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"GRAVITATIONAL N-BODY PROBLEM"
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Yale Univ.

Summary

The first phase of this research was aimed at developing an integration program for the dynamical system consisting of n ($n \geq 3$) point masses moving under their mutual gravitational influence. This development was accomplished and the program is described in Part 1 of this report. In addition to this, a comprehensive analytical and experimental investigation was conducted to clarify the problems arising from the singularities present in the dynamical system. This work is described in Part 2 of this report. Thirdly a series of visits to research centers of universities were made to avoid any possible repetitions and to receive the benefits of the experiences of other investigators. A review of the status of the field of research on the n-body gravitational problem is described in Part 3 of this report. Part 4 discusses future research and personnel performing the work.

Part 1. Integration program for n gravitational bodies

General

The n-body integration program prepared under the sponsorship of this Grant is a 6th order Cowell integration scheme, coded in Fortran IV. The computation is performed on the IBM 7094 computer with a 32,768 word memory in double precision.

The integration method is known as "predictor-corrector", without the explicit use of a difference table. The equations have been obtained from the standard difference-table formulae. They use only the ordinates ("leading edge" or "function column") of a difference table. The advantage is the saving of memory-space.

Step-sizes

The program uses individual and variable step-sizes for every body. The step-sizes are so aligned that when any body is iterated, every other body with an equal or smaller step-size is also iterated. All step-sizes may be expressed by 2^{-m}, where m is an integer. If at an integration step certain bodies are not iterated, their coordinates are extrapolated by means of Lagrangian formulae. These formulae are stored in the memory for the range from 0 to 1 in intervals of 1/128. If a smaller interval is needed, the formula is computed during the integration. This is rarely necessary and happens only when the step-size of the particle being extrapolated is 2^8 (=256) times larger than the smallest step-size used. Never must an extrapolation be performed further than one step-size past the range of the stored ordinates.

Halving-doubling

To halve the interval of integration for a particle, the intermediate values are found by Lagrangian interpolation formulