tag{94}$$

so $E(T) = O(1)$ until $E_r \to 1$ as shown by (89), and δ is chosen to make the approximation (94) adequate. Alternatively, when $E = T$ in $T > 0$, $E_r = r\delta$, so

$$\frac{\Sigma_{r+1} - \Sigma_r}{\delta} = \frac{1 + \delta\lambda + r\delta - \Sigma_r}{\varepsilon + \delta\lambda} \quad , \tag{95}$$

which is order $(1/\varepsilon)$ for $E \lesssim 1$ until $\Sigma_r \to 1 + r\delta$, assuming $\delta \lesssim \varepsilon$, confirmed by

(91). An adequate approximation will therefore require $\delta \ll \varepsilon$ until $\varepsilon^{-1}e^{-T/\varepsilon} = O(1)$. During computation of a nonlinear problem it would seem necessary to start with $\delta \ll \varepsilon$, unless boundary conditions clearly impose an order unity $\underset{\sim}{\Sigma}$, and allow increasing δ, subject to $\| \underset{\sim}{\Sigma}_{r+1} - \underset{\sim}{\Sigma}_r \| / \delta = O(1)$ as time proceeds. Contact problems will include some strain variation prescription, which for some velocity ranges will imply a rapid stress rise on the time scale $T = \varepsilon$, at least initially.

Recall that the magnitude of ε depends on the elastic strain associated with the maximum stress reached compared to a strain magnitude at peak stress, and may therefore not be too small for high strain-rate, large peak stress, applications. Effects of significant nonlinearity could also modify estimates made with the above linear model, and experience with the nonlinear computation is required to choose an optimum strategy for the finite difference scheme.

BOUNDARY-VALUE PROBLEMS

Let X_1, X_2 be dimensionless coordinates with a length unit ℓ representing the stress variation length scale in the application; for example, a plate thickness, or width, or maximum contact span. Define dimensionless displacement $\underset{\sim}{U}$ and velocity $\underset{\sim}{V}$ by

$$\underset{\sim}{u} = e_o \ell \underset{\sim}{U}, \quad \underset{\sim}{V} = \dot{\underset{\sim}{U}} = \frac{t_m}{e_o \ell} \underset{\sim}{v}, \tag{96}$$

$$\text{so } E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) .$$

Figure 1 shows vertical sections or horizontal planes respectively for some idealized plane strain or plane stress contact-problems, in which straining in the horizontal plane or vertical plane respectively is neglected. In each case an ice plate of uniform (dimensionless) thickness or width 2, with stress free horizontal surfaces or lateral edges and rear edge, impacts with a wall with initial (dimensionless) velocity V_0 in the negative X_1 direction, then decelerates as the contact force increases. Relative to non-Newtonian axes fixed with respect to the midpoint of the rear edge, the equilibrium equations are

$$\frac{\partial \Sigma_{11}}{\partial X_1} + \frac{\partial \Sigma_{12}}{\partial X_2} + \frac{\rho \ell^2 e_o}{\sigma_o t_m^2}\dot{V} = 0, \quad \frac{\partial \Sigma_{12}}{\partial X_1} + \frac{\partial \Sigma_{22}}{\partial X_2} = 0 \quad , \qquad (97)$$

where ρ is the ice density, and the wall moves with velocity V(T) into the ice plate. The dimensionless contact force per unit horizontal or vertical span (force/$\sigma_o \ell$) is

$$F = -\frac{2\rho \ell^2 L e_o}{\sigma_o t_m^2}\dot{V} \quad \chi = -\frac{2\rho \ell^2 L}{\epsilon G_o t_m^2}\dot{V}, \qquad (98)$$

so the body force in $(97)_1$ is -F/2L. Note that $\dot{V} = \frac{t_m^2}{e_o \ell}\frac{dv}{dt}$ will be order unity or less if $\ell \sim 10$m, $t_m \lesssim 2$s, $e_o \sim 0.01$, when $|dv/dt| \lesssim 0.02\text{ms}^{-2}$, which are common conditions, and the ~~dimension~~ dimensionless body force $(\rho\ell/\sigma_o)$ dv/dt $\sim 10^{-4}$ for $\sigma_o = 20 \times 10^5$ Nm^{-2}, $|dv/dt| = 0.02\ \text{ms}^{-2}$, and decreases as σ_o increases and $|dv/dt|$ decreases. Commonly this body force will be negligible in comparison with stress gradients in the contact area, which may be a useful simplification since V(T) is part of the solution.

Figure 1a shows contact with a wall parallel to the front edge. Let the friction coefficient be μ (constant). The stress ~~face~~ free boundary conditions are

$$\begin{aligned} X_2 &= \pm 1, \quad 0 < X_1 < L: \quad \Sigma_{12} = 0, \quad \Sigma_{22} = 0; \\ X_1 &= L, \quad -1 < X_2 < 1: \quad \Sigma_{12} = 0, \quad \Sigma_{11} = 0. \end{aligned} \qquad (99)$$

The contact conditions are

$$X_1 = 0, \quad -1 < X_2 < 1: \qquad \Sigma_{12} = -\mu\Sigma_{11}\operatorname{sgn} X_2, \qquad (100)$$
$$U_1(T) = \int_0^T V(T')dT',$$

so U_1 is independent of X_2, but its variation with T is part of the solution. Since the rear edge $X_1 = L$ is stationary by choice of axes, there is an auxiliary condition

$$U_1(L,0,T) = 0 \text{ or } U_1(0,0,T) = -\int_0^L E_{11}\Big|_{X_2=0} dX_1, \qquad (101)$$

which is used in place of $(100)_2$. Values of E_{11}, through Σ_{11}, Σ_{22}, at time T_{r+1} can be prescribed by the iteration process (71), and in turn the body force is given by $\ddot{U}_1$, (0,0,T), using, say, a quadratic fitted at three previous times.

Figure 1b shows an increasing contact zone as the ice pushes into an inclined wall at angle θ to the front edge, but the small strain and small rotation approximation can only apply through to complete contact if θ << 1. If the ice plate has already taken up the configuration shown in Figure 1c, and starts moving from rest, then the small deformation approximation is valid for arbitary θ. Further, if the ice is at rest in the partial contact shown in Figure 1b, then the small deformation theory can be applied for initial movement in which the contact zone changes by order of the small strain. A friction condition on the inclined wall depends on the direction of relative motion. For θ = 0, Figure 1a, the ice slides away from the symmetry line due to the lateral expansion (suppressed) because of the stress free lateral surfaces, so the tangential friction changes direction at the symmetry line (100). This, of course, introduces a stress discontinuity, unfortunate for numerical calculation - a continuous velocity dependent friction condition would be more satisfactory for numerical calculation. For example, (100) could be replaced by

$$X_1 = 0: \quad \Sigma_{12} = -\mu\Sigma_{11}u_2, \qquad (102)$$

where the friction depends linearly on both normal pressure and sliding velocity, and is continuously zero at the symmetry line. The corresponding condition applied on the inclined wall avoids the difficulty associated with not knowing a priori at which point the slip changes direction as θ increases from zero. Beyond some value of θ we would expect the friction to act in the negative X_2-direction over the entire contact zone, not just in a zone adjacent to $X_2 = 1$. The additional contact condition is

$$\dot{U}_1 \cos\theta + \dot{U}_2 \sin\theta = V \cos\theta \quad , \tag{103}$$

and the noncontact surfaces are again stress free.

Further plane stress problems describing idealized horizontal deformation configurations are shown in Figures 2, 3, and 4. Figure 2 shows an ice plate pushing against a rigid structure of arbitary profile of smoothly varying large radius of curvature; that is, $|g'(X_2)| << 1$. The coordinates are fixed relative to the midpoint of the rear edge so equilibrium is given by (97) and conditions (101) apply. Stress free boundary conditions are

$$X_2 = \pm H, \quad 0 < X_1 < L: \qquad \Sigma_{12} = \Sigma_{22} = 0, \tag{104}$$

$$X_1 = L, \quad -H < X_2 < H: \qquad \Sigma_{12} = \Sigma_{11} = 0, \tag{105}$$

$$X_1 = 0, \quad |X_2| \geq W(T): \qquad \Sigma_{12} = \Sigma_{11} = 0, \tag{106}$$

while the contact conditions, to first order, are

$$X_1 = 0, \quad |X_2| < W(T): \qquad U_1 = d(T) - g(X_2), \quad \Sigma_{12} = -\mu\Sigma_{11} U_2, \tag{107}$$

if the continuous velocity dependent friction condition is used, where d(T) is the indentation depth (dimensionless) relative to the rear edge; that is, $\dot{d}(T) = V(T)$. The length unit ℓ is a semi-span magnitude at maximum contact, so $W(T) < 1$, but W(T) is part of the solution, as well as d(T) or equivalently F(T). As in the elastic formulation, d(T) can be eliminated from the boundary conditions by replacing (107) by a stress-type condition

$$X_1 = 0, \quad |X_2| < W(T): \quad \frac{\partial U_1}{\partial X_2} = -g'(X_2) \quad , \qquad (108)$$

but here F(T) remains in the body force. The semi-span W(T) is governed by smooth separation at the end point, normal and shear stress continuously zero and

$$X_1 = 0, \quad X_2 = \pm W(T): \quad \frac{\partial^2 U_1}{\partial X_2^2} + g''(X_2) \gtrless 0 \quad . \qquad (109)$$

As in the previous problem, $(101)_2$ with iteration determines $d(T_{r+1})$ and $F(T_{r+1})$ if $W(T_{r+1})$ is prescribed. An initial estimate for $W(T_{r+1})$ can be made by extrapolating from previous time steps, with some alternative starting procedure at T= 0. Since the numerical solution satisfies equilibrium and continuous stress gradients (second derivatives of displacement) within the approximation limits, an incorrect $W(T_{r+1})$ must be revealed by failure of the separation condition adjacent to the contact zone. If the $W(T_{r+1})$ estimate is too large, it is plausible that the correspondingly "reduced contact pressures" for the given deceleration will lead to "reduced displacement" outside the contact zone, and hence overlap. If the $W(T_{r+1})$ estimate is too small, since the contact pressure is zero at the edge we can conjecture that the increased contact pressures necessary to balance the given deceleration lead to a larger negative gradient near the edge, and so again to smaller stress and displacements just outside the contact zone, and hence overlap. Thus, while (109) will be too sensitive for numerical testing, a criterion for judging the $W(T_{r+1})$ estimate is

$$U_1(0,\ X_2,\ T_{r+1}) + g(X_2) - d(T_{r+1}) > 0 \quad \text{for } |X_2| > W(T_{r+1}) \quad , \ (110)$$

applied in some small region adjacent to the contact zone.

In each of the above problems, the rear edge $X_1 = L$ could also be subjected to a driving stress $\Sigma_{11}(L,\ X_2,\ T) = -P\ (X_2,T)$ in place of $(99)_2$ or $(105)_2$.

Figure 3 represents a static ice plate in which a rigid structure is frozen, and then the ice plate is set in motion, for example by water currents which impose a velocity $V(T)$ in the X_1 direction through basal shear traction. In generalized plane stress we can identify this with a body force, precisely that shown in $(97)_1$, so with respect to Newtonian axes fixed on the structure there is equilibrium governed by (97). Here we regard $V(T)$ as prescribed.

