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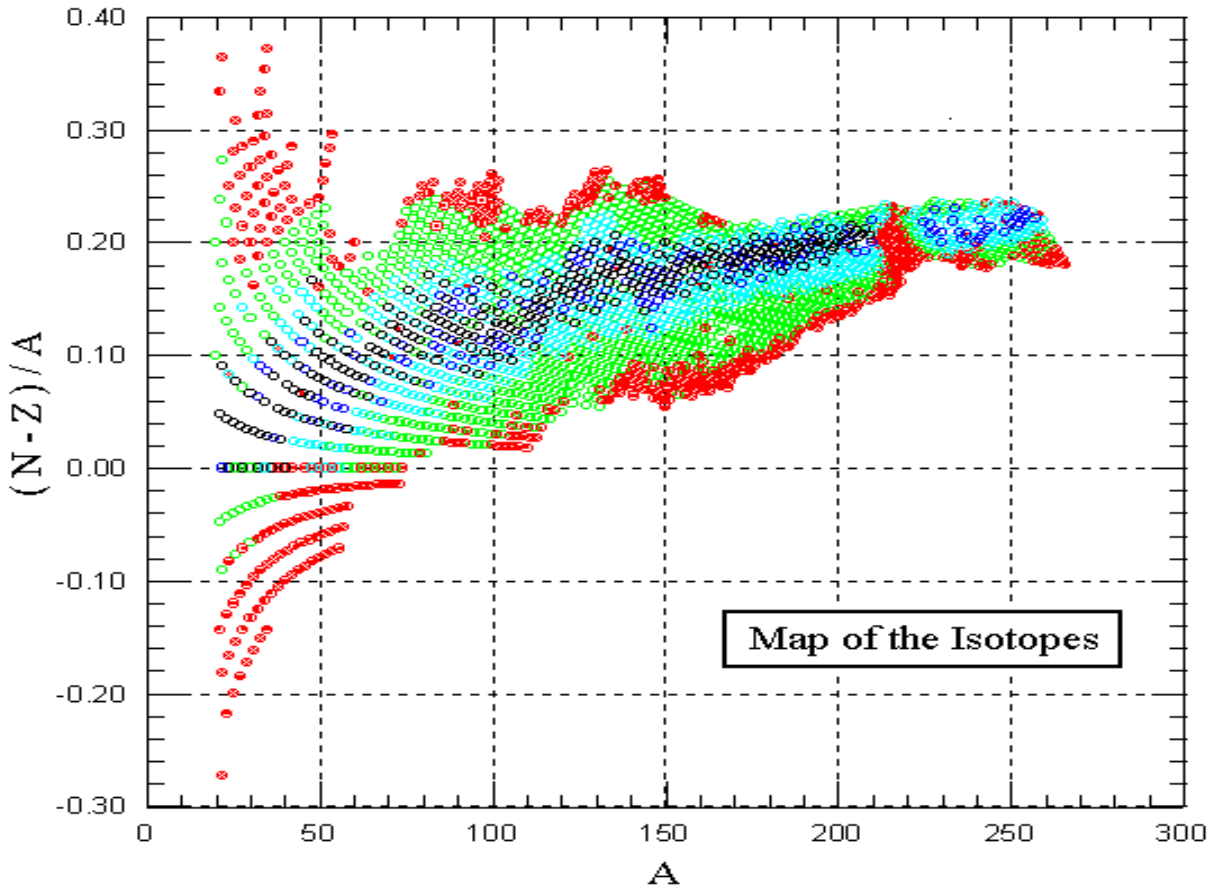
ANL/NDM-156

## A Demonstration of the Lognormal Distribution

**Donald L. Smith**

July 2003

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# A Demonstration of the Lognormal Distribution [a]

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July 2003

**Keywords**

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SIMULATION. Monte-Carlo. Probability. Uncertainty. Errors. Lognormal distribution. Experimental perturbations. Experimental modeling.

