A hybrid approach for state estimation: combining moving horizon estimation and particle filtering

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Outline

1. State Estimation of Linear Systems
   - Limitations
2. Moving Horizon Estimation (MHE)
3. Particle Filtering
4. Combining Particle Filtering and MHE
5. Conclusions

The conditional density function

For the linear, time invariant model with Gaussian noise,

$$x(k+1) = Ax + Bu + Gw$$
$$y = Cx + v$$

$$w \sim N(0, Q) \qquad v \sim N(0, R) \qquad x(0) \sim N(\overline{x}_0, Q_0)$$

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We can compute the conditional density function exactly

$$p_{x|Y}(x|Y(k-1)) = N(\widehat{x}^-, P^-) \qquad \text{(before } y(k)\text{)}$$
$$p_{x|Y}(x|Y(k)) = N(\widehat{x}, P) \qquad \text{(after } y(k)\text{)}$$

Mean and covariance before and after measurement

Forecast

$$\widehat{x}^-(k+1) = A\widehat{x} + Bu \qquad \text{(estimate)}$$

$$P^-(k+1) = APA' + GQG' \qquad \text{(covariance)}$$

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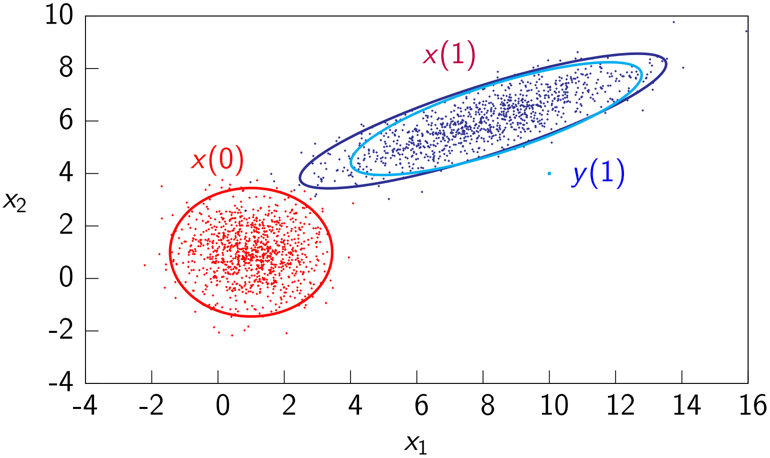
Correction

$$\widehat{x} = \widehat{x}^- + L(y - C\widehat{x}^-) \qquad \text{(estimate)}$$

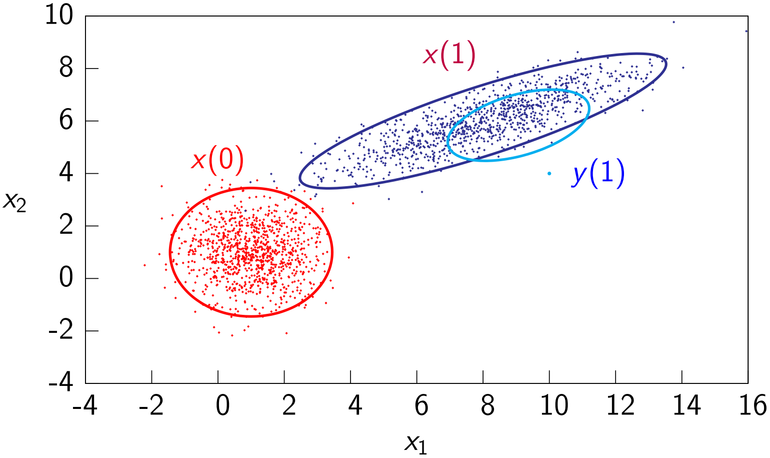
$$L = P^- C'(R + CP^- C')^{-1} \qquad \text{(gain)}$$