Consider the circular disc shown in Figure 3 with unit dimensionless radius, and let the distant plate boundary be circular with radius L for convenience. In plane polar coordinates (r,θ), the physical components of displacement, strain, and stress, satisfy the strain-displacement and equilibrium equations

$$E_{rr} = \frac{\partial U_r}{\partial r}, \quad E_{\theta\theta} = \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}, \quad E_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right) \quad ; \ (111)$$

$$\frac{\partial \Sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \Sigma_{r\theta}}{\partial \theta} + \frac{\Sigma_{rr} - \Sigma_{\theta\theta}}{r} + \frac{\rho \ell^2 e_o}{\sigma_o t_m^2}\dot{V}\cos\theta = 0,$$

$$\frac{1}{r}\frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \Sigma_{r\theta}}{\partial r} + 2\frac{\Sigma_{r\theta}}{r} - \frac{\rho \ell^2 e_o}{\sigma_o t_m^2}\dot{V}\sin\theta = 0; \quad (112)$$

and stress-strain relations are unchanged apart from the subscript translation $1 \to r$, $2 \to \theta$. For numerical purposes it may be more convenient to keep a retangular plate and use rectangular coordinates. Boundary conditions are a stress free plate boundary and prescribed displacements on the disc boundary given by

$$r = 1: \quad \dot{U}_1 = -V, \; \dot{U}_2 = 0, \quad \text{or} \quad \dot{U}_r = -V\cos\theta, \; \dot{U}_\theta = V\sin\theta. \qquad (113)$$

Finally, consider an ice plate region in uniform stress

$$\Sigma_{11} = N_1, \quad \Sigma_{22} = N_2, \quad \Sigma_{12} = S, \qquad (114)$$

into which an elastic inclusion shown as a circular disc in Figure 4, is carefully embedded and the ice refrozen onto its edge without disturbing the far field (L >> 1). The local stress field is changed, and the elastic disc is stressed and deformed. It is supposed that $N_1 + N_2 < 0$ so that the disc is under compression when frozen in at time T = 0. If the elastic properties are given and N_1, N_2, S are prescribed, the coupled plate-disc problem with continuous traction and displacement (bonded interface) is well posed, and the tractions and displacements at the interface can be calculated. Suppose that the common interface displacements can be measured in the disc:

$$U_r(1,\theta,T) = \hat{U}_r(\theta,T), \quad U_\theta(1,\theta,T) = \hat{U}_\theta(\theta,T). \qquad (115)$$

Then the elastic disc solution can be determined, giving the tractions on the interface. In turn, coupled with the plate edge tractions N_1, N_2, S, these determine the viscoelastic plate solution, and in particular $U_r(1,\theta,T)$, $U_\theta(1,\theta,T)$. If N_1, N_2, S are the correct field values, these interface displacements would match (115). Since the problem of prescribed displacements and tractions on r = 1 is not a well posed viscoelastic problem, N_1, N_2, S cannot be determined directly. A sequence of trial N_1, N_2, S may lead to correct interface matching, but alternatively, prescribing displacement on the plate edge (say zero relative to initial stressed state) and displacement or traction on the disc plates interface, may lead to closer matching with interface tractions or displacements respectively.

Note that equilibrium in the non-Newtonian frame given by (97), modifies the spatial equation (85), at time T_{r+1} to

$$\frac{\partial(\hat{\Sigma}_{11})_{r+1}}{\partial X_1} + \frac{\partial(\hat{\Sigma}_{12})_{r+1}}{\partial X_2} + b_1 = \frac{1-\lambda}{\lambda}\frac{\rho \ell^2 \varepsilon_0}{\sigma_0 t_m^2}\dot{V}_r - \frac{\rho \ell^2 \varepsilon_0}{\sigma_0 t_m^2}\left(\frac{V_{r+1}-V_r}{\lambda\delta}\right), \qquad (116)$$

and similar additions will arise from the polar coordinates equations (112).

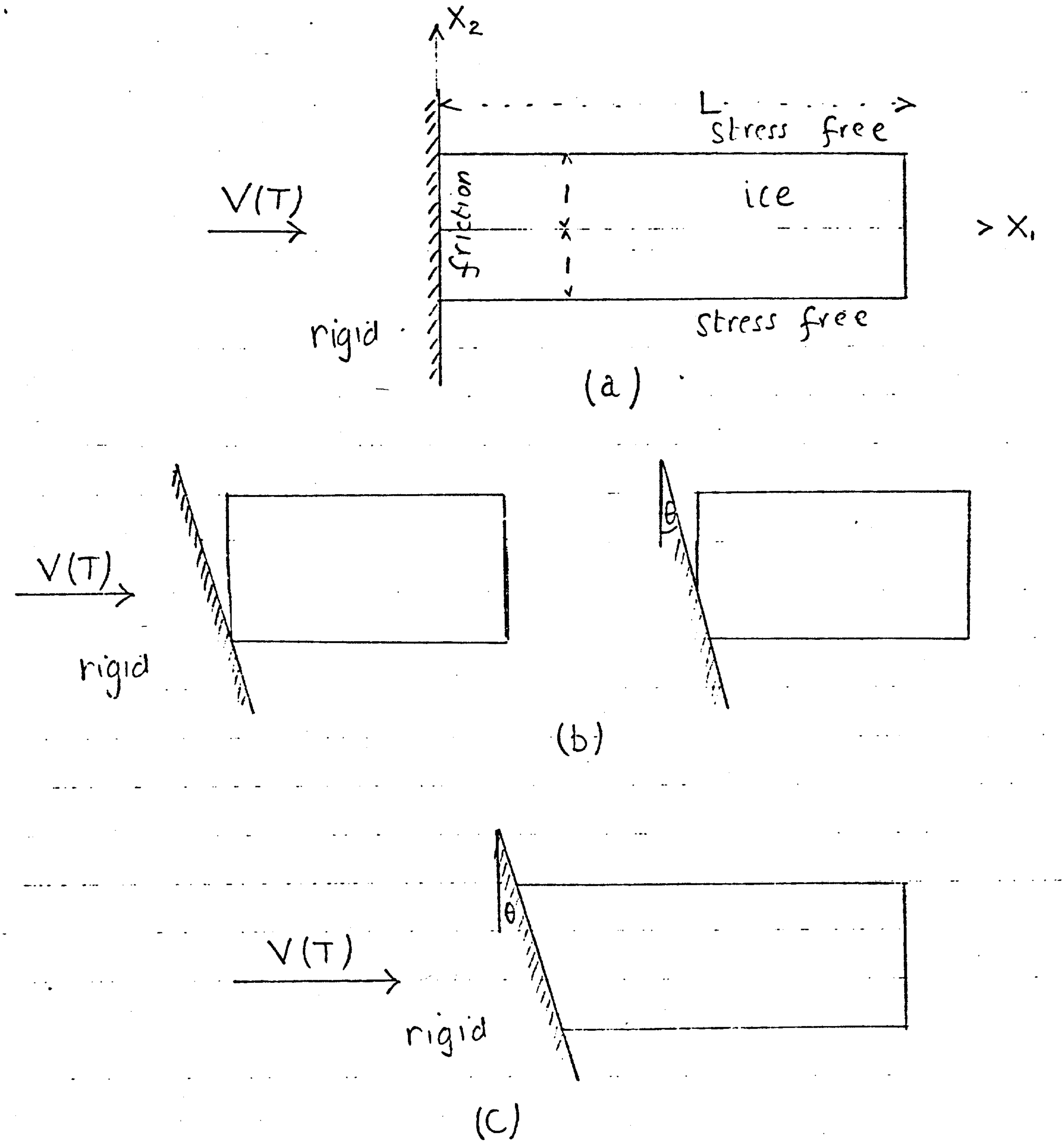

Fig. 1 Plane strain or stress / contact configurations

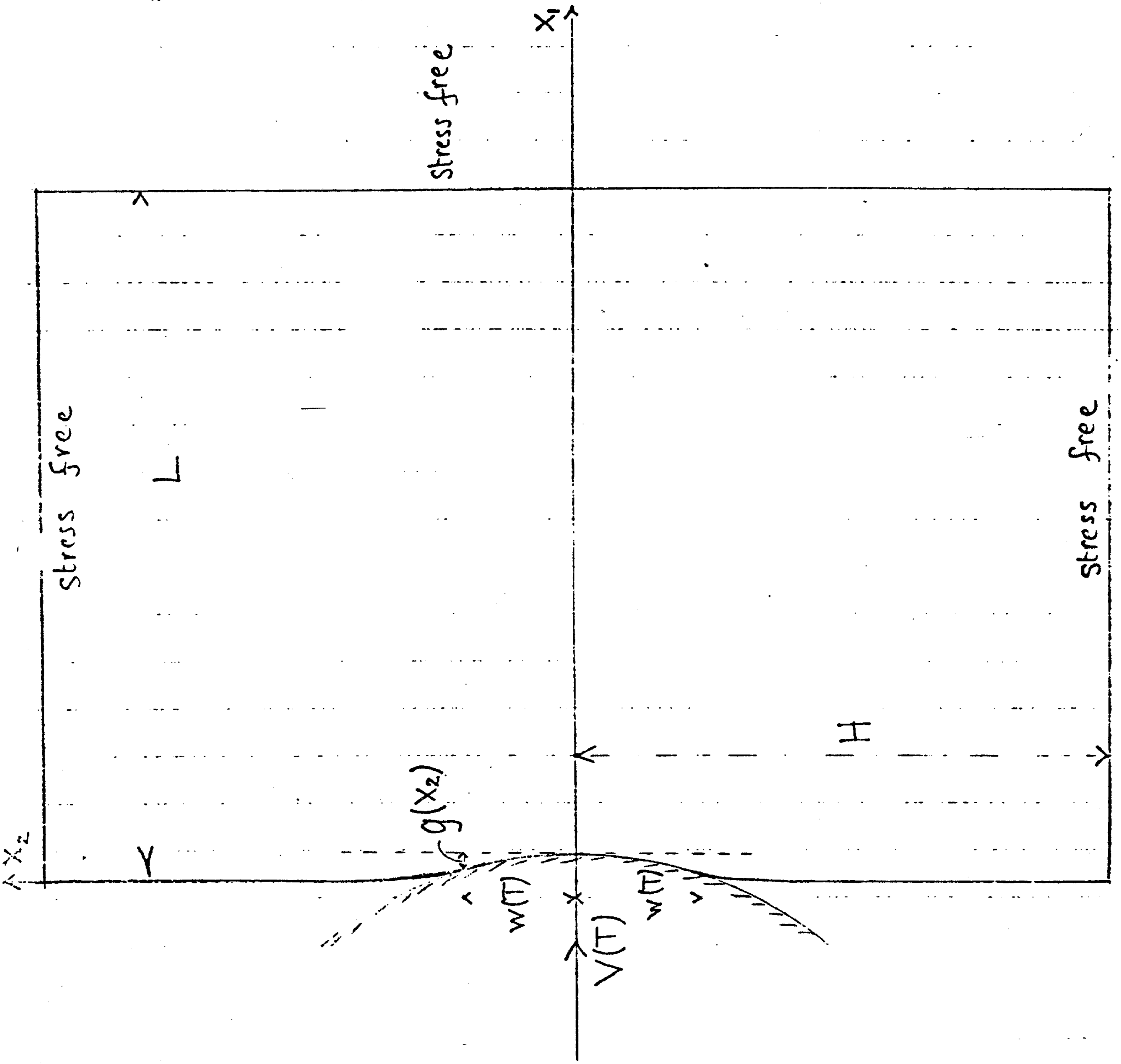

Fig 2   Plane stress configuration for smooth indentation

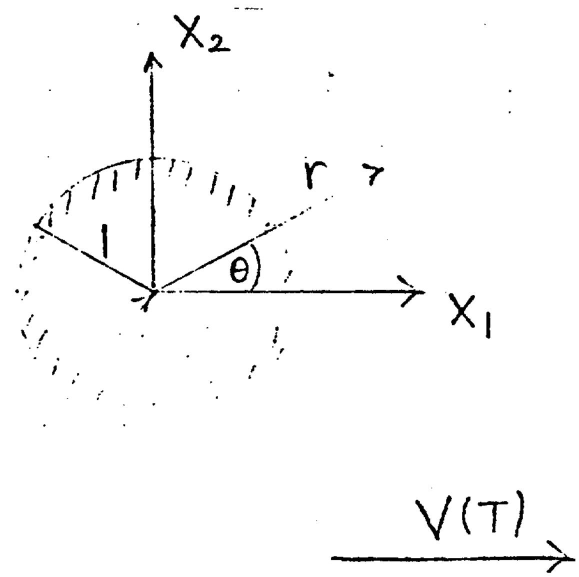

Fig. 3. Rigid inclusion frozen into plate which is set in motion.

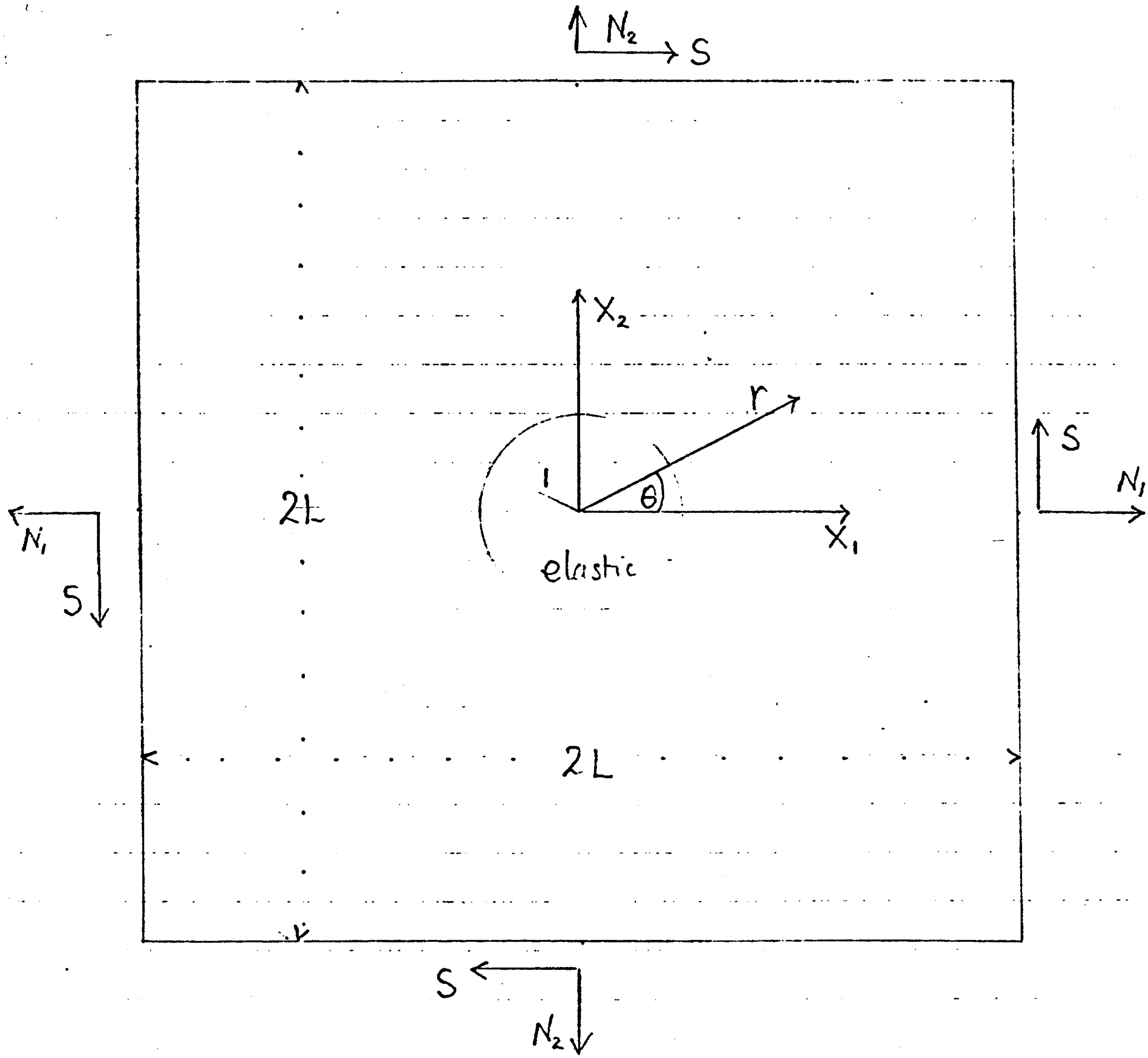

Fig. 4. Elastic inclusion frozen into stressed plate.

Part I

Small Deformation of a Non-Linear Viscoelastic Solid

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1. Introduction
2. Viscoelastic solid model
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Part II

Uni-axial stress

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1. Viscoelastic and elastic models

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1. Viscoelastic and elastic models

Uni-axial stress relations for the small deformation non-linear visceolastic model are obtained by setting $\Sigma_{12} = \Sigma_{22} = 0$, $E_{12} = 0$, $E_{33} = E_{22} = E_2$, and $\Sigma_{11} = \Sigma$ $E_{11} = E$, in the Part I equations (40) and (45) - (50). Adding (48) and (49) gives the uni-axial relation

$$\Phi\dot{E} + 2\Omega E = (\frac{2}{3} + \frac{\Phi}{9K})\varepsilon\dot{\Sigma} + (\frac{2}{3}\Psi + \frac{2\varepsilon\Omega}{9K})\Sigma \quad . \tag{1}$$

From (40),

$$E + 2E_2 - e_o(E^2 + 2E_2^2) = \frac{\varepsilon\Sigma}{3K} , \tag{2}$$

which has the solution

$$E_2 = -\tfrac{1}{2}E + \frac{\varepsilon\Sigma}{6K} \tag{3}$$

when e_o is neglected in comparison with unity. The stress and strain invariants (58) and (59) become

$$\hat{J} = \frac{1}{3}\Sigma^2, \quad \hat{I} = \frac{1}{3}(E-E_2)^2 = \frac{3}{4}E^2 - \frac{\varepsilon\Sigma E}{6K} + \frac{\varepsilon^2\Sigma^2}{108K} . \tag{4}$$

If we define a Poisson's ratio

$$\nu = -\frac{E_2}{E} = \frac{1}{2} - \frac{\varepsilon\Sigma}{6KE} , \tag{5}$$

then since E is not constant at constant stress Σ, and Σ is not linear in time at constant strain-rate $\dot{E}$, then ν varies with time in both these tests, even though compression is elastic, independent of time.

Unloading from any state to zero stress gives an instantaneous elastic strain change followed by relaxation

$$\dot{E} = -\frac{2\Omega E}{\Phi}, = -f(E) \tag{6}$$

since Ω and Φ are evaluated at $\Sigma = 0$; that is, at $\hat{J} = 0$, $\hat{I} = \frac{3}{4}E^2$. If a model is assumed in which the ratio Ω/Φ depends only on $\hat{I}$, independent of $\hat{J}$, then the unloading data function $f(E)$ determines Ω/Φ completely. Now, if $\dot{E} = F(\Sigma, E)$ is the measured response at constant stress Σ, then by (1) and the definition (6) of $f(E)$,

$$F(\Sigma,E) = \frac{2\Psi}{\Phi}\Sigma + \frac{\varepsilon\Sigma}{9K}\frac{f(E)}{E} - f(E). \tag{7}$$

If, further, the response at constant strain-rate $\dot{E} = w$ determines the modulus

$$\dot{E} = w: \quad \frac{d\Sigma}{dE} = \frac{1}{\varepsilon}Y(\Sigma,E), \tag{8}$$

and the strain response $E(\Sigma, w)$ allows an inversion of $w = W(\Sigma, E)$, then (1) leads to the relation

$$(\frac{2}{3} + \frac{\Phi}{9K})Y = \Phi(1 - \frac{F}{W}) , \tag{9}$$

which determines Φ . Hence (7) determines Ψ and Ω is given by (6). The normalisation supposes Φ , Ω , Ψ are order unity functions, and the definition (8) of order unity Y takes into account the initial modulus magnitude ε^{-1}. By equation (48) of Part I, ε^{-1} is precisely the initial shear modulus in the normalised variables, equivalent to an instantaneous elastic stress jump - strain jump ratio.

An elastic relation is obtained from (1) by setting $\Omega = \Psi = 0$ and $\Phi = \Phi(\hat{J}) = \Phi(\frac{1}{3}\Sigma^2)$, then

$$\dot{E} = (\frac{2}{3\Phi} + \frac{1}{9K})\varepsilon\dot{\Sigma} . \tag{10}$$

Define

$$\chi'(\Sigma) = (\frac{2}{3\Phi} + \frac{1}{9K})\varepsilon , \tag{11}$$

so (10) becomes a strain-stress relation

$$E = \chi(\Sigma) , \quad \chi(0) = 0 \tag{12}$$

where $\chi'(0) = O(\varepsilon)$ since $\Phi(0) = O(1)$. A significant non-linearity and order unity modulus at order unity normalised stress Σ requires $\chi'(1) = O(1)$, and a simple model with these features is

$$\chi'(\Sigma) = \varepsilon A[1 + a\Sigma^2] , \tag{13}$$

where

$$A = \varepsilon^{-1}\chi'(0) = O(1) \ , \quad a = O(\varepsilon^{-1}) \ . \tag{14}$$

The linear elastic case is simply $a = 0$.

Effects of non-monotonic strain-rate response at constant stress, reflected by the primary decelerating creep, secondary (stationary) creep, and accelerating tertiary creep illustrated in TR Figs 1 and 3, can be explored by using an appropriate linear viscoelastic relation. Solution of boundary-value problems can then exploit linear analysis, and in particular correspondence methods associated with linear elastic analyses. Of course, the significant non-linear dependence of minimum strain-rate and time to minimum on the applied constant stress is lost, and similarly that of the maximum stress and time to maximum on an applied constant strain-rate. In our normalised variables the stress unit σ_o is a maximum (shear) stress and the strain unit is the relatively uniform strain e_o (~ 0.01) arising at the maximum stress at constant strain-rate, and at the minimum strain-rate at constant stress. The linear model should aim to retain these features. An isotropic linear viscoelastic material in which compression is purely elastic has the equivalent strain and stress formulations

$$\hat{\underset{\sim}{E}}(T) = J_o \underset{\sim}{S}(T) + \int_{-\infty}^{T} J'(T-T')\underset{\sim}{S}(T')dT' \ , \tag{15}$$

$$\underset{\sim}{S}(T) = 2g_o\hat{\underset{\sim}{E}}(T) + \int_{-\infty}^{T} 2g'(T-T')\hat{\underset{\sim}{E}}(T')dT' , \tag{16}$$

$$\mathrm{tr}\,\underset{\sim}{\Sigma} = 3k_o \mathrm{tr}\,\underset{\sim}{E} , \tag{17}$$

where $\underset{\sim}{S}$ and $\hat{\underset{\sim}{E}}$ are the stress and strain deviators

$$\underset{\sim}{S} = \underset{\sim}{\Sigma} - \frac{1}{3}\mathrm{tr}\,\underset{\sim}{\Sigma}\,\underset{\sim}{1} , \quad \hat{\underset{\sim}{E}} = \underset{\sim}{E} - \frac{1}{3}\mathrm{tr}\,\underset{\sim}{E}\,\underset{\sim}{1} . \tag{18}$$

K is the elastic bulk modulus, $g(T)$ is the relaxation function in shear and $J(T)$ is the creep function in shear, both $g(T)$ and $J(T)$ vanish in $T < 0$, and $g_o = g(0+)$, $J_o = J(0+)$. The use of $2g(T)$ in the relaxation form (16) allows direct correspondence with a constant elastic shear modulus g.

In uni-axial stress, (15) and (17) give

$$E(T) = \frac{2}{3}(J_o + \frac{1}{6k_o})\Sigma(T) + \frac{2}{3}\int_{-\infty}^{T} J'(T-T')\Sigma(T')dT' , \tag{19}$$

which has the same form as the shear relation (15), but the similar integral expression for $\Sigma(T)$ in terms of $E(T)$ involves a composite kernel, not simply $g(T)$. Recall that both g_o and k_o are $O(\varepsilon^{-1})$, and hence $J_o = O(\varepsilon)$, but an incompressibility approximation sets $(1/k_o) = 0$. In the latter case, the creep function is $3g(T)$.

Retaining compressibility, define the creep function in uni-axial stress by

$$Z(T) = \frac{1}{9k_o} + \frac{2}{3}J(T) \quad , \tag{20}$$

and let $Y(T)$ be the corresponding relaxation function, then

$$E(T) = Z_o\Sigma(T) + \int_{-\infty}^{T} Z'(T-T')\Sigma(T')dT' \quad , \tag{21}$$

$$\Sigma(T) = Y_oE(T) + \int_{-\infty}^{T} Y'(T-T')E(T')dT' \quad , \tag{22}$$

are equivalent uni-axial relations, with $Y_o = O(\varepsilon^{-1})$, $Z_o = O(\varepsilon)$.

Now the creep function $Z(T)$ is the normalised strain response to unit stress Σ in $T > 0$; that is, to a constant stress σ_o which reflects the maximum stress of concern. The corresponding strain-rate response is

$$\Sigma = H(T) \; : \quad \dot{E}(T) = Z'(T) \quad , \tag{23}$$

which is required to have the forms shown in TR Figs 1 and 3 while strain remains small, that is, while $E \leqslant O(1)$. Thus

$$\left.\begin{aligned} Z'(O) = R_o > O \;, \quad Z''(T) < O \text{ in } O < T < T_m \;, \\ Z''(T_m) = O, \; Z'(T_m) = R_m > O, \quad Z(T_m) = Z_m = O(1) \;, \\ Z''(T) > O \text{ in } T > T_m, \quad Z'(T) \to R_e \text{ as } T \to \infty \;, \end{aligned}\right\} \tag{24}$$

where the asymptotic limit, representing unbounded strain, is included as a convenient model extrapolation beyond the applicable small strain range. An alternative extrapolation is required if the application involves maintained stress for times greatly exceeding T_m.

Similarly, by (22), the stress response to unit applied strain-rate $\dot{E}$ in $T > 0$ is

$$E = T\,H(T) \;:\; \Sigma(T) = \int_0^T Y(T')\,dT' = Q(T) \,, \tag{25}$$

which reaches a maximum $Q(T_M) = Q_M$ at $T = T_M$ and $E = E_M = O(1)$; that is, at strain e_o. Hence the model requires

$$Q_M = O(1) \,, \quad T_M = O(1) \,. \tag{26}$$

The required form of $\Sigma(T)$ is shown in TR Fig. 4, implying

$$\left.\begin{aligned} &Y(T) > 0 \quad \text{in} \quad 0 < T < T_M \,, \\ &Y(T_M) = 0 \,, \quad Y(T) < 0 \quad \text{in} \quad T > T_M \,, \\ &Y(T) \to 0- \,, \quad Q(T) \to Q_E > 0 \quad \text{as} \quad T \to \infty \,. \end{aligned}\right\} \tag{27}$$

We also have the inequalities

$$0 < R_m < R_e < R_o \,, \quad 0 < Q_E < Q_M \,, \tag{28}$$

and would like

$$Q_E = O(1) \tag{29}$$

to avoid a dramatic reduction of stress at constant strain-rate as the maximum stress is passed.

The most simple creep function exhibiting the finite strain-rate as $T \to \infty$ and compatible with the derivative properties (24) is

$$Z(T) = Z_o\{\gamma T + 1 + \sum_{i=1}^{2} j_i(1-e^{-c_i T})\}H(T) \,, \tag{30}$$

where $\gamma Z_o = R_e$. The two exponential terms incorporate two characteristic creep times c_1^{-1} and c_2^{-1}, and at least two such terms are necessary for $Z''(T_m) = 0$ at finite T_m:

$$j_1 c_1^2 e^{-c_1 T_m} + j_2 c_2^2 e^{-c_2 T_m} = 0 \,. \tag{31}$$

Given Z_o and R_e, there are four parameters j_1, j_2, c_1, c_2, which, in principle, may be chosen to fit four other physical properties; for example, R_o, R_m, Z_m, and T_m. Thus,

$$R_o = Z_o(\gamma + j_1 c_1 + j_2 c_2) \,, \tag{32}$$

$$R_m = Z_o(\gamma + j_1 c_1 e^{-c_1 T_m} + j_2 c_2 e^{-c_2 T_m}) \quad , \tag{33}$$

$$Z_m = Z_o\{\gamma T_m + 1 + j_1(1 - e^{-c_1 T_m}) + j_2(1 - e^{-c_2 T_m})\}, \tag{34}$$

in conjunction with (31). Equations (31) - (34) are a highly non-linear implicit system for j_1, j_2, c_1, c_2, which raises questions about uniqueness and numerical stability and accuracy. In particular, magnitudes of starting values for the solution may be necessary, and not obvious in view of the small parameter ε in the model. It is also necessary that the relaxation function $Y(T)$ determined by the creep function $Z(T)$ has the required properties (25) - (29). An analysis of $Z(T)$ and $Y(T)$ simultaneously (Morland, unpublished work) determines the restrictions on the parameters in (30) and associated parameters in $Y(T)$, together with their magnitudes.

Inversion of (21) or (22) to determine the $Z(T)$, $Y(T)$ relation - each determines the other - is conveniently derived by Laplace transforms. Let $\overline{f}(s)$ be the transform of $f(T)$, where $f(T)$ vanishes in $T < 0$ and $f_o = f(0+)$, then

$$\mathcal{L}[f] = \overline{f}(s) = \int_o^\infty f(t)e^{-sT}\,dT \ , \quad \overline{f'(T)} = s\overline{f}(s) - f_o \ , \tag{35}$$

and the transform of a convolution integral is

$$\overline{f}(s)\overline{g}(s) = \mathcal{L}\int_o^T f(T-T')g(T')dT' \ . \tag{36}$$

Hence (21) and (20) give for vanishing Σ and E in $T < 0$:

$$\overline{E}(s) = s\overline{Z}(s)\overline{\Sigma}(s) = \tilde{Z}(s)\overline{\Sigma}(s) \ , \tag{37}$$

$$\overline{\Sigma}(s) = s\overline{Y}(s)\overline{E}(s) = \tilde{Y}(s)\overline{E}(s) \ , \tag{38}$$

so that the transformed material functions satisfy

$$\tilde{Z}(s)\tilde{Y}(s) = 1 \quad \text{or} \quad s\overline{Z}(s)\overline{Y}(s) = \frac{1}{s} \ , \tag{39}$$

which, using the results (35) and (36), gives the Volterra integral relations

$$Z_o Y(T) + \int_o^T Z'(T-T')Y(T')dT'$$

$$= Y_o Z(T) + \int_o^T Y'(T-T')Z(T')dT' = H(T) \ . \tag{40}$$

In particular,

$$Z_o Y_o = 1 \ . \tag{41}$$

For the creep function (30),

$$Z_o^{-1}\tilde{Z}(s) = \frac{Y}{s} + 1 + \frac{j_1 c_1}{s+c_1} + \frac{j_2 c_2}{s+c_2} \ , = \frac{P_3(s)}{sP_2(s)} \tag{42}$$

where $P_2(s)$ and $P_3(s)$ are respectively quadratic and cubic polynomials, with s^2 and s^3 coefficients respectively unity. Hence

$$s\overline{Y}(s) = \tilde{Y}(s) = Y_o s \sum_{i=1}^{3} \frac{y_i}{s+b_i} \ , \quad \sum_{i=1}^{3} y_i = 1 \ , \tag{43}$$

where $-b_i$ $(i = 1,2,3)$ are the roots of

$$P_3(s) = s^3 + s^2(c_1+c_2+\gamma+j_1c_1+j_2c_2) + s(c_1c_2+\gamma c_1+\gamma c_2+j_1c_1c_2+j_2c_1c_2) + \gamma c_1c_2 = 0 \ . \tag{44}$$

As $s \to \infty$, $Z_o^{-1}\tilde{Z}(s) \to 1$, hence $Y_o^{-1}\tilde{Y}(s) \to 1$, which implies the condition $(43)_2$. Inverting (43):

$$Y(T) = Y_o \sum_{i=1}^{3} y_i e^{-b_iT} \ , \tag{45}$$

and in turn

$$Q(T) = Y_o \sum_{i=1}^{3} \frac{y_i}{b_i} (1 - e^{-b_iT}) \ . \tag{46}$$

The required $Y(T)$ and $Q(T)$ properties imply that the roots $-b_i$ must have negative real part, or preferably that the b_i are real and positive. There are no simple explicit conditions on j_1, j_2, c_i, c_2, which guarantee this or the various other properties, but starting with the relaxation form (45) and its properties allows more explicit statements about the parameters y_i, b_i $(i = 1,2,3)$ and j_i, c_i $(i = 1,2)$. Such results, together with an analysis of parameter magnitudes necessary to achieve the large initial modulus in conjunction with order unity Q_M, T_M, Z_M, have been determined (Morland, unpublished work), and allow the construction of a variety of models compatible with both constant stress and constant strain-rate response. In addition, known parameter magnitudes will improve starting values for the numerical solution of equations like (31) - (34) to correlate with prescribed features.

2. Impact of ice plate with rigid wall

Consider the plane stress problem shown in Fig. 1a, but with perfect slip on the wall $X = X_1 = 0$; there is then a uni-axial stress solution compatible with the momentum equations and all boundary conditions. In the non-Newtonian axes fixed with respect to the rear edge of the plate, the momentum equations (97) of Part I reduce to

$$\frac{\partial \Sigma}{\partial X} + \frac{\rho \ell^2 e_o}{\sigma_o t_m^2} \dot{V} = 0 , \qquad (47)$$

where

$$V(T) = \left. \frac{\partial U}{\partial T} \right|_{X=0} \qquad (48)$$

is the plate rear edge velocity in the negative X direction in the Newtonian frame fixed with respect to the wall, and $U(X,T)$ is the particle displacement normal to the wall. Boundary conditions on the rear edge are

$$X = L : \Sigma = 0 \Leftrightarrow E = 0 , \quad U = 0 , \qquad (49)$$

where

$$E = \frac{\partial U}{\partial X} . \qquad (50)$$

Stress and displacement are zero until the moment of impact, $T = 0$, so that $\dot{V}(0) = 0$ by (47), recalling that any discontinuity propagation is supposed to be completed on a shorter time scale, thus

$$T = 0 : \quad \Sigma = U = \dot{V} = 0 , \quad V = V_o , \tag{51}$$

where V_o is the uniform plate velocity before impact.

After impact, the plate compresses and decelerates until the rear edge is at rest, then may partly or fully rebound. It is interesting to derive first the simple analytic linear elastic solution, which introduces comparison values of the time and depth of full penetration, when the plate is brought to rest, and the maximum contact pressure and strain which occur at that time. For the linear elastic material,

$$\Sigma = YE \tag{52}$$

where Y is the constant (normalised) Young's modulus. If the initial modulus Y_o is used, then $Y = 0(\varepsilon^{-1})$, if a modulus at $\Sigma = 1$ is used, then $Y = 0(1)$. Eliminating Σ from (47) through (50) and (52) gives

$$\frac{\partial^2 U}{\partial X^2} = - \frac{\rho \ell^2 e_o}{Y \sigma_o t_m^2} \dot{V}(T) , \tag{53}$$

and integrating subject to the end conditions (49) shows that

$$U = - \frac{\rho \ell^2 e_o}{2 Y \sigma_o t_m^2} \dot{V}(T) (L-X)^2 , \tag{54}$$

and

$$\Sigma = \frac{\rho \ell^2 e_o}{\sigma_o t_m^2} \dot{V}(T)(L-X), < 0 \quad . \tag{55}$$

Now (54) and (48) yield

$$V + \frac{\rho \ell^2 L^2 e_o}{2Y\sigma_o t_m^2} \ddot{V} = 0 \quad , \tag{56}$$

subject to initial conditons (51), and hence

$$V = V_o \cos\left(\frac{\pi}{2} \frac{T}{T_e}\right), \qquad T_e = \frac{\pi \ell L}{2t_m} \left(\frac{\rho e_o}{2Y\sigma_o}\right)^{\frac{1}{2}} \tag{57}$$

where T_e is the normalised penetration time required for the plate to come to rest. In terms of the uni-axial wave speed in the material, the actual penetration time is

$$t_e = t_m T_e = \frac{\pi}{2^{3/2}} \frac{\ell L}{c_e} \quad , \text{ where } \quad c_e = \left(\frac{Y\sigma_o}{\rho e_o}\right)^{\frac{1}{2}} \tag{58}$$

and $\ell L/c_e$ is the wave travel time down the plate (of length ℓL) . Thus t_e is approximately equal to the wave travel time, and is independent of the impact speed $v_o = e_o \ell V_o / t_m$.

The penetration depth is

$$d_e = e_o \ell U(O,T_e) = \frac{2e_o \ell T_e V_o}{\pi} = \frac{2v_o t_e}{\pi} = \frac{\ell L}{2^{\frac{1}{2}}} \frac{v_o}{c_e} , \qquad (59)$$

and the stress and strain at time t_e are

$$-\sigma_e = -\sigma_o \Sigma = \frac{4T_e V_o \sigma_o}{L\pi} = 2^{\frac{1}{2}} \rho c_e v_o = 2^{\frac{1}{2}} \frac{Y\sigma_o}{e_o} \frac{v_o}{c_e} , \qquad (60)$$

$$-e_e = - e_o E = 2^{\frac{1}{2}} \frac{v_o}{c_e} , \qquad (6=)$$

where $Y\sigma_o/e_o$ is the physical uni-axial modulus. That is, the maximum strain magnitude is given by v_o/c_e. Alternatively, equating the elastic strain energy at time t_e with the initial kinetic energy gives

$$\tfrac{1}{2}\rho L \ell v_o^2 = \tfrac{1}{2} \frac{Y\sigma_o}{e_o} \int_o^{\ell L} e^2 dx , \qquad (62)$$

which implies

$$(e^2)_{mean} = \frac{1}{\ell L} \int_o^{\ell L} e^2 dx = (\frac{v_o}{c_e})^2 . \qquad (63)$$

The velocity solution (57) shows as expected that V becomes $-V_o$ at time $T = 2T_e$ when Σ and U become zero and contact is lost; that is, the plate rebounds with velocity V_o.

For the non-linear elastic material, (52) is replaced by (12). Integrating (47) subject to the end condition (49) again gives the linear stress distribution (55), so that (12) implies

$$\frac{\partial U}{\partial X} = E = \chi(\Sigma) \ , \tag{64}$$

and hence

$$U = -\frac{1}{\lambda^2 \dot{V}(T)} \int_0^{\Sigma} \chi(\Sigma')d\Sigma' \ , \tag{65}$$

where

$$\lambda^2 = \frac{\rho \ell^2 e_o}{\sigma_o t_m^2} \ , \tag{66}$$

and at $X = 0$, $\Sigma = \lambda^2 \dot{V}(T)L, = \Sigma_o(T)$ say. Thus

$$V(T) = \frac{\partial U(0,T)}{\partial T} = \frac{\ddot{V}(T)}{\lambda^2 \dot{V}^2(T)} \left\{ \int_0^{\Sigma_o(T)} \chi(\Sigma')d\Sigma' - \Sigma_o(T)\chi[\Sigma_o(T)] \right\}$$

$$= -\frac{\ddot{V}(T)}{\lambda^2 \dot{V}^2(T)} \int_0^{\Sigma_o(T)} \Sigma' \chi'(\Sigma')d\Sigma' \ . \tag{67}$$

Define

$$I(\Sigma_o) = \int_o^{\Sigma_o} \Sigma'\chi'(\Sigma')d\Sigma' , \qquad (68)$$

which is determined by the elastic relation (12), then (67) becomes

$$\frac{I(\Sigma_o)}{\Sigma_o}\frac{d\Sigma_o}{dV} + \lambda^2 V = 0 , \qquad (69)$$

where

$$T = 0 : \quad V = V_o , \quad \Sigma_o = 0 . \qquad (70)$$

Thus

$$\lambda^2(V_o^2 - V^2) = 2\int_o^{\Sigma_o} \frac{I(\Sigma_o')}{\Sigma_o'} d\Sigma_o' = H(\Sigma_o) , \qquad (71)$$

where, since $\chi(\Sigma)$ is odd in Σ, positive in $\Sigma > 0$, $I(\Sigma_o)$ and $H(\Sigma_o)$ are positive and even in Σ_o, and

$$I(\Sigma_o) \sim \Sigma_o^2 , \quad H(\Sigma_o) \sim \Sigma_o^2 \quad \text{as} \quad \Sigma_o \to 0 . \qquad (72)$$

Now

$$\Sigma_o = \lambda^2 L\dot{V} = H^{-1}[\lambda^2(V_o^2 - V^2)] < 0 , \qquad (73)$$

so that

$$T = -\frac{\lambda L}{2}\int_0^{\lambda^2(V_o^2-V^2)} \frac{d\xi}{(\lambda^2 V_o^2-\xi)^{\frac{1}{2}} H^{-1}(\xi)} , \quad (V_o \geq V \geq 0) \qquad (74)$$

determines $V(T)$ implicitly for $0 \leq T \leq T_E$ where $V(T_E) = 0$, with symmetric rebound on $T_E \leq T \leq 2T_E$, and then Σ is given by (55) and U by (65).

The linear elastic solution is recovered by setting

$$\chi'(\Sigma) = 1/Y , \; I(\Sigma_o) = H(\Sigma_o) = \Sigma_o^2/2Y, \; H^{-1}(\xi) = -(2Y\xi)^{\frac{1}{2}}, \qquad (75)$$

when (74) gives $T(V=0) = T_e$ as required. For the non-linear model (13),

$$\left.\begin{aligned} H(\Sigma_o) &= \tfrac{1}{8}\varepsilon A(4\Sigma_o^2 + a\Sigma_o^4) , \\ H^{-1}(\xi) &= -\left(\tfrac{2}{a}\right)^{\frac{1}{2}}\{[1 + \tfrac{2a\xi}{\varepsilon A}]^{\frac{1}{2}} - 1\}^{\frac{1}{2}} , \end{aligned}\right\} \qquad (76)$$

where strong non-linearity is described by $A = 0(1)$, $a = 0(\varepsilon^{-1})$. Numerical quadrature is required to complete the solution and determine the penetration time T at $V = 0$ and the associated depth of penetration and stress.

For the linear viscoelastic material with uni-axial relation (21) or (22), we use the transformed relation (37) or (38) in conjunction with the transforms of (47) - (50) and initial conditions (51); thus

$$\frac{\partial\bar{\Sigma}}{\partial X} + \lambda^2(s\bar{V} - V_o) = 0 \quad , \tag{76}$$

$$\bar{V} = s\bar{U}(0,s), \quad \bar{U}(L,s) = 0 \ , \quad \bar{E}(L,s) = 0 \ , \tag{77}$$

$$\frac{\partial\bar{U}}{\partial X} = \bar{E} = \tilde{Z}\,\bar{\Sigma} \quad . \tag{78}$$

Eliminating $\bar{\Sigma}$ between (76) and (78) gives the transform of (53) with Y replaced by $\tilde{Y} = 1/\tilde{Z}$:

$$\frac{\partial^2\bar{U}}{\partial X^2} = -\lambda^2\tilde{Z}(s\bar{V} - V_o) \quad , \tag{79}$$

and hence

$$\bar{U} = -\tfrac{1}{2}\lambda^2\,\tilde{Z}(s\bar{V} - V_o)(L - X)^2 \quad . \tag{80}$$

Now combining with $(77)_1$ gives

$$\bar{V} = \frac{\lambda^2L^2V_o s}{2\tilde{Y} + \lambda^2L^2s^2} \quad , \tag{81}$$

so that $\bar{V} \sim V_o/s$ as $s \to \infty$ $(\tilde{Y} \to Y_o)$, which implies $V \to V_o$ as $T \to 0$ as required. The linear elastic case is recovered by setting $\tilde{Y} = Y$, constant Y , when (81) inverts to (57). For the model (30) leading to the transform (42),

$$\overline{V} = \frac{V_o \lambda^2 L^2 P_3(s)}{\lambda^2 L^2 s P_3(s) + 2Y_o P_2(s)} , \tag{82}$$

which has the structure and inversion

$$\overline{V} = V_o \sum_{i=1}^{4} \frac{q_i}{s + \alpha_i} , \quad V = V_o \sum_{i=1}^{4} q_i e^{-\alpha_i T} , \quad \sum_{i=1}^{4} q_i = 1 . \tag{83}$$

The roots $(-\alpha_i)$, $i = 1,\ldots,4$ of the denominator of the expression (82) must be determined numerically once the model (30) is prescribed, but a physically sensible solution implies that $\mathrm{Re}(\alpha_i) > 0$ for each root, where the roots are real or occur in complex conjugate pairs. The time $T = T_V$ when $V(T_V) = 0$, $\dot{V} < 0$ in $0 \le T \le T_V$, must be determined numerically, and then "rebound" in $T > T_V$, $V < 0$, investigated to see if contact is lost or if U remains positive. The maximum stress no longer coincides with maximum strain, so will not occur at maximum penetration. A detailed numerical solution for a model of the form (30) compatible with the main features of constant stress and constant strain-rate response would provide guidelines to viscoelastic effects on maximum compression and stress for different impact velocities.

Part III

Linear viscoelastic plane stress problems

Contents

1. Effects of ice plate movement on fixed embedded disk

2. Correlation of far-field ice plate stresses with boundary displacements of an embedded elastic disk

1. Effects of ice plate movement on fixed embedded disk

Consider the plane stress situation illustrated in Fig. 3 where a rigid disk is frozen into an ice plate, which at time $t = 0$ is set in motion with prescribed velocity $v(t)$ in the positive x_1 direction, with $v(0) = 0$. The driving forces, ocean and atmosphere effects, are modelled by a specific body force $\dot{v}$ in the positive x_1 direction with respect to fixed Newtonian axes with origin at the disk centre. Neglecting wave effects, the equilibrium equations (112) of Part I in normalised variables with respect to the polar coordinates (r,θ) are

$$\frac{\partial \Sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \Sigma_{r\theta}}{\partial \theta} + \frac{\Sigma_{rr}-\Sigma_{\theta\theta}}{r} + \lambda^2\dot{V}\cos\theta = 0 \ ,$$

$$\frac{1}{r}\frac{\partial \Sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \Sigma_{r\theta}}{\partial r} + 2\frac{\Sigma_{r\theta}}{r} - \lambda^2\dot{V}\sin\theta = 0 \ , \tag{84}$$

where λ^2 is defined by (66). The traction free plate edge requires

$$r = L : \quad \Sigma_{rr} = \Sigma_{r\theta} = 0 \ , \quad 0 \le \theta \le 2\pi \ , \tag{85}$$

and the frozen on fixed disk requires

$$r = 1 : \quad \dot{U}_r = - V\cos\theta \ , \ \dot{U} = V\sin\theta \ , \ 0 \le \theta \le 2\pi, \tag{86}$$

during the initial period in which displacements and strains remain small, and bonding does not fail. We are considering the situation when

$$\delta = 1/L << 1 \; ; \tag{87}$$

that is, the plate radius is much greater than the embedded disk radius. This is the most simple prototype problem to model the effects on a fixed structure frozen into a large ice plate which is set in motion. The strain-displacement relations are

$$E_{rr} = \frac{\partial U_r}{\partial r} \, , \; E_{\theta\theta} = \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} \, , \; 2E_{r\theta} = \frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} \, . \tag{88}$$

The system is completed by adjoining the stress-strain relations, and initial conditions in the viscoelastic case, namely, that the displacement and stress is zero at time $T = 0+$; impulsive motion is not considered.