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# A Demonstration of the Lognormal Distribution [a]

by

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## ABSTRACT

Although its applicability is not universal, the lognormal distribution is frequently used to represent physical parameters that are inherently positive and are not well known, i.e., that have large errors. In particular, arguments drawn from Bayesian statistics suggest that the lognormal function is the optimal choice of a probability distribution for those cases where the only knowledge possessed about a particular parameter is an estimate of its mean value and associated error. In this report it is demonstrated by a simulation exercise that the lognormal function represents very well the outcomes of repeated measurements that are subjected to a variety of multiplicative disturbances that, by their very nature, preclude the observation of negative values. In the present study, a simplified model of measurement is defined and it is then subjected to numerical analysis using the Monte Carlo method. There is no attempt to prove rigorously that the lognormal distribution results as the unique consequence of assumptions about the defined model, nor are other more complex models of measurement investigated. In any event, it is not possible to produce a proof by the use of anecdotal information, e.g., by analyzing the results of repeated Monte Carlo trials. Nevertheless, the simulation study described here does provide strong evidence that for many practical situations the lognormal distribution can be used to represent the probable outcome of measurements of positive quantities.

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# 1. Introduction

Most parameters that represent physical quantities are either inherently positive or can be redefined to be positive, e.g., as in the case of temperature. Mankind's attempts to determine precise, enduring values for these parameters through experimentation or other means are inevitably thwarted by a number of perturbing factors. As a consequence, repeated measurements by a particular investigator, or determinations by various investigators, inevitably lead to varying results. It is useful to have an understanding of the underlying probability functions that govern measurement processes since then it may be possible to predict the impact of known perturbations and to estimate the amount of effort that might be required to determine a particular parameter to some desired level of accuracy.

In earlier communications [SNW2001,SNW2002] the arguments supporting the use of the lognormal distribution to represent fundamental physical quantities that are inherently positive and characterized only to the extent of possessing estimates of their mean values and uncertainties were discussed. This previous work relied upon results from earlier mathematical investigations by Jeffreys [Jef1939], Shannon [Sha1949], Jaynes [Jay1983], and Froehner [Fro1997]. Furthermore, it was demonstrated through Monte Carlo simulation that physical parameters related to primary ones by a variety of well-defined mathematical relationships are quite well represented by lognormal functions. Illustrative examples from radioactivity decay, radiation transport, and astrophysics [Hix+1999] were considered.

The focus in the present work is somewhat different. Here we are not concerned with uncertainty propagation in the conventional sense. Instead, we examine the impact of natural random perturbations associated with realistic measurement procedures developed to ascertain values for single parameters. We do not resort to the Bayesian arguments used to support the lognormal function but rather actually demonstrate how distributions that can be well represented by the lognormal function emerge to describe the outcomes of repeated attempts to determine values for an individual parameter.

Section 2 of this report offers a brief review of the properties of the lognormal distribution. Section 3 describes the simplified model of measurement that was chosen for the present simulation exercise. Section 4 gives details about the Monte Carlo simulation procedure employed. Section 5 provides results from the numerical exercises. Section 6 interprets and discusses these results. Finally, Section 7 offers some conclusions drawn from the present study.

## 2. Brief Review of the Lognormal Distribution

Earlier communications from this laboratory discuss in considerable detail the motivations for using the lognormal function rather than the normal (Gaussian) function in certain applied situations [SNW2001,SNW2002]. These will not be repeated. However, it is useful to summarize the properties of the lognormal function in this report for the convenience of the reader, since some of the key formulas have been employed in the present study [Smi1991].

For a single random variable, this distribution takes the normalized form

$$p(Q) = (2\pi S^2 Q^2)^{-½} \exp [- (ln\ Q - M)^2/2S^2] \quad (Q > 0) . \tag{1}$$

The mean value and standard deviation are obtained by using the formulas

$$m = \exp[M + (S^2/2)] \tag{2}$$

and

$$s^2 = m^2 [\exp (S^2) - 1] = \exp(2M + 2S^2) – \exp(2M + S^2). \tag{3}$$

Conversely, if the mean value m and standard deviation s for this distribution are given, then those parameters M and S that characterize the lognormal function can be derived from the expressions

$$S^2 = ln\ [1 + (s^2/ m^2)] \tag{4}$$

and

$$M = ln\ m – (S^2/2) . \tag{5}$$

When the probability function for Q is assumed to be lognormal, then the distribution for y = *ln* Q is normal [Smi1991]. The converse is also true. In this sense, these two functions are conjugates of each other. The lognormal function clearly exhibits the desired property of non-negativity over the variable range for which it is defined (Q > 0), no matter how large a standard deviation is involved.

