THE GSD95 GEOID MODEL FOR CANADA

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Abstract: This paper describes the basic theory, assumptions and approximations behind the computation of the Canadian GSD95 geoid model. It discusses the remove-restore approach and describes Helmert's second condensation method. This latter topic shows the sequence of the gravity reductions to be applied to the surface gravity measurements in order to comply with the Stokes theory. It introduces a series of approximations to the Stokes theory in order to evaluate the geoid in a practical way. Finally, it discusses the comparison between the GSD95 geoid model and GPS-levelling data.

1. Introduction

The use of a geoid model in combination with the Global Positioning System (GPS) provides orthometric heights (H) (i.e. heights above the geoid along the plumbline). This is accomplished by subtracting a geoid height (N) from a GPS-observed ellipsoidal height (h): H = h - N. Since June, 1992, Canadian GPS users have been using the GSD91 geoid model (Véronneau and Mainville, 1992) in order to determine geoid heights. Because of the acquisition of additional data (gravity and a Digital Elevation Model), the Geodetic Survey Division of Natural Resources Canada have adopted a new model and made it available to the GPS and satellite altimetry users. This new model is distributed under the name of GSD95.

The GSD95 model is computed in three parts using a remove-restore technique (Rapp and Rummel, 1975). The first part of the model is computed from the OSU91A geopotential model (Rapp et al., 1991). The second part is the contribution of the local gravity measurements reduced by the global field (OSU91A). GSD95 uses some 1.5 million surface gravity measurements (land and oceans) coming from the Geodetic Survey Division, the U.S. National Geodetic Survey and the U.S. Defence Mapping Agency. In addition, the shipboard measurements are augmented by satellite altimetry-derived gravity data determined by Sandwell and Smith (1994). These data are used to fill gaps in the shipboard data. Finally, the third part is the primary indirect effect on the geoid, which is due to the condensation of the masses above the geoid according to Helmert’s second condensation method (Heiskanen and Moritz, 1967). The GSD95 model covers most of Canada. It extends from latitudes 41°N to 72°N and from longitudes 218°E to 314°E. The interval of the grid is 5 minutes of arc.

2. The Remove-Restore Method and Helmert’s Second Condensation Approach

The determination of a geoid model using Stokes’s approach requires global gravity coverage. The collection of such a set of gravity data is difficult, if not impossible, for two main reasons: 1) gravity still has to be measured in many mainly remote regions of the world and 2) a wealth of gravity information is still classified. Furthermore, the collection of all of the observed gravity values in a unified horizontal and vertical datum from different agencies along with the required storage space could also be a major problem for many organisations. However, these inconveniences can be avoided partly by using the remove-restore technique. The Stokes integral (Heiskanen and Moritz, 1967 eq. 2-163b) is written as follows:

$$N = \frac{R}{4\pi\gamma}\int\int_{\sigma} \Delta g_{gm}\, S(\psi)\, d\sigma + \frac{R}{4\pi\gamma}\int\int_{\sigma_0} (\Delta g - \Delta g_{gm})\, S(\psi)\, d\sigma + \varepsilon_N \quad . (1)$$

The first term is the contribution to the gravity anomalies from a global geopotential model (Dg_{gm}), the second term is the contribution from the surface gravity measurements (Dg), less the geopotential contribution for the computational area s_0, and the third term is the truncation error, i.e. the geoid height information missing from gravity data outside the computational region ($s - s_0$). The truncation error is neglected for the computation, but it can be significant. An indication of its magnitude can be measured by determining two geoid models with different size s_0. The truncation error for GSD95 reaches several decimetres along the edges and produces a 20-cm leakage across Canada.

Stokes's integral for the determination of the geoid holds only when the disturbing potential is a harmonic function outside the geoid. Therefore, the gravity anomalies have to refer to the geoid and no masses should be present above the geoid. Thus, these two conditions have to be satisfied by suitable gravity reductions. The procedure followed in this paper is the Helmert's second condensation method (Heiskanen and Moritz, 1967). This method is applied in three steps: 1) remove all masses outside the geoid (AT); 2) restore these masses as a condensed layer onto the geoid (AcT); and 3) downward continue gravity from point P on the topography to a point P0 on the geoid (Dg), along the plumb line.