$$P = P^- - LCP^- \qquad \text{(covariance)}$$

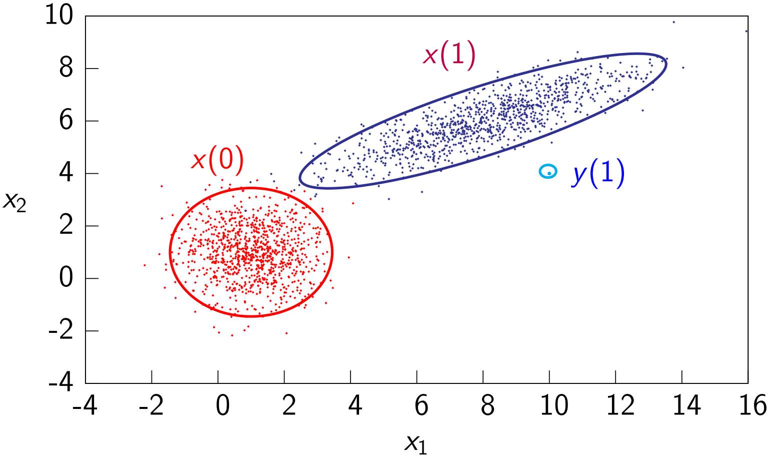
Large R, ignore the measurement, trust the forecast



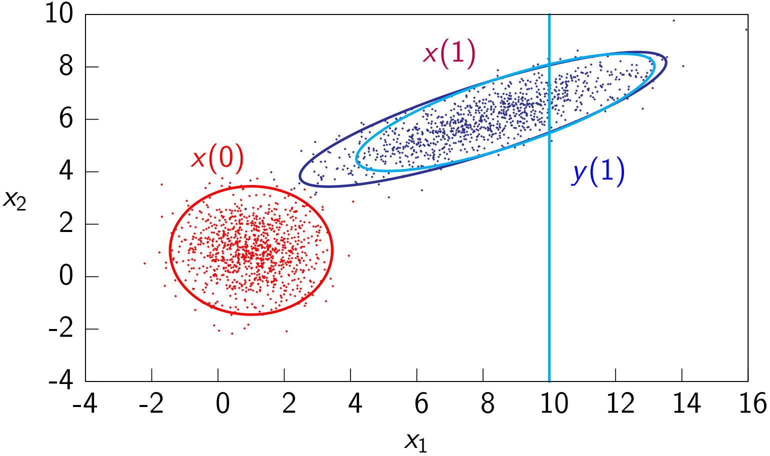
Medium R, blend the measurement and the forecast



Small R, trust the measurement, override the forecast



Large R, y measures x_1 only



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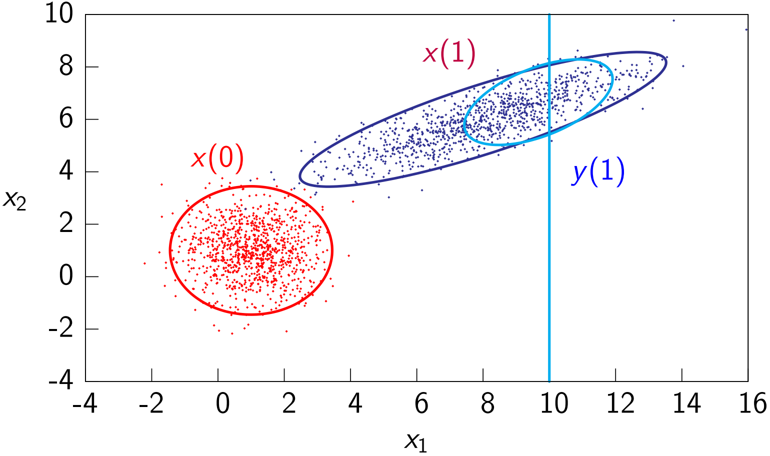