It is well known that the normal probability function applies to situations where a random variable is influenced by many additive disturbances. This, in fact, is the basis for the well-known Central Limit Theorem [Smi1991]. Consequently, it is reasonable to suppose that a series of positive multiplicative disturbances might form the basis for the lognormal distribution. A series of multiplicative disturbances for parameter Q corresponds to a series of additive disturbances for *ln* Q, so we are led to invoke the relationship between the distributions for Q and *ln* Q described above in supporting this

conclusion. In fact, this indeed is the case and the core of the present study is a direct demonstration of this phenomenon.

## 3. A Simplified Model of Measurement

Let Q be a physical quantity that is to be determined. Suppose that $Q_0 > 0$ is viewed as representing the "true value" of Q. This precise value is elusive due to the uncertainty of "real life" attempts of humans to determine it. Thus, for demonstration purposes let us assume that attempts to determine Q precisely are thwarted by n independent, random perturbations. Furthermore, to insure that no attempt to determine Q ever results in a negative number (which is physically unreasonable), let us assume that these physical disturbances can be represented by inherently positive multiplicative factors $f_i$ (i=1,n). Thus, in this model any particular experiment should yield the result

$$Q = Q_0 \, \Pi_{i=1,n} \, f_i \, . \tag{6}$$

Furthermore, we require that the mean values $<f_i>$ for all i=1,n are equal to unity, since were it otherwise this would indicate a bias (systematic error). For convenience, we choose to avoid complicating the discussion by getting involved in the distinctions between random and systematic errors.

For present purposes, it is assume that $f_i$ falls in the range $1-\alpha_i < f_i < 1+\alpha_i$ and that within these ranges each $f_i$ is governed by a continuous uniform distribution [Smi1991]. For convenience, we refer to $\alpha_i$ as the perturbation parameter for the $i^{th}$ perturbation. Clearly $0 < \alpha_i < 1$, otherwise $f_i$ would become zero or negative. The assumption of a continuous uniform distribution need not be a limiting one, but it makes sense to assume that the uniform distribution applies in many practical situations because, from a Bayesian point of view, it is the correct distribution for any parameter that is known to fall within a certain range but for which absolutely no other information is available [Jay1983,Fro1997]. For the limits mentioned above, the mean value is

$$m_{0i} = <f_i> = 1 \tag{7}$$

and the standard deviation is

$$\sigma_i = [ \, \alpha_i^2 \, / \, 3]^{1/2} \, . \tag{8}$$

We could assume that for each $f_i$ there is a distinct value $\alpha_i$. Although this feature surely provides a model closer to reality, since the various sources of disturbance need not have the same magnitude, it results in more complicated numerical analysis. The additional complexity adds little new insight to justify the computational penalty. Therefore, it has been assumed that $\alpha_i = \alpha$. Thus the same perturbation parameter is chosen for each i=1,n. Furthermore, one must assume that there is some practical limit for the product $n\alpha$ as n increases. That is as the number n of perturbations becomes larger, the value $\alpha$ needs to decrease.

If the perturbations above are numerous and each relatively small, then the constraints on complexity imposed by the model culminating in Eq. (6) could be eased. Thus, we can consider a more general model relationship of the form

$$Q = Q_0 \, F(\mathbf{x};\mathbf{f}) \, , \tag{9}$$

where **f** is the vector of perturbations described above, **x** is a the vector representing other constant (non-random) parameters of the model, and F is a function which defines the experimental model. As discussed above, we require that $F(\mathbf{x};\mathbf{1}) = 1$, where the vector **1** symbolizes that each element in the array **f** has the value unity. The following approximation to Q, based on a first-order Taylor series approximation to Eq. (9), can then be considered

$$Q \approx Q_0 \, F(\mathbf{x};\mathbf{1}) \, [\, 1 + \textstyle\sum_{i=1,n} (\partial F/\partial f_i)_0 \, \Delta f_i] \approx Q_0 \, F(\mathbf{x};\mathbf{1}) \, \textstyle\prod_{i=1,n} [1 + (\partial F/\partial f_i)_0 \, \Delta f_i] \, . \tag{10}$$

The subscript zero implies calculation of the indicated partial derivative for the condition $\mathbf{f} = \mathbf{1}$. The differential increment $\Delta f_i$ correspond to a perturbation of $f_i$ from unity (positive or negative) with a maximum variation of magnitude $\alpha_i$. This clearly brings us to the same point as is indicated by Eq. (6). Thus, the present study deals solely with Eq. (6), with the tacit understanding that more complex models could be devised. Evidently, these would lead us to reach the same conclusions.