These reductions are summarized in the following equation:

$$\Delta g^{H} = g_{p} - A_{T} + A_{T}^{c} - D_{g} - \gamma_{q}, \quad (2)$$

where g_p is a surface gravity measurement and g_q is the normal gravity on the ellipsoid. A_T consists of three terms as follows:

$$A_{T} = A_{B} + A_{t} + A_{atm}, \quad (3)$$

where A_B is a Bouguer plate reduction, A_t is the terrain correction, which is a refinement to A_B by accounting the deviation of the topography from the Bouguer plate and A_{atm} is the atmospheric gravity correction. Similarly, the attraction of the condensed masses $A^c{}_T$ is split into three terms which will be represented by the superscript c later in the paper.

Helmert anomalies can be introduced into the Stokes integral because they refer to the geoid with no masses present above it. However, the computation of the Stokes formula using Helmert anomalies results in the cogeoid. A primary indirect effect has to be added to the cogeoid to take into consideration the effect of the condensation process on the geoid. Finally, the geoid height can be expressed as:

$$N = N_1 + \frac{R}{4\pi\gamma}\int\int_{\sigma_0}(\Delta g^H - \Delta g_{gm})\,S(\psi)\,d\sigma + N_3 + e_N \quad , (4)$$

where N_1 is a geoid height derived from a global geopotential model, N_3 is the primary indirect effect on the geoid produced by the condensation of the masses above the geoid, and e_N is the truncation error. The second term on the right hand side will be represented by N_2 in the following section.

3. Assumptions for the Computation of GSD95

The geoid model cannot be computed as proposed in section 3 because the integral terms N_1 and N_2 cannot be evaluated analytically. Therefore, a series of assumptions and approximations are applied to equation (4) in order to determine the GSD95 geoid model.

The theory assumes that the reference ellipsoid has the same potential as the geoid and the same mass as the earth. By taking into consideration this assumption, the geoid model will be biased by a constant N_0. In addition, it is assumed that the centre of the ellipsoid coincides with the centre of gravity of the Earth. Thus, the GSD95 will not be compatible with the North American Datum of 1983 (NAD83).

The term N_1 is computed from a spherical harmonic expansion as follows:

$$N_1 = N_{gm} = \frac{GM}{r\gamma}\sum_{n=2}^{360}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n}(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\bar{P}_{nm}(\sin\phi) \quad , (5)$$

where g is the normal gravity on the GRS80 reference ellipsoid at the geodetic latitude of the computation point, *r* is the geocentric radial distance of the computation point projected on the ellipsoid, *a* is the semi-major axis, *f* is the latitude and *l* is the longitude. The fully normalized harmonic coefficients, C_{nm} and S_{nm}, are obtained from the OSU91A geopotential model to degree and order 360 (Rapp et al., 1991). Finally, P_{nm} are the fully normalized associated Legendre polynomials.

The determination of N_2 is the most complex in equation (4). It involves two important approximations: 1) the gravity reductions are applied in a planar approximation and 2) the gravity anomalies are linearly correlated to the topography. The former allows the determination of A_B, A^c_B, A_t and A^c_t which can be expressed as:

$$A_B = A_B^c = 2\pi G\rho H \quad , (6)$$

$$A_t \approx -\frac{1}{2} G\rho \int\int_{\Omega} \frac{(H - H_p)^2}{l^3} d\Omega \quad , (7)$$

$$A_t^c \approx G\rho H_p \int\int_{\Omega} \frac{H - H_p}{(l^2 + H_p^2)^{3/2}} d\Omega \quad , (8)$$

respectively. In the above equations, *H* is the orthometric height, *r* is the mean density of the topographic masses (2.67 g/cm^3) and *l* is the horizontal distance between the computation point *P* on the topography and the integration point. Through the years, the terrain corrections A_t were computed in the Canadian Western Cordillera and a few specific areas in eastern Canada using a variety of methods such as template (Bible, 1962), rectangular prisms (Nagy, 1966), surface fitted cones and more recently triangular prisms with inclined tops (Ferland, 1984). In addition, terrain corrections are available for all gravity measurements in the conterminous U.S.A.