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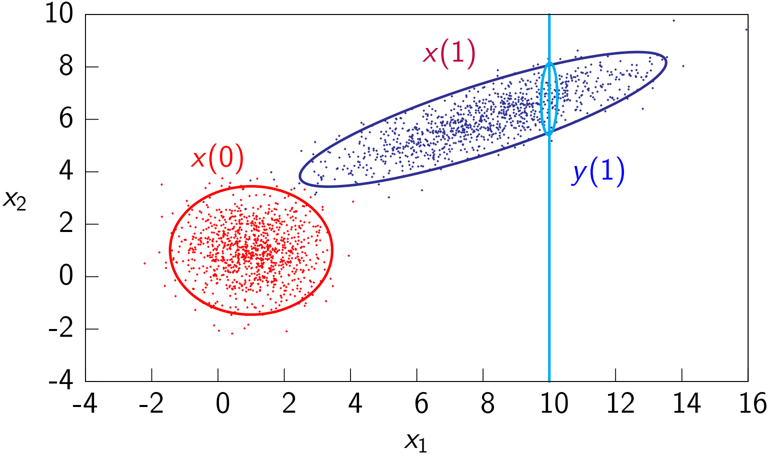

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  - Projecting the unconstrained KF estimates to the feasible region is an ad hoc solution that does not satisfy the model.
- What about nonlinear models?
  - Almost all physical models in chemical and biological applications are nonlinear differential equations or nonlinear Markov processes.
  - Linearizing the nonlinear model and using the standard update formulas (extended Kalman filter) is the standard industrial approach.

Extended Kalman filter — assessment

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Many of these difficulties arise from its use of linearization.

Julier and Uhlmann (2004).

Alternative approaches

Options for handling constraints and nonlinearity in state estimation

1. Optimization (moving horizon estimation (MHE))
2. Sampling (particle filtering)

Full information estimation

Nonlinear model, Gaussian noise,

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Maximizing the conditional density function

$$\max_{X(T)} p_{X|Y}(X(T)|Y(T))$$

Equivalent optimization problem

Using the model and taking logarithms

$$\min_{X(T)} V_0(x_0) + \sum_{j=1}^{T-1} L_w(w_j) + \sum_{j=0}^{T} L_v(y_j - h(x_j))$$

subject to $x(j+1) = F(x, u) + w \quad (G(x, u) = I)$

$$V_0(x) := -\log(p_{x_0}(x))$$

$$L_w(w) := -\log(p_w(w)) \qquad L_v(v) := -\log(p_v(v))$$

Arrival cost and moving horizon estimation

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Optimization problem

$$\min_{X(T-N:T)} \underbrace{V_{T-N}(x_{T-N})}_{\text{arrival cost}} + \sum_{j=T-N}^{T-1} L_w(w_j) + \sum_{j=T-N}^{T} L_v(y_j - h(x_j))$$

subject to $x(j+1) = F(x, u) + w$.

Arrival cost approximation

The statistically correct choice for the arrival cost is the conditional density of $x_{T-N}|Y(T-N-1)$

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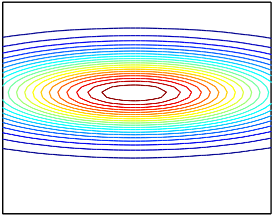
Arrival cost approximations (Rao et al., 2003)

- uniform prior (and large N)
- EKF covariance formula
- MHE smoothing

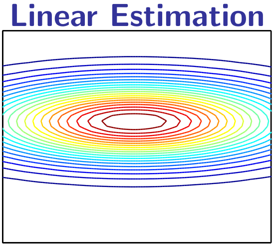
The challenge of nonlinear estimation

Linear Estimation



Estimation Possibilities:

The challenge of nonlinear estimation



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1. *one* state is the optimal estimate
2. *infinitely many* states are optimal estimates (unobservable)

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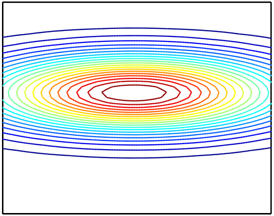

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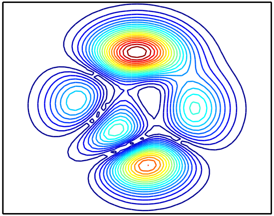

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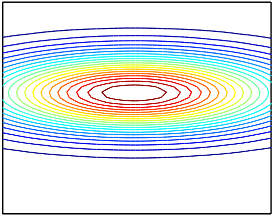

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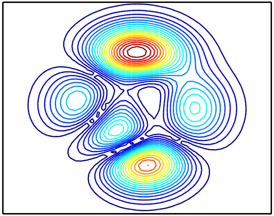

Estimation Possibilities:

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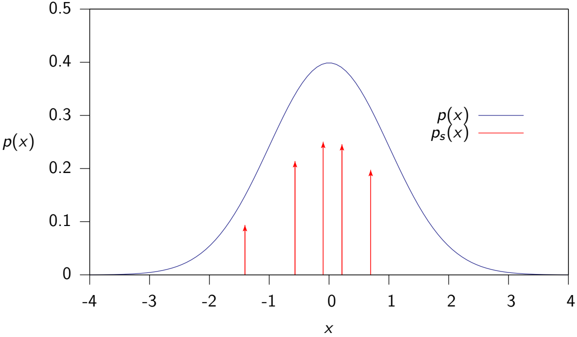
2. *infinitely many* states are optimal estimates (unobservable)
3. *finitely many* states are locally optimal estimates

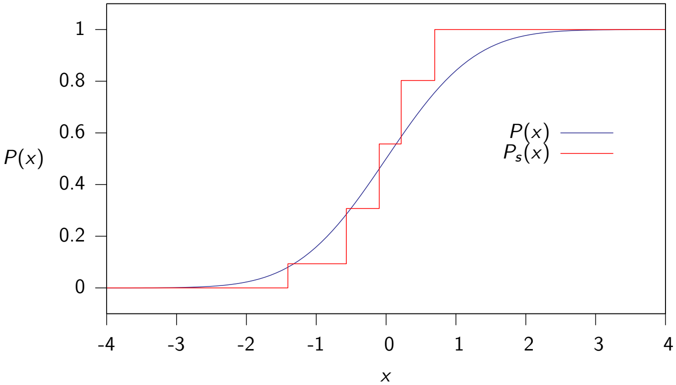
Particle filtering — sampled densities

$$p_s(x) = \sum_{i=1}^{s} w_i \delta(x - x_i) \qquad x_i \text{ samples (particles)} \qquad w_i \text{ weights}$$



Exact density $p(x)$ and a sampled density $p_s(x)$ with five samples for $\xi \sim N(0, 1)$

Convergence — cumulative distributions



Corresponding exact $P(x)$ and sampled $P_s(x)$ cumulative distributions

Importance sampling

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Both $\overline{p}_s(x)$ and $p_s(x)$ are *unbiased* and *converge* to $p(x)$ as sample size increases (Smith and Gelfand, 1992).

Importance sampled particle filter (Arulampalam et al., 2002)

$$p(x(k+1)|Y(k+1)) = \{x_i(k+1), \overline{w}_i(k+1)\}$$

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$$w_i(k+1) = w_i(k)\,\frac{p(y(k+1)|x_i(k+1))p(x_i(k+1)|x_i(k))}{q(x_i(k+1)|x_i(k), y(k+1))}$$

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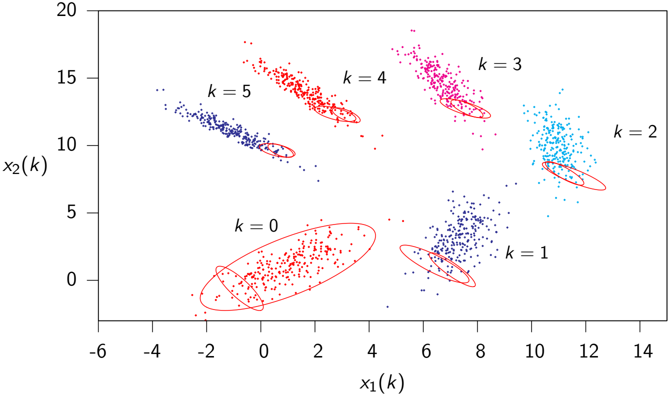
The importance sampled particle filter *converges* to the conditional density with increasing sample size. It is *biased* for finite sample size.

Research challenge — placing the particles

- Optimal importance function (Doucet et al., 2000). Restricted to linear measurement $y = Cx + v$.
- Resampling
- Curse of dimesionality

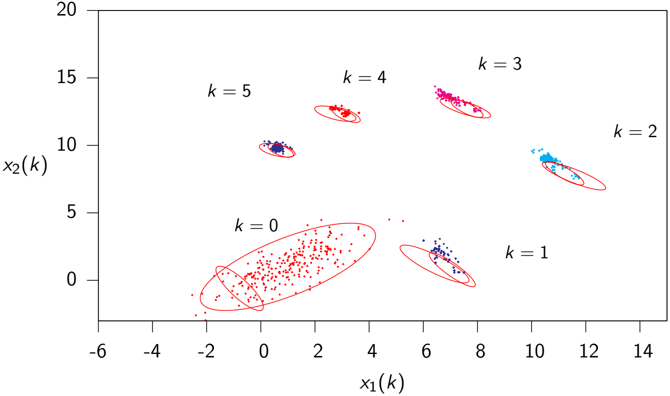
Optimal importance function



Particles' locations versus time using the optimal importance function; 250 particles.

Ellipses show the 95% contour of the true conditional densities before and after measurement.

Resampling



Particles' locations versus time using the optimal importance function with resampling; 250 particles.

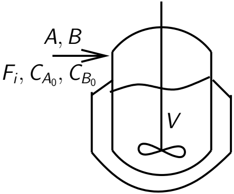
The MHE and particle filtering hybrid approach

Hybrid implementation

- Use the MHE optimization to locate/relocate the samples
- Use the PF to obtain fast state estimates between MHE optimizations

Application: Semi-Batch Reactor

- Reaction: $2A \rightarrow B$
- $k = 0.16$
- Measurement is $C_A + C_B$
- $x_0 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$

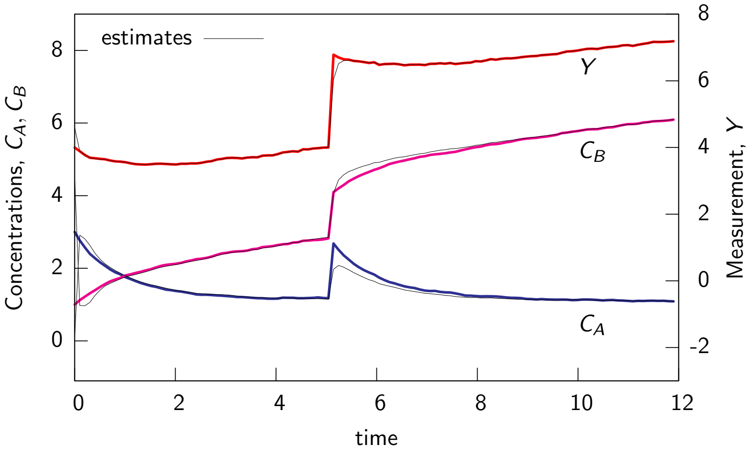


$$\frac{dC_A}{dt} = -2kC_A^2 + \frac{F_i}{V} C_{A_0} \qquad \Delta t = 0.1$$
$$\frac{dC_B}{dt} = kC_A^2 + \frac{F_i}{V} C_{B_0}$$

- Noise covariances $Q_w = \text{diag}\left(0.01^2, 0.01^2\right)$ and $R_v = 0.01^2$
- **Bad Prior:** $\bar{x}_0 = \begin{bmatrix} 0.1 & 4.5 \end{bmatrix}^T$ with a large P_0
- **Unmodelled Disturbance:** C_{A_0}, C_{B_0} is pulsed at $t_k = 5$