## 4. Monte Carlo Simulation Procedure

Simulation by the Monte Carlo procedure is fairly straightforward. A sequence of "histories" denoted by the index "k" is investigated. For a specific k, a collection of perturbation factors $f_{ik}$ for i=1,n is generated by random selection in the interval $1-\alpha < f_{ik} < 1+\alpha$ according to the continuous uniform distribution. Thus,

$$Q_k = Q_0 \, \Pi_{i=1,n} \, f_{ik} \tag{11}$$

is the value of Q produced in the $k^{th}$ history. The total number of histories pursued is assumed to be K, i.e., k=1,K is the range of compiled histories. The mean value for Q that can be deduced from this exercise is

$$m \approx \sum_{k=1,K} Q_k / K \tag{12}$$

The corresponding standard deviation is

$$s \approx [\sum_{k=1,K} (m - Q_k)^2 / K]^{1/2} \ . \tag{13}$$

Approximate equality comes about because of the fact that K is finite. The larger the value of K the closer these Monte Carlo results will approach the true parameters of the underlying distribution. There is no disputing the fact that this Monte Carlo procedure will generate an empirical distribution function. The issue to be examined here is whether for all practical purposes this distribution can be approximated by or actually approaches a lognormal function.

In order to proceed with the numerical analysis, it is necessary to establish a grid structure so that all the obtained $Q_k$ values can be assigned to their appropriate grid intervals. This can be done best by selecting values $Q_L$ and $Q_H$ such that for every $Q_k$

$$Q_L < Q_k < Q_H \ . \tag{14}$$

This can be done quite easily after the K sampling cycles have been completed and the results are tallied. Then, one decides on the number of grid intervals N. Each of these equal intervals will have width

$$\Delta Q = (Q_H - Q_L)/N \ . \tag{15}$$

The grid midpoints are defined by the formula

$$Q_j = Q_L + (j - ½)[(Q_H - Q_L)/N] \quad (j=1,N) \ . \tag{16}$$

It is important to select N so that N << K. Then one can expect to obtain sufficiently large tallies in the dominant grid intervals to provide some insight into the nature of the probability distribution resulting from the Monte Carlo sampling exercise.

## 5. Numerical Results

For the purpose of the present simulation exercise, the values n = 10, K = 1000, and N = 10 were selected. Furthermore, it was assumed, with no loss of generality, that $Q_0 = 1$. Calculations were then carried out for values of $\alpha = 0.1$, 0.2, 0.3, and 0.5. All of these calculations were performed using an EXCEL spreadsheet developed especially for this purpose. The procedure followed for a specific value of $\alpha$ is indicated below.

The first worksheet was used to generate a ten-column by 1000-row array of random numbers in the range (a,b) using the EXCEL library function RAND()*(b-a)+a. The constants a and b were selected to be consistent with the selected values of $\alpha$. For example, for $\alpha = 0.1$, a = 0.9 and b = 1.1. The mean value is always unity, but the standard deviation depends on a and b. Values for these parameters are given for each considered value of $\alpha$ in Table 1. Within each of the 1000 rows, the random values in the above-mentioned ten columns were multiplied to generate a value that was then stored in the $11^{th}$ column. These values correspond to the various $Q_k$, in accordance with Eq. (11). A total of 1000 of them are generated by this procedure. This column of values was pasted into a fresh worksheet for further analysis. Values of m and s were then determined from this array using the EXCEL library functions AVERAGE and STDEV, respectively. From these, parameters S and M were calculated using Eqs. (4) and (5), respectively. The 1000 random values in the array were then sorted from smallest to largest using a data sorting routine of EXCEL. The selected grid values of Q were included in the sorting process to conveniently define the grid boundaries within the table. The tallies of random values within each of the ten grid limits were determined and collected into a separate region of the worksheet for further processing and eventual plotting. The parameters M and S were used to define a corresponding lognormal function, according to Eq. (1). The only difference is that a multiplicative constant C was introduced in order to facilitate comparison between the discrete probability distribution generated by Monte Carlo simulation and the lognormal function.

The results of this analysis are shown in Tables 2-5 and Figs. 1-4. It is seen that the agreement is remarkably good given the general coarseness of the grid structure and limited number of traced histories. Note that for $\alpha = 0.1$, 0.2, and 0.3, Eq. (12) led to $m \approx 1$ as expected. However, due to the limited number of histories and very skewed nature of the distribution obtained for $\alpha = 0.5$, the present procedure led to $m \approx 0.91$ in that case.