In plane stress, $\Sigma_{zz} = \Sigma_{rz} = \Sigma_{\theta z} = 0$, the transforms of the linear viscoelastic relations (15) - (17) give

$$\begin{gathered} \bar{E}_{r} = \tilde{J}\bar{\Sigma}_{r\theta} \, , \quad \bar{E}_{rr} - \bar{E}_{\theta\theta} = \tilde{J}(\bar{\Sigma}_{rr} - \bar{\Sigma}_{\theta\theta}) \, , \\ \bar{E}_{rr} + \bar{E}_{\theta\theta} + \bar{E}_{zz} = (\bar{\Sigma}_{rr} + \bar{\Sigma}_{\theta\theta})/3k_o \, , \quad \bar{E}_{rr} - \bar{E}_{zz} = \tilde{J}\bar{\Sigma}_{rr} \, , \end{gathered} \tag{89}$$

and hence

$$\bar{E}_{r\theta} = \tilde{J}\bar{\Sigma}_{r\theta} \, , \quad \bar{E}_{rr} = \tilde{n}\bar{\Sigma}_{rr} - \tilde{m}\bar{\Sigma}_{\theta\theta} \, , \quad \bar{E}_{\theta\theta} = \tilde{n}\bar{\Sigma}_{\theta\theta} - \tilde{m}\bar{\Sigma}_{rr} \, , \tag{90}$$

where

$$\tilde{n} = (1 + 6k_o\tilde{J})/9k_o, \quad \tilde{m} = (3k_o\tilde{J} - 1)/9k_o , \tag{91}$$

and $\overline{E}_{zz}$ is then evaluated by the last relation of (89). The linear elastic relations are recovered by setting $\tilde{J} = J_o = 1/2g_o$, constant, when $\tilde{n}$ and $\tilde{m}$ are constants, and the relations (90) invert directly to linear relations between the strain and stress components. These are equivalent to the relations (50) of Part I. Now time enters the equilibrium equations (84) only as a parameter in the prescribed $\dot{V}(T)$, which transforms to $s\overline{V}(s)$, and in the boundary velocities (86) which transform to $s\overline{U}_r$ and $s\overline{U}_\theta$ respectively. Hence the system of transformed viscoelastic equations is identical to the elastic system transform in the spatial coordinates when the correspondence

$$\tilde{J} \rightarrow J_o = 1/2g_o \quad (\tilde{m} \rightarrow m , \quad \tilde{n} \rightarrow n) \tag{92}$$

is made, so the transform of the viscoelastic solution is simply that of the elastic solution with J_o replaced by $\tilde{J}$, which can then be inverted (Conventional Correspondence Principle). An approximate elastic solution can be constructed exploiting the small parameter δ defined by (87).

The stress magnitude on the disk is determined by the prescribed plate acceleration $\dot{v}$, since the total body force acting on the plate is balanced by the reaction of the fixed disk. Neglecting δ^2 compared to unity, the total body force is $\rho\pi\ell^2 L^2 \dot{v}$, and a measure of mean traction on the disk is therefore

$$\sigma_{mean} = \frac{\rho\pi\ell^2 L^2 \dot{v}}{2\pi\ell} = \tfrac{1}{2}\rho\ell L^2 \dot{v} \ . \qquad (93)$$

For example, if the disk radius is 10m and $\dot{v} = 2 \times 10^{-2} ms^{-2}$, then

$$\sigma_{mean} \sim 10^4 \rightarrow 10^6 Nm^{-2}, \text{ as } L \sim 10 \rightarrow 100 \ . \qquad (94)$$

As L or $\dot{v}$ increases, σ_{mean} increases as $L^2\dot{v}$, and the local maximum stress (tension and compression) will exceed σ_{mean}.

A solution of the equilibrium equations (84) can be expressed in terms of a biharmonic stress function $\phi(r,\theta)$:

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0 \ , \qquad (95)$$

where

$$\Sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \lambda^2\dot{V}r\cos\theta \ ,$$

$$\Sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} - \lambda^2\dot{V}r\cos\theta \ , \quad \Sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \ . \qquad (96)$$

(see, for example, Chapter 4 of Theory of Elasticity by

S. Timoshenko and J. N. Goodier, McGraw Hill, 1951). The body force contribution gives $\Sigma_{rr} \propto \cos\theta$ on $r = L$, which must be balanced by the stress function contribution to make the edge free of traction, which suggests that $\phi \propto \cos\theta$. A separable solution

$$\phi = \hat{\phi}(r)\cos\theta \tag{97}$$

of (95) gives

$$\hat{\phi} = Ar^3 + Br + \frac{C}{r} + Dr\ell n\, r , \tag{98}$$

$$\Sigma_{rr} = \cos\theta\{2Ar - \frac{2C}{r^3} + \frac{D}{r} - \lambda^2\dot{V}r\} , \tag{99}$$

$$\Sigma_{\theta\theta} = \cos\theta\{6Ar + \frac{2C}{r^3} + \frac{D}{r} - \lambda^2\dot{V}r\} , \tag{100}$$

$$\Sigma_{r\theta} = \sin\theta\{2Ar - \frac{2C}{r^3} + \frac{D}{r}\} , \tag{101}$$

which do not allow Σ_{rr} and $\Sigma_{r\theta}$ to vanish simultaneously on $r = L$. Additional ϕ contributions proportional to $\theta\cos\theta$ or $\theta\sin\theta$ could be investigated, but a direct examination of the net force to be supplied by the disk reaction is more fruitful.

Consider the net force (F_X, F_Y) due to the stress field (99) - (101) on a circle of radius r:

$$\begin{aligned} F_X &= \int_0^{2\pi} (\Sigma_{rr}\cos\theta - \Sigma_{r\theta}\sin\theta)\, r d\theta = -\pi r^2\lambda^2\dot{V} , \\ F_Y &= \int_0^{2\pi} (\Sigma_{rr}\sin\theta + \Sigma_{r\theta}\cos\theta)\, r d\theta = 0 , \end{aligned} \tag{102}$$

and only the body force terms contribute – the ϕ terms are self-equilibrating over all circles. Hence, on $r = L$,

$$F_X = -\pi L^2 \lambda^2 \dot{V} = -\frac{F_B}{1-\delta^2} \tag{103}$$

where F_B is the total body force on the plate. We therefore require an additional stress field which annuls the body force contribution $\Sigma_{rr} = -\lambda^2 \dot{V} L$, $\Sigma_{r\theta} = 0$, on $r = L$, and hence the net force F_X given by (103), and which supplies over $r = 1$ the extra reaction of the plate on the disk (in the positive X_1 direction)

$$F_B + \pi\lambda^2 \dot{V} = \pi L^2 \lambda^2 \dot{V}\ , \ = P \ \text{ say }; \tag{104}$$

that is, a net force P in the negative X_1 direction on the plate. Viewed on the length scale L, this extra net force supplied by the disk bonding over $r = 1$ looks like a point force P at the origin, so we can anticipate that such an additional point force stress field, which is bounded over the plate, will provide an approximate solution. This solution is now constructed and is in fact exact, and allows simple lead order expressions when δ^2 is neglected in comparison with unity.

The stress field for a point force P at the origin along $\theta = \pi$ (T and G, page 113) is

$$\Sigma_{rr} = \frac{P\cos\theta}{4\pi r}\{4 - (1-\nu)\} ,$$

$$\Sigma_{\theta\theta} = -\frac{P\cos\theta}{4\pi r}(1-\nu) , \quad \Sigma_{r\theta} = -\frac{P\sin\theta}{4\pi r}(1-\nu) , \tag{105}$$

where ν is Poisson's ratio and

$$m = \nu n , \quad J_o = (1+\nu)n . \tag{106}$$

Terms with the factor $(1-\nu)$, equivalent to the D terms in (99) - (101), contribute zero net force over r = constant, while $\Sigma_{rr} = P\cos\theta/\pi r$ contributes a net force P in the positive X_1 direction as required. However, the terms with factor $(1-\nu)$ are necessary for compatibility with continuous U_θ at $\theta = \pm\pi$; that is, they imply a discontinuity in U_θ which balances that given by the purely radial field $\Sigma_{rr} = P\cos\theta/\pi r$. Addition of further D terms in (99) - (100), giving such a discontinuity, is not acceptable, so combining A, C, and body force terms with the point force solution (105) gives a stress field of the form

$$\Sigma_{rr} = \frac{P\cos\theta}{4\pi L}\{4[\frac{L}{r} - \frac{r}{L}] + (1-\nu)[-\frac{L}{r} + a\frac{r}{L} + c\frac{L^3}{r^3}]\} \tag{107}$$

$$\Sigma_{r\theta} = \frac{P\sin\theta}{4\pi L}(1-\nu)\{-\frac{L}{r} + a\frac{r}{L} + c\frac{L^3}{r^3}\} , \tag{108}$$

$$\Sigma_{\theta\theta} = \frac{P\cos\theta}{4\pi L}\{-4\frac{r}{L} + (1-\nu)[-\frac{L}{r} + 3a\frac{r}{L} - c\frac{L^3}{r^3}]\} . \tag{109}$$

The edge condition $\Sigma_{rr} = \Sigma_{r\theta} = 0$ on $r = L$ is now satisfied by setting

$$c = 1 - a \,, \tag{110}$$

leaving one free parameter to satisfy two displacement conditions (86) on $r = 1$, if possible. By construction, the net force over $r = 1$ is $P(1-\delta^2) = F_B$ in the positive X_1 direction as required.

By the relations (88) - (90) and (106),

$$\frac{1}{n}\frac{\partial U_r}{\partial r} = \Sigma_{rr} - \nu\Sigma_{\theta\theta}, \quad \frac{1}{nr}\left(\frac{\partial U_\theta}{\partial \theta} + U_r\right) = \Sigma_{\theta\theta} - \nu\Sigma_{rr} \,,$$
$$\frac{1}{n}\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right) = 2(1+\nu)\Sigma_{r\theta} \,. \tag{111}$$

The rigid body displacement corresponding to zero stress which satisfies $U_\theta = 0$ on $\theta = 0$, a required symmetry condition, is

$$U_r = -\frac{nP}{4\pi} b\cos\theta \quad , \quad U_\theta = \frac{nP}{4\pi} b\sin\theta \quad , \tag{112}$$

which is compatible with the boundary condition (86) and can be added to the integral of (111) for the stress field (107) - (109), introducing a second free parameter b.