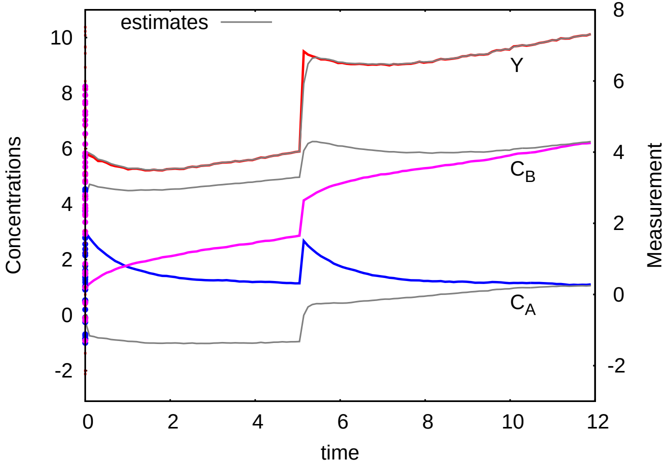
Using only MHE

- MHE implemented with $N = 15 (t = 1.5)$ and a smoothed prior
- MHE recovers robustly from bad priors and unmodelled disturbances



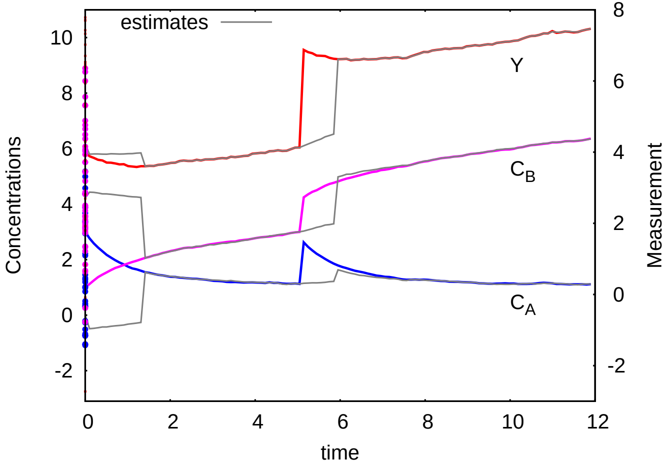
Using only particle filter

- Particle filter implemented with the Optimal importance function: $p(x_k|x_{k-1}, y_k)$, 50 samples, Resampling
- The PF samples never recover from a bad $\bar{x}_0$ and is unreliable



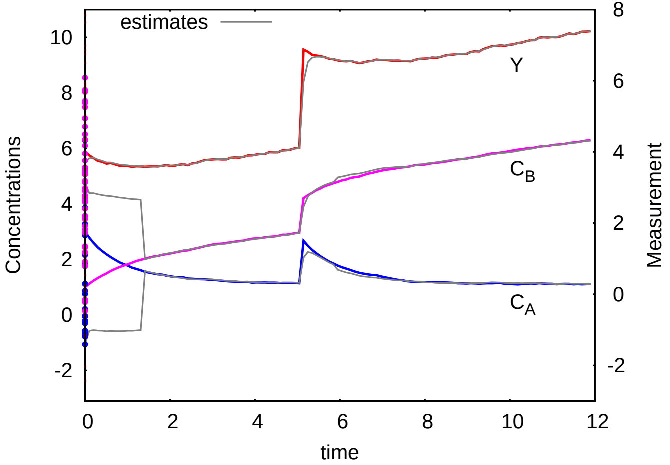
MHE/PF hybrid with a simple importance function

- Importance function for PF: $p(x_k|x_{k-1})$, 50 samples
- The PF samples recover from a bad $\bar{x}_0$ and the unmodelled disturbance only after the MHE relocates the samples



MHE/PF hybrid with an optimal importance function

- The optimal importance function: $p(x_k|x_{k-1}, y_k)$, 50 samples
- MHE relocates the samples after a bad $\bar{x}_0$, but samples recover from the unmodelled disturbance without needing the MHE



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- Hybrid MHE/PF methods can combine these complementary strengths.

Future challenges

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  We must decide on the subset of states to reconstruct from the data – an additional part to the modeling question.
- Nonlinear systems produce multi-modal densities. We need better solutions for handling these multi-modal densities in real time.

Acknowledgments

- Professor Bhavik Bakshi of OSU for helpful discussion.
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Further Reading I

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