An interesting observation can be gleaned from a close inspection of Tables 1-5. The standard deviation of the distribution for Q generated by Monte Carlo simulation is approximately equal to the square root of ten times the square of the standard deviation of the uniform distribution governing each of the perturbing factors. This apparent quadratic summing of errors is a well-known phenomenon in statistics, so it is very revealing to see it demonstrated once again in the present exercise. What is also remarkable is that this effect occurs even though the ten contributing perturbation factors included in the present

Monte Carlo demonstration exercises are governed by uniform distributions while the composite distribution for Q is evidently lognormal, or at least very nearly so.

## 6. Discussion

A simulation demonstration such as the present one does not definitively prove that the lognormal function is the appropriate choice for a probability distribution in the analysis of inherently positive physical quantities. Counterexamples are not hard to come by. For example, it is well known that the appropriate distribution for rare, discrete events is the Poisson distribution, and that the binomial distribution describes the statistics of small discrete samples under conditions frequently encountered in nature. However, in those situations where the physical quantity is positive, can assume any value (not discrete), and is known only to the extent that it's mean value and uncertainty can be estimated, the present exercise suggests that the lognormal function is well suited.

The present model involves a large number of repeated determinations of a physical quantity under conditions where the measured value can be perturbed by a number of factors that can be represented by multiplicative constants. In reality, matters are generally more complicated than this. However, it is remarkable that distributions that appear to be quite well represented by lognormal functions emerge from the simulation process even when the disturbances are quite large.

Finally, it could be argued that the Central Limit Theorem should be invoked to prove what has been demonstrated in the present investigation. One simply considers the logarithms of the physical quantities, transforms the multiplicative disturbances into additive ones and ends up with a situation similar to the one described by Smith [Smi1991]. This contention is certainly true. However, a direct demonstration of the lognormal distribution serves to strengthen one's conviction that this function is an appropriate choice for the analysis of inherently positive physical data in situations such as those frequently encountered in nuclear science.

## 7. Conclusions

A model in which the measurement of inherently positive physical quantities is accompanied by a series of multiplicative disturbances leads, via Monte Carlo simulation, to the conclusion that the lognormal probability function is extremely useful for analysis of the results of common experimental measurements. This result is very useful because from a practical point of view in nuclear science it is generally impossible to repeat a measurement a sufficient number of times to discern the nature of the underlying distribution by empirical means alone. One is forced to suggest the functional form for the distribution based either on postulates or more fundamental theorems, as is the Bayesian approach, or by direct demonstration via the Monte Carlo method as has been done in the present investigation.

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## Tables

The parameters in the tables below are defined as follows:

Q: Value of the measured quantity at the midpoint of the grid interval

P(Q): Tally of Monte Carlo histories falling in the indicated grid interval

LogNorm(Q): Lognormal function calculated at midpoint value of Q

a: Lower limit of the uniform distribution for $f_i$

b: Upper limit for the continuous uniform distribution for $f_i$

m0: Mean value of continuous uniform distribution for $f_i$ (equals unity)

sig: Standard deviation of continuous uniform distribution for $f_i$ [Eq. (8)]

m: Q distribution mean value from Monte Carlo sampling [Eq. (12)]

s: Q distribution standard deviation from Monte Carlo sampling [Eq. (13)]

M: Parameter of lognormal function calculated from m and s [Eq. (5)]

S: Parameter of lognormal function calculated from m and s [Eq. (4)]

C: Normalization constant for lognormal function

Table 1: Parameters of the continuous uniform sampling distributions for $f_i$

| | α | | | |
|---|---|---|---|---|
| | 0.1 | 0.2 | 0.3 | 0.5 |
| a = | 0.9 | 0.8 | 0.7 | 0.5 |
| b = | 1.1 | 1.2 | 1.3 | 1.5 |
| m0 = | 1 | 1 | 1 | 1 |
| sig = | 0.057735 | 0.11547 | 0.173205 | 0.288675 |