For (107) - (109),

$$\frac{1}{n}\frac{\partial U_r}{\partial r} = \frac{P\cos\theta}{4\pi L}\{(3+2\nu-\nu^2)\frac{L}{r} + (1-\nu)[(1-3\nu)a-4]\frac{r}{L} + (1-\nu^2)c\frac{L^3}{r^3}\} \quad , \tag{113}$$

$$\frac{1}{nr}(\frac{\partial U_\theta}{\partial \theta} + U_r) = \frac{P\cos\theta}{4\pi L}\{-(1+\nu)^2\frac{L}{r} + (1-\nu)[(3-\nu)a-4]\frac{r}{L} - (1-\nu^2)c\frac{L^3}{r^3}\} \quad , \tag{114}$$

$$\frac{1}{n}(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}) = \frac{P\sin\theta}{4\pi L}(1-\nu^2)\{-2\frac{L}{r} + 2a\frac{r}{L} + 2c\frac{L^3}{r^3}\}. \tag{115}$$

Integrating (113),

$$U_r = \frac{nP\cos\theta}{4\pi}\{(3+2\nu-\nu^2)\ell nr + \tfrac{1}{2}(1-\nu)[(1-3\nu)a-4](\frac{r}{L})^2 - \tfrac{1}{2}(1-\nu^2)c(\frac{L}{r})^2\}, \tag{116}$$

and now from (114)

$$U = \frac{nP\sin\theta}{4\pi}\{-(3+2\nu-\nu^2)\ell nr - (1+\nu)^2 + \tfrac{1}{2}(1-\nu)[(5+\nu)a-4](\frac{r}{L})^2 - \tfrac{1}{2}(1-\nu^2)c(\frac{L}{r})^2\} \quad . \tag{117}$$

Substituting (116) and (117) confirms that the compatibility condition (115) is satisfied.

Adding the rigid body displacements (112) to (116) and (117) and applying the boundary conditions (86) in the form

$$r = 1 : \quad U_r = -q\cos\theta \quad , U_\theta = q\sin\theta \quad , \quad q(T) = \int_0^T (V(T')dT' , \tag{118}$$

gives

$$\frac{4\pi q}{nP} = b - \tfrac{1}{2}(1-\nu)\delta^2[(1-3\nu)a-4] + \tfrac{1}{2}(1-\nu^2)c\delta^{-2} ,$$

$$= b - (1+\nu)^2 + \tfrac{1}{2}(1-\nu)\delta^2[(5+\nu)a-4] - \tfrac{1}{2}(1-\nu^2)c\delta^{-2} . \tag{119}$$

Hence, subtracting to eliminate q and b, and using the relation (110),

$$a(1 + \frac{3-\nu}{1+\nu}\delta^4) = 1 + (\frac{1+\nu}{1-\nu})^2\delta^2 + \frac{4\delta^4}{1+\nu} = O(1) ,$$

$$c(1 + \frac{3-\nu}{1+\nu}\delta^4) = - \delta^2(\frac{1+\nu}{1-\nu} + \delta^2) = O(\delta^2) , \tag{120}$$

which are independent of the displacement magnitude q. Thus, the stresses are simply proportional to the acceleration $\dot{V}(T)$ through P in the elastic case when ν is constant, but for a viscoelastic plate are given by convolutions of $\dot{V}(T)$ with various time dependent material functions. The parameter b which enters the rigid body displacement (112) is now given by (119), and depends explicitly on $\dot{V}(T)$ and $q(T)$ in the elastic case, and on various convolutions in the viscoelastic case.

The stress at the disk interface is

$$\Sigma_{rr} = \frac{P\cos\theta}{4\pi}\{4(1-\delta^2) + (1-\nu)[-1 + a\delta^2 + c\delta^{-2}]\} ,$$

$$= \tfrac{1}{2}\lambda^2 L^2 \dot{V}\cos\theta\{1 + O(\delta^2)\} , \quad (121)$$

$$\Sigma_{r\theta} = \frac{P\sin\theta}{4\pi}(1-\nu)\{-1 + a\delta^2 + c\delta^{-2}\} ,$$

$$= -\tfrac{1}{2}\lambda^2 L^2 \dot{V}\sin\theta\{1 + O(\delta^2)\} , \quad (122)$$

$$\Sigma_{\theta\theta} = \frac{P\cos\theta}{4\pi}\{-4\delta^2 + (1-\nu)[-1 + 3a\delta^2 - c\delta^{-2}]\} ,$$

$$= \tfrac{1}{2}\nu\lambda^2 L^2 \dot{V}\cos\theta\{1 + O(\delta^2)\} , \quad (123)$$

and hence the lead order tractions Σ_{rr}, $\Sigma_{r\theta}$ are independent of the elastic or viscoelastic properties, but the lead order hoop stress is proportional to Poisson's ratio ν, or a convolution of the associated $\nu(T)$ in the viscoelastic case. For an incompressible plate $\nu = \frac{1}{2}$, $n = \frac{2}{3}J_o$ or $\tilde{n} = \frac{2}{3}\tilde{J}$, the stresses (107) - (109) are proportional to $\dot{V}$ and identical for both elastic and viscoelastic shear, and the displacements are proportional to J_oP or the convolution of $J(T)$ and $\dot{P}(T)$ respectively. With the elastic dilatation, viscoelastic shear, model

$$\tilde{\nu}(s) = \frac{3k_o\tilde{J}(s) - 1}{6k_o\tilde{J}(s) + 1} , \quad (124)$$

so that both $\tilde{n}(s)$ and $\tilde{\nu}(s)$ are rational functions for the model in Part II, and the viscoelastic transforms corresponding to (107) - (109) and (112), (116), (117), are rational functions which invert to sums of exponentials.

Note that the maximum normalised interface stress is $|\frac{1}{2}\lambda^2 L^2 \dot{V}|$, which corresponds to a physical stress $\frac{1}{2}\rho \ell L^2 \dot{V}$ which is the estimated mean (93).

Complex variable methods and conformal mapping would allow solutions for alternative disk shapes, for example, an embedded elliptic inclusion, to investigate effects of curvature and aspect ratio. Exploiting the large plate condition $\delta << 1$ may lead to relatively simple lead order approximation for the body force, point force, plus correction stress fields construction. Inversion of the viscoelastic transforms will show how the ratio of acceleration and viscoelastic time scales influences the interface stresses.

2. Correlation of far-field ice plate stresses with boundary displacements of an embedded elastic disk

Consider the configuration shown in Fig. 4 when the axes OX_1X_2 are principal axes of the uniform far field stress. Then, as $r \to \infty$:

$$\Sigma_{xx} \to \Sigma_1 \ , \quad \Sigma_{yy} \to \Sigma_2 \ , \quad \Sigma_{xy} \to 0 \ ,$$

$$\left.\begin{aligned} \Sigma_{rr} &\to \tfrac{1}{2}(\Sigma_1 + \Sigma_2) + \tfrac{1}{2}(\Sigma_1 - \Sigma_2)\cos 2\theta \ , \\ \Sigma_{r\theta} &\to -\tfrac{1}{2}(\Sigma_1 - \Sigma_2)\sin 2\theta \ , \\ \Sigma_{\theta\theta} &\to \tfrac{1}{2}(\Sigma_1 + \Sigma_2) - \tfrac{1}{2}(\Sigma_1 - \Sigma_2)\cos 2\theta \ , \end{aligned}\right\} \qquad (125)$$

where Σ_1, Σ_2 are the principal stresses at infinity. We suppose that the plate is initially in compression at uniform stress $\Sigma_1 < 0$, $\Sigma_2 < 0$, and that the far field is not disturbed by the embedding of the elastic disk. Let the elastic disk be initially stress free and have radius $1 + \delta$ $(0 \le \delta << 1)$. Consider two embedding problems:

(A) Hole of unit radius cut in plate and disk simultaneously $(T = 0)$ embedded and bonded (frozen on),

(B) Hole of unit radius cut in plate, contracts as boundary stress becomes zero during instantaneous elastic response of plate, then $(T = 0)$ disk embedded and bonded (frozen on).

The bonding implies that no tangential slip occurs in $T > 0$, and that the interface can support tension if required. Thus the interface conditions on $r = 1$ are continuity of U_r and U_θ and of Σ_{rr} and $\Sigma_{r\theta}$ in $T > 0$. Procedure (B) allows a tangential displacement $U_\theta^{(1)}$ at the plate boundary $r = 1$ during the instantaneous stress relaxation before the disk is embedded and bonded.

Let $\underset{\sim}{\Sigma}^d$, $\underline{U}^d$ denote stress and displacement in the disk, and $\underset{\sim}{\Sigma}^p$, $\underline{U}^p$ the stress and displacement in the plate additional to the uniform field $\underset{\sim}{\Sigma}^o$, $\underline{U}^o$ in $T < 0$. Setting

$$\tfrac{1}{2}(\Sigma_1 + \Sigma_2) = -P\ ,\quad \tfrac{1}{2}(\Sigma_1 - \Sigma_2) = S\ , \tag{126}$$

then

$$\begin{aligned} \Sigma^o_{rr} &= -P + S\cos 2\theta\ , \\ \Sigma^o_{r} &= -S\sin 2\theta\ , \\ \Sigma^o_{\theta\theta} &= -P - S\cos 2\theta\ , \end{aligned} \tag{127}$$

and $\underline{U}^o$ does not influence the subsequent deformation. For procedure (A), the interface conditions at $r = 1$ for $T > 0$ are

$$\Sigma^d_{rr} = \Sigma^o_{rr} + \Sigma^p_{rr}\ ,\quad \Sigma^d_{r\theta} = \Sigma^o_{r\theta} + \Sigma^p_{r\theta}\ , \tag{128}$$

$$U^d_r + \delta = U^p_r\ ,\quad U^d_\theta = U^p_\theta\ , \tag{129}$$

while in procedure (B) the displacement conditions (129) are replaced by

$$U_r^d + \delta = U_r^p \quad , \quad U_\theta^d = U_\theta^p - U_\theta^{(1)} \quad . \tag{130}$$

That is, in procedure (B), the extra plate stress $\underset{\sim}{\Sigma}^p$ is split into an instantaneous relaxation $\underset{\sim}{\Sigma}^{(1)}$ to make the hole boundary stress free, plus a reloading $\underset{\sim}{\Sigma}^{(2)}$ when the disk is embedded, with corresponding displacements $\underline{U}^{(1)}$ and $\underline{U}^{(2)}$. The radial displacement of a boundary point is the sum of $U_r^{(1)}$ and $U_r^{(2)}$, namely U_r^p, but tangential displacement subsequent to bonding is $U_\theta^{(2)} = U_\theta^p - U_\theta^{(1)}$. We also require, from (125) and (127),

$$\Sigma_{rr}^p \; , \quad \Sigma_{r\theta}^p , \quad \Sigma_{\theta\theta}^p \to 0 \quad \text{as} \quad r \to \infty \quad . \tag{131}$$

The solution for both problems (A) and (B) are determined, and are inverted to express P and S in terms of the displacements of the disk boundary.