The second approximation in N_2 is related to the downward continuation of the gravity point *P* down to the geoid. The downward continuation can be considered as consisting of a normal and anomalous part; therefore, it can be expressed as (Heiskanen and Moritz, 1967; eq. 2.215):

$$D_g = D_\gamma + D_{\Delta g} \quad . (9)$$

The normal part is calculated using the second-order normal gravity gradient as derived in Heiskanen and Moritz (1967; eq. 2.123):

$$D_\gamma \approx -2\frac{\gamma}{a}(1 + f + m - 2f\sin^2\varphi)H + 3\frac{\gamma}{a^2}H^2 \quad , (10)$$

where a, f and m are related to the GRS80 reference ellipsoid.

The anomalous part assumes a linear correlation between gravity anomalies and topography. It is determined from equation (2-217) in Heiskanen and Moritz (1967) by substituting Dg by 2pGrH:

$$D_{\Delta g} = \frac{\partial \Delta g}{\partial h}H_p \approx G\rho H_p \int\int_{\Omega} \frac{H - H_p}{l^3} d\Omega \quad . (11)$$

For the determination of the GSD95 geoid model, it is assumed that the difference between equations (8) and (11) is negligible. However, this represents a crude approximation because H^2_p at the denominator of equation (8) can be larger than l. The discrepancy between the two equations can be at the mGal level.

The effect of the atmosphere ($-A_{atm} + A^c_{atm}$) on gravity is obtained from Wichiencharoen (1982):

$$\delta g_a \approx 0.8658 - 9.727 x10^{-5} H + 3.482 x10^{-9} H^2 \quad . (12)$$

This correction removes the atmosphere outside the earth by shifting it vertically into the interior of the geoid.

As a result of these approximations, Helmert anomalies can be obtained from:

$$\Delta g^{H} \approx g_p - \gamma_q - D_r - A_t + \delta g_a \text{. (13)}$$

Because gravity measurements are spaced irregularly across Canada, refined Bouguer anomalies are predicted at a five-minute of arc interval from surrounding measurements using least-squares collocation. These predicted values become Helmert anomalies by adding a Bouguer plate where the thickness is determined from a Digital Elevation Model (DEM). This procedure can be expressed as follows:

$$\Delta g_c^{H} \approx C[g_p - \gamma_q - D_r - A_t + \delta g_a - 2\pi G\rho H - \Delta g_{gm}] + 2\pi G\rho H_{DEM} \text{, (14)}$$

where *C* is the least-squares collocation operator. The estimated gravity anomalies are determined from 20 observations (the 5 closest measurements within a radius of 50 km in each quadrant). The gravity anomalies Dg_{gm} are determined from a spherical harmonic expansion:

$$\Delta g_{gm} = \frac{GM}{r^2}\sum_{n=2}^{360}(n-1)\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\bar{P}_{nm}(\sin\phi) \text{. (15)}$$

Finally, the GSD95 geoid model for Canada is computed from the following equation:

$$N \approx N_{gm} + \frac{R}{4\pi\gamma}\int\int_{\sigma_0}\Delta g_c^{H} S(\psi)\, d\sigma - \frac{\pi G\rho}{\gamma}H^2 \text{, (16)}$$

where the last term on the right hand side is the first term of the primary indirect effect on the geoid in a planar approximation as formulated by Wichiencharoen (1982). The integral is evaluated numerically using a six-band 2D-FFT (Forsberg and Sideris, 1993).

4. Comparison to GPS-Levelling Data

Since 1983, the Geodetic Survey Division has conducted GPS campaigns across Canada for the purpose of improving the framework of the national network, monitoring crustal deformation, calibrating instrumentations and validating geoid models. The GPS-levelling campaigns are carried out by collecting GPS measurements at selected bench marks 30 km apart for a period of approximately 4 hours. The early campaigns used single frequency receivers. However, throughout the series of GPS-levelling campaigns, there was significant improvement in the technology of the GPS instruments and software.

For each GPS campaign, the geodetic coordinates (f, l, h) are obtained by least-squares adjustment with minimum constraint; and the fixed station is either a station of the Canadian Active Control System (CACS) (Kouba and Popelar, 1994) or a station of the Canadian Base Network (CBN). The CACS and CBN are high accuracy networks. The geodetic coordinates are expressed in the ITRF93 (epoch 1995.5) reference system for the GRS80 ellipsoid. The error estimates of the ellipsoidal heights (Table 4.1) are formal estimates and they are too optimistic. The comparison of ellipsoidal heights obtained from independent campaigns indicates several discrepancies significantly larger than the formal estimates.