Table 2: Results of the simulation exercise for $\alpha = 0.1$

LogNorm(Q) = (C/S/Q)*EXP(-(LN(Q)-M)*(LN(Q)-M)/2/S/S)

| Q | P(Q) | LogNorm(Q) |
|---|---|---|
| 0.575 | 14 | 6.506860154 |
| 0.725 | 102 | 102.347509 |
| 0.875 | 286 | 293.896979 |
| 1.025 | 321 | 314.6657809 |
| 1.175 | 177 | 182.8035318 |
| 1.325 | 73 | 71.48264588 |
| 1.475 | 18 | 21.46998248 |
| 1.625 | 7 | 5.391810664 |
| 1.775 | 2 | 1.198132375 |
| 1.925 | 0 | 0.244932553 |
| Sums | 1000 | 1000.008165 |

| | | | | |
|---|---|---|---|---|
| m | = | 1.01022 | QL = | 0.5 |
| s | = | 0.186286 | QH = | 2 |
| S | = | 0.182862 | Delta Q = | 0.15 |
| M | = | -0.00655 | | |
| C | = | 59.846 | | |

Table 3: Results of the simulation exercise for $\alpha = 0.2$

LogNorm(Q) = (C/S/Q)*EXP(-(LN(Q)-M)*(LN(Q)-M)/2/S/S)

| Q | P(Q) | LogNorm(Q) |
|---|---|---|
| 0.42 | 73 | 52.946301 |
| 0.66 | 231 | 252.5928675 |
| 0.9 | 288 | 295.0779327 |
| 1.14 | 206 | 201.2752937 |
| 1.38 | 102 | 107.6363908 |
| 1.62 | 57 | 51.15944864 |
| 1.86 | 28 | 22.96928491 |
| 2.1 | 8 | 10.05979345 |
| 2.34 | 4 | 4.37553549 |
| 2.58 | 3 | 1.909650227 |
| Sums | 1000 | 1000.002499 |

| | | | | |
|---|---|---|---|---|
| m | = | 0.997575 | QL = | 0.3 |
| s | = | 0.370052 | QH = | 2.7 |
| S | = | 0.359062 | Delta Q = | 0.24 |
| M | = | -0.06689 | | |
| C | = | 95.905 | | |

Table 4: Results of the simulation exercise for $\alpha$ =0.3

LogNorm(Q) = (C/S/Q)*EXP(-(LN(Q)-M)*(LN(Q)-M)/2/S/S)

| Q | P(Q) | LogNorm(Q) |
|---|---|---|
| 0.35 | 184 | 183.1844028 |
| 0.75 | 365 | 387.6857527 |
| 1.15 | 258 | 231.7070529 |
| 1.55 | 98 | 108.5820365 |
| 1.95 | 51 | 48.50070507 |
| 2.35 | 25 | 21.84017039 |
| 2.75 | 7 | 10.09717136 |
| 3.15 | 5 | 4.818867057 |
| 3.55 | 5 | 2.375551602 |
| 3.95 | 2 | 1.208135928 |
| Sums | 1000 | 999.9998463 |

| | | | | | |
|---|---|---|---|---|---|
| m | = | 1.006481 | QL | = | 0.15 |
| s | = | 0.562553 | QH | = | 4.15 |
| S | = | 0.521402 | Delta Q | = | 0.4 |
| M | = | -0.12947 | | | |
| C | = | 158.7476 | | | |

Table 5: Results of the simulation exercise for $\alpha$ =0.5

LogNorm(Q) = (C/S/Q)*EXP(-(LN(Q)-M)*(LN(Q)-M)/2/S/S)

| Q | P(Q) | LogNorm(Q) |
|---|---|---|
| 0.5 | 695 | 769.6201866 |
| 1.5 | 200 | 160.7785245 |
| 2.5 | 64 | 43.23146796 |
| 3.5 | 30 | 14.85635096 |
| 4.5 | 8 | 6.022154161 |
| 5.5 | 0 | 2.745184517 |
| 6.5 | 1 | 1.366314752 |
| 7.5 | 1 | 0.728206699 |
| 8.5 | 0 | 0.410046989 |
| 9.5 | 1 | 0.24157757 |
| Sums | 1000 | 1000.000015 |

| | | | | | |
|---|---|---|---|---|---|
| m | = | 0.912657 | QL | = | 0 |
| s | = | 0.920956 | QH | = | 10 |
| S | = | 0.837997 | Delta Q | = | 1 |
| M | = | -0.44251 | | | |
| C | = | 337.2201 | | | |