First treat the plate as elastic with the constants n, ν of the previous section, and let the disk have corresponding constants n_d, ν_d. We seek equilibrium solutions of (84) in the absence of the body force term $(V = 0)$ for the plate and the disk such that Σ_{rr}^d, Σ_{rr}^p, and $\Sigma_{r\theta}^d$, $\Sigma_{r\theta}^p$ incorporate the $\cos 2\theta$ and $\sin 2\theta$ terms necessary to satisfy the traction conditions (128) on $r = 1$, and generate U_r^d, U_r^p and

U^d_θ , U^p_θ with the same θ dependence on $r = 1$ to satisfy (129). In the disk, $\underset{\sim}{\Sigma}^d$ is bounded at $r = 0$, and in the plate $\underset{\sim}{\Sigma}^p \to \underset{\sim}{0}$ as $r \to \infty$. Suitable stress fields can be constructed by symmetric and separable stress function solutions analogous to those given by T & G, pp. 58 - 80, namely

$$\Sigma^d_{rr} = - \cos 2\theta \, A_d + C_d \, ,$$
$$\Sigma^d_{r\theta} = \sin 2\theta (A_d + B_d r^2) \, , \qquad (132)$$
$$\Sigma^d_{\theta\theta} = \cos 2\theta (A_d + 2B_d r^2) + C_d \quad ;$$

$$\Sigma^p_{rr} = - \cos 2\theta \left(\frac{A_p}{r^4} + 2 \frac{B_p}{r^2}\right) + \frac{C_p}{r^2} \quad ,$$
$$\Sigma^p_{r\theta} = - \sin 2\theta \left(\frac{A_p}{r^4} + \frac{B_p}{r^2}\right) \, , \qquad (133)$$
$$\Sigma^p_{\theta\theta} = \cos 2\theta \, \frac{A_p}{r^4} - \frac{C_p}{r^2} \quad ;$$

which satisfy conditions at $r = 0$ and as $r \to \infty$. Traction continuity (128) gives the three relations

$$C_d = - P + C_p \quad ,$$
$$A_d = - S + A_p + 2B_p \, , \qquad (134)$$
$$A_d + B_d = - S - A_p - B_p \quad .$$

Integrating the stress-displacement relations (111) subject to the symmetry conditions

$$U_\theta = 0 \quad \text{on} \quad \theta = 0, \quad \pi/2 \ , \quad \pi \ , \tag{135}$$

gives

$$U_r^d = - n_d \cos 2\theta \{(1+\nu_d)A_d r + \tfrac{2}{3}\nu_d B_d r^3\} + n_d(1-\nu_d)C_d r \ ,$$

$$U_\theta^d = n_d \sin 2\theta \{(1+\nu_d)A_d r + \tfrac{1}{3}(3+\nu_d)B_d r^3\} \ ; \tag{136}$$

and

$$U_r^p = n \cos 2\theta \{\tfrac{1}{3}(1+\nu)A_p r^{-3} + 2B_p r^{-1}\} - n(1+\nu)C_p r^{-1} \ ,$$

$$U^p = n \sin 2\theta \{\tfrac{1}{3}(1+\nu)A_p r^{-3} - (1-\nu)B_p r^{-1}\} \ . \tag{137}$$

The stress field $\underset{\sim}{\Sigma}^{(1)}$ required to make $r = 1$ stress free is given by the field (133) with coefficients A_1, B_1, C_1, in place of A_p, B_p, C_p, which satisfy (134) with $A_d = B_d = C_d = 0$. Thus

$$C_1 = P \ , \quad B_1 = 2S \ , \quad A_1 = -3S \ , \tag{138}$$

and hence the instantaneous tangential displacement on $r = 1$ is

$$U_\theta^{(1)} = - n_o S \sin 2\theta (3-\nu_o) \ , \tag{139}$$

where n_o, ν_o are the instantaneous elastic parameters if the plate is viscoelastic. When the far field stress is an isotropic pressure, $S = 0$, $U_\theta^{(1)} = 0$.

The displacement conditions (129) of problem (A) give

$$\left.\begin{aligned} R_r &= n_d(1-\nu_d)C_d = -\delta - n(1+\nu)C_p \,, \\ R_\theta &= -n_d\{(1+\nu_d)A_d + \tfrac{2}{3}\nu_d B_d\} = n\{\tfrac{1}{3}(1+\nu)A_p + 2B_p\} \,, \\ \textcircled{H} &= n_d\{(1+\nu_d)A_d + \tfrac{1}{3}(3+\nu_d)B_d\} = n\{\tfrac{1}{3}(1+\nu)A_p - (1-\nu)B_p\} \,, \end{aligned}\right\} \quad (140)$$

where we have written

$$U_r^d(1,\theta) = R_r + R_\theta \cos 2\theta \,, \quad U_\theta^d(1,\theta) = \textcircled{H} \sin 2\theta \,. \quad (141)$$

Thus, if the disk boundary displacement is measured, compatible with the distribution (141), then R_r, R_θ, $\textcircled{H}$, are prescribed quantities, functions of time when the plate is viscoelastic. The six relations (134) and (140) determine $A_d, B_d, C_d, A_p, B_p, C_p$, and hence R_r, R_θ, $\textcircled{H}$, given P and S, or alternatively, given compatible R_r, R_θ, $\textcircled{H}$, they determine the far field stress P and S. The condition (130) of problem (B), with expression (139), replaces $(140)_3$ by

$$\textcircled{H} = n_d\{(1+\nu_d)A_d + \tfrac{1}{3}(3+\nu_d)B_d\} = n\{\tfrac{1}{3}(1+\nu)A_p - (1-\nu)B_p\} + n_o S(3-\nu_o) \,. \quad (142)$$

By $(134)_1$ and $(140)_1$,

$$C_p = C_d + P\ , \quad C_d\{n_d(1-\nu_d) + n(1+\nu)\} = -\delta - n(1+\nu)P, \tag{143}$$

determine C_p and C_d, or

$$-P = \frac{R_r + \delta}{n(1+\nu)} + \frac{R_r}{n_d(1-\nu_d)} \tag{144}$$

determines P in terms of R_r. Equations $(134)_{2,3}$ and $(140)_{2,3}$ are 4 relations for A_d, B_d, A_p, B_p, independent of C_d and C_p, so S can be expressed in terms of (H), with analogous results for problem (B) when (142) replaces $(140)_3$. Both sets are conveniently expressed in matrix form, respectively

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & \dfrac{2\nu_d}{3(1+\nu_d)} & \frac{1}{3}j & \dfrac{2j}{1+\nu} \\ 1 & \dfrac{3+\nu_d}{3(1+\nu_d)} & -\frac{1}{3}j & \dfrac{j(1-\nu)}{1-\nu} \end{bmatrix} \begin{bmatrix} A_d \\ B_d \\ A_p \\ B_p \end{bmatrix} = S \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ or } S \begin{bmatrix} -1 \\ -1 \\ 0 \\ n_o(3-\nu_o) \end{bmatrix} \tag{145}$$

where

$$j = \frac{n(1+\nu)}{n_d(1+\nu_d)} \ . \tag{146}$$

The matrix determinant is

$$\Delta = j\{1 + \frac{3j(3-\nu)}{1+\nu}\} + \frac{1}{3(1+\nu_d)}\{\frac{12j(3+\nu\nu_d)}{1+\nu} + 3 - \nu_d\} > 0 , \quad (147)$$

so by Kramer's rule

$$\Delta B_d = \begin{vmatrix} 1 & -1 & -1 & -2 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & \frac{1}{3}j & 2j/(1+\nu) \\ 1 & n_o(3-\nu_o) & -\frac{1}{3}j & j(1-\nu)/(1+\nu) \end{vmatrix}$$

$$= n_o(3-\nu_o)(1 - j + \frac{4j}{1+\nu}) , \quad (148)$$

with n_o set zero for problem (A).

Hence, for problem (A),

$$B_d = 0 , \quad (149)$$

and the relations (145) are simplified. Continuing with (A), the first two relations of (145) show that

$$A_p = -\frac{3}{2}B_p , \qquad A_d = -S + \frac{1}{2}B_p , \quad (150)$$

while the last two relations confirm $(150)_1$ and give

$$A_d = -\frac{j(3-\nu)}{2(1+\nu)} B_p , \quad (151)$$

and hence

$$S = \frac{1 + \nu + j(3-\nu)}{2(1+\nu)} B_p \quad , \tag{152}$$

so that B_p, A_d, A_p are determined by S, all zero if $S = 0$. Alternatively, for given $\textcircled{H}$, $(140)_3$ with (151), (152), gives

$$S = - \textcircled{H} \frac{1 + \nu + j(3-\nu)}{n(1+\nu)(3-\nu)} . \tag{153}$$

For problem (B), B_d is given by (147) and (148), then

$$A_p = -\frac{3}{2} B_p - \frac{1}{2} B_d \ , \quad A_d = - S + \frac{1}{2} B_p - \frac{1}{2} B_d \ ,$$

$$S\{1 + \frac{1}{2} n_o (3-\nu_o)\} = \frac{1 + \nu + j(3-\nu)}{2(1+\nu)} B_p \quad , \tag{154}$$

and

$$\textcircled{H} = \frac{1}{6} n_d (3-\nu_d) B_d - \frac{S\{- jn(1+\nu)(3-\nu) + \frac{1}{2} n_o n (3-\nu_o)(1+\nu)^2\}}{j\{1 + \nu + j(3-\nu)\}} . \tag{155}$$

determines S if $\textcircled{H}$ is given.

Viscoelastic plate solutions are given by the Laplace transforms of the above relations with n replaced by $\tilde{n}(s)$,

ν replaced by $\tilde{\nu}(s)$ etc. Since P and S are independent of T, we find, for example, that in problem (A) by (144):

$$- \frac{P}{s} = \frac{\overline{R}_r(s) + \delta}{\tilde{J}(s)} + \frac{\overline{R}_r(s)}{n_d(1-\nu_d)} . \tag{156}$$

Thus, the measured $R_r(T)$ must be compatible with (156) for all time. In particular, the instantaneous response is given by

$$- P = \frac{R_r(0+) + \delta}{J_o} + \frac{R_r(0+)}{n_d(1-\nu_d)} . \tag{157}$$

Since the initial elastic response of the non-linear viscoelastic model of ice also has high modulus and gives an infinitesimal strain, (157) will apply with J_o simply the instantaneous elastic modulus in shear. Similar conclusions apply to the elastic relations (153) and (155) applied at $T = 0+$.

Concluding Remarks

The solution of ice force problems on time scales in which the ice responds as a highly non-linear viscoelastic material, solid or fluid models, will require numerical methods. This report has formulated small strain approximations for solids of differential type, and the complete slow flow equations in plane stress and plane strain, when time scales are much larger than wave travel times through the region of interest. Implicit finite difference schemes for time-stepping reduce the problems to a sequence of non-homogeneous elastic problems, which could be approached by finite difference or finite element methods. Possibly the combined space and time variable domain can be treated by finite element methods. As integral type models are constructed to describe the viscoelastic response of ice, these should be investigated in the context of boundary-value problem formulation, since the Volterra integrals may be more tractable to fast and accurate numerical solution. An approximate integral type model matching the main features of uni-axial compression data should be available soon, and will require extrapolation to a frame indifferent tensor relation for multi-axial loading, using plausible bi-axial response in the absence of detailed data.

Further development of linear viscoleastic models which match the main featuresof the non-linear response at constant stress and constant strain-rate at chosen levels would be useful. Such a uni-axial correlation, together with assumptions of isotropy and incompressibility, fully determines the response to multi-axial loading. A key feature is the high initial elastic modulus compared to stress-strain ratios during creep, but analysis has indicated the relative magnitudes of various material parameters needed to meet this requirement in conjunction with other features. Given such a linear model, a variety of contact problems could be solved to investigate the influence of loading time scale in comparison with material time scale, particularly on maximum stresses reached. For example, the one-dimensional impact problem described in Part II could be solved in detail, and also the embedded disk problem described in Part III. Such solutions will also provide a valuable test scheme for the stability and accuracy of the time-iteration numerical procedures in the non-linear numerical algorithms. In addition, solutions of simple problems for a highly non-linear elastic model, reflecting the dramatic change of modulus, would test the non-linear aspects of the spatial problem.

It may also be useful to examine loading-unloading-reloading responses of both linear and non-linear viscoelastic models at different strain-rates, in particular at small strain-rates, to demonstrate that in the stress-strain domain (uni-axial), eliminating time, they exhibit yield and hystereris features associated with rate-independent plasticity theory. In particular, the sensitivity to small (non-zero) strain-rate can be examined. Here, small strain-rate implies loading times long compared to the material characteristic time. This aspect arose from discussions with Dr. R. Norgren of Shell, and the results would provide some basis for the construction of a viscoplastic model incorporating a yield concept if this proves necessary.