The orthometric heights (H) used in this paper are derived from the Canadian levelling network referred to as NAVD88 (Oct. 95) (Mainville and Bérubé, 1996). The datum is realized by holding fixed a single point in Rimouski, Québec. This adjustment indicates that the water level in the vicinity of Vancouver (Pacific Ocean) is higher than the water level near Halifax (Atlantic Ocean) by 88 cm.

An indication of the precision of GSD95 can be evaluated independently against geoid heights derived from GPS/levelling data. The discrepancies can be expressed as:

$$\varepsilon = (h_{GPS} - H_{NAVD88(Oct95)}) - N_{GSD95} \quad (17)$$

and are summarized in Table 4.1.

The discrepancies shown in Table 4.1 are the mean differences between the GSD95 model and geoid heights derived from GPS/Levelling data. The discrepancies are not expected to be zero because the levelling network has not been constrained on the geoid. However, they should be constant across Canada, if no systematic error existed. Unfortunately, this is not true as shown by the discrepancies between western and eastern Canada. The presence of systematic errors is being investigated in levelling data, terrain corrections (not computed in most of eastern Canada) and the possibility of a gravity bias in the shipboard data.

The standard deviation of the discrepancies e is given before and after fitting out the systematic biases and tilts by using a four-parameter transformation (Sideris, 1993), which absorbs long wavelength errors in GPS measurements, levelling data and geoid heights. From Table 4.1, the standard deviation of the discrepancies ranges from 7 to 40 cm. The large standard deviation for the Trans-Canada project is due to a systematic error in GSD95 between western and eastern Canada. For British Columbia, South Alberta and West Québec campaigns, a systematic error on the ellipsoidal heights was found by comparing them to common stations of the CBN. Therefore, the standard deviation after the transformation is more representative. In addition, the standard deviation of most of the campaigns could be improved if better GPS measurements would be available, especially for the Yukon, British Columbia and Great Slave Lake campaigns.

Table 4.1: Comparison of the GSD95 geoid model to GPS-levelling data across Canada.

Project	Year	Sigma h (cm)	No of Points	Discrepancies (m)	Sigma h-H-N* (cm)
Yukon Territory	1987	7	60	-1.47	20 (12)
British Columbia	1989	14	183	-1.40	20 (14)
Great Slave Lake	1986	9	68	-1.44	12 (9)
North Alberta	1990	1	24	-1.40	7 (4)
Central Alberta	1990	2	32	-1.46	7 (4)
South Alberta	1990	3	108	-1.46	15 (4)
West Québec	1993	2	50	-0.56	11 (5)
East Québec	1993	2	46	-0.30	8 (6)
Maritimes	1992	5	27	-0.36	7 (6)
Trans-Canada	1991	6	148	-0.96	40 (15)

*: The results in parenthesis indicate the standard deviations after fitting out systematic biases and tilts by using a four-parameter transformation.

A better external estimate of the precision of GSD95 will be available when the CBN project will be completed across Canada. The accuracy of the CBN stations is estimated at 3 to 5 cm (95% confidence level). In addition, most of the CBN stations will be tied by first order levelling to the national network. As of mid-1996, the GPS component of the CBN project has been completed in the Maritimes, Québec, Alberta and Saskatchewan.

5. Conclusion

This paper describes the theory, assumptions and approximations used for the computation of the Canadian GSD95 geoid model. The model is computed by the remove-restore approach, where the global contribution is determined from the OSU91A geopotential model. The surface gravity measurements, reduced by a planar approximation according to Helmert's second condensation method, allow the refinement of the geoid heights in Canada. Comparison of GSD95 to geoid heights derived from GPS and levelling data has shown that a 5 cm to 10 cm precision over tens of kilometres is achievable. However, the development of a geoid model with a precision better than 5 cm (2 s) will require new data and reevaluation of the approximations used. To achieve this precision, a three-year plan has been prepared at the Geodetic Survey Division under the leadership of Dr. S. Pagiatakis (1996). Under this three-year plan, the main role of Geodetic Survey Division is to collect, assess and validate the integrity of the data required for the computation of the Canadian geoid model. On the other hand, the development of the theoretical aspects of geoid modeling is under the leadership of Prof. M.G. Sideris (The University of Calgary) and Prof. P. Vanicek (The University of New Brunswick). This collaboration between the three institutions will culminate with the development of a new Canadian Gravimetric Geoid model for the year 2000 (CGG2000).

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