## Figures

Some explanation is required in order to help the reader to understand Figs. 1 – 4 below. The plots that appear in Figs. 1-4 are derived from the numerical values in Tables 2-5, respectively. The plotted solid square "data" points are the integer values for the Monte Carlo tallies of histories assigned to each of the ten defined intervals. These points are plotted at values of Q corresponding to the midpoints of these intervals. Note that the sum of the tally values is 1000 since 1000 histories were traced in each of these simulations. The value for the corresponding lognormal function is plotted at the same value of Q, but the data point itself is not shown. Instead these "invisible" points are connected with solid straight lines (no smoothing). These lines serve as an "eye guide" for the plotted lognormal function. The floating normalization constant for the lognormal function was chosen so that the sum of the ten calculated values of the lognormal function is approximately equal to 1000.

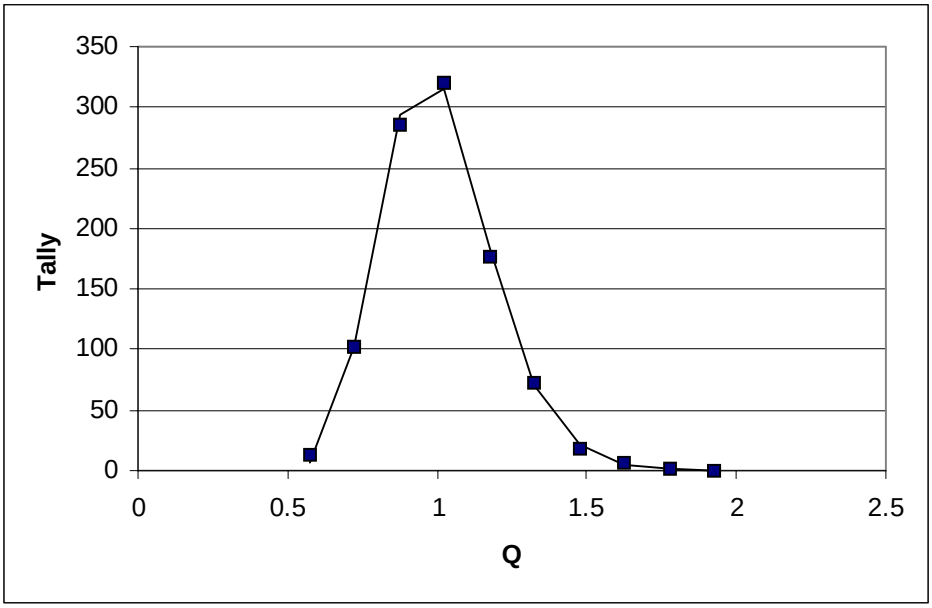


Figure 1: Simulation results for $f_i = 0.9 \rightarrow 1.1$ ($\alpha = 0.1$)

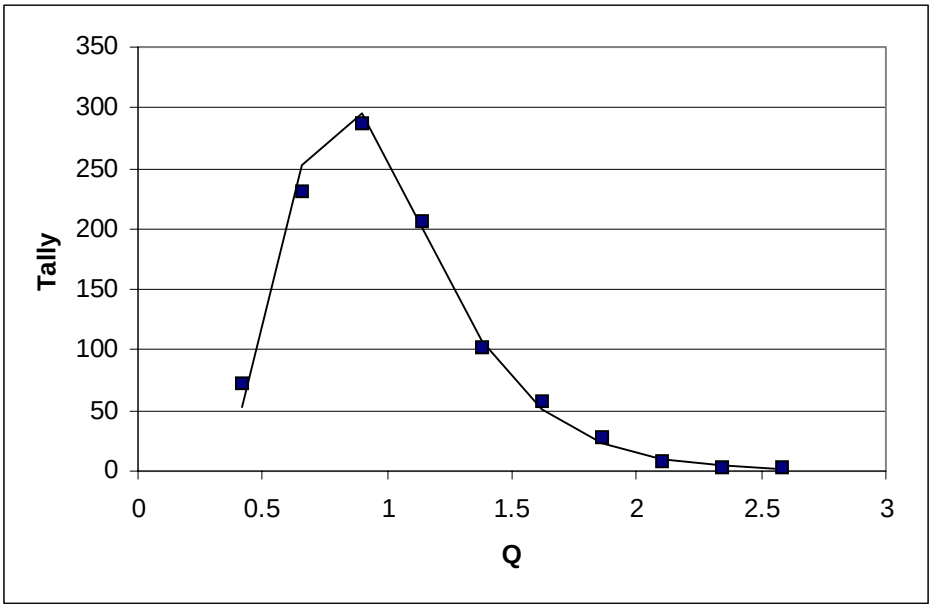


Figure 2: Simulation results for $f_i = 0.8 \rightarrow 1.2$ ($\alpha = 0.2$)

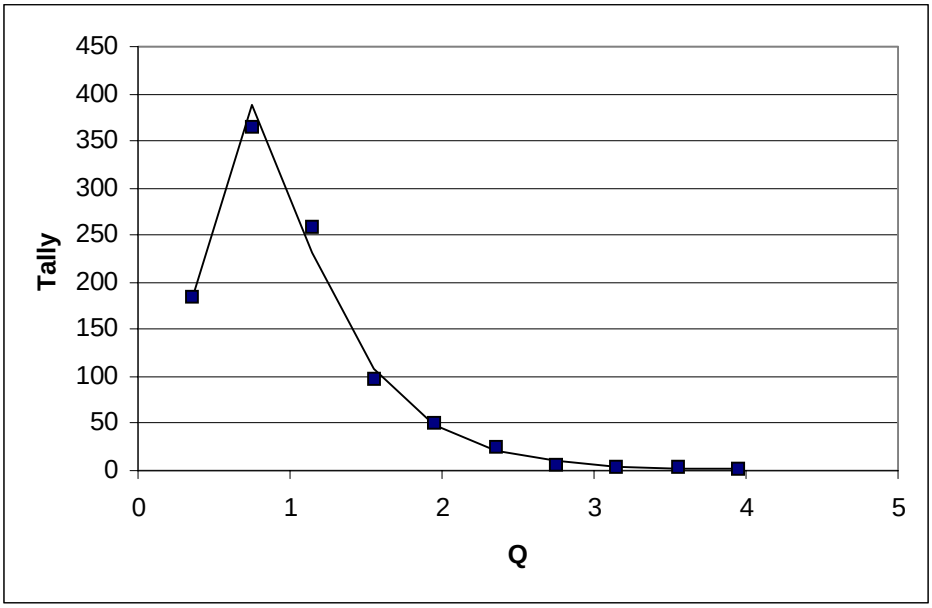


Figure 3: Simulation results for $f_i = 0.7 \rightarrow 1.3$ ($\alpha = 0.3$)

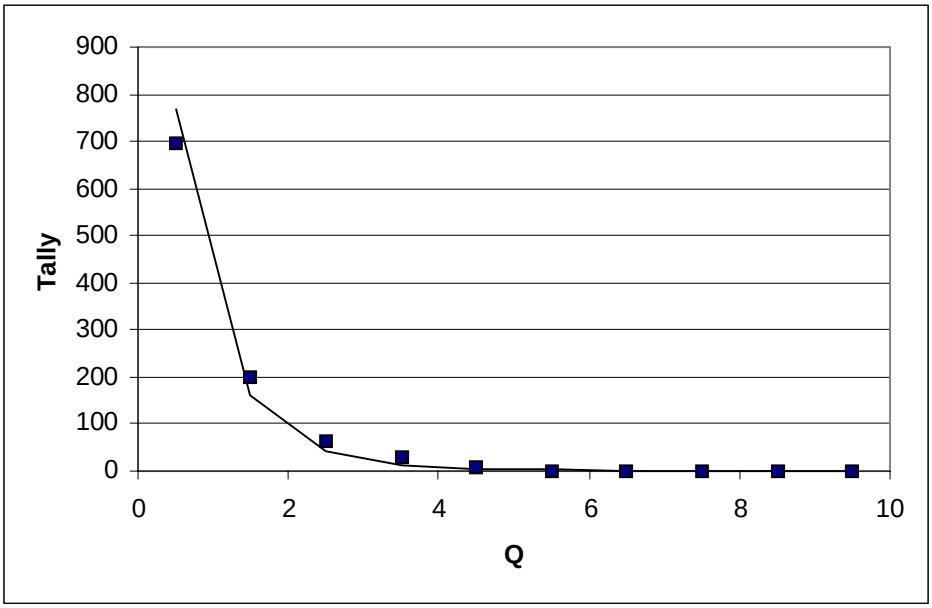


Figure 4: Simulation results for $f_i = 0.5 \rightarrow 1.5$ ($\alpha = 0.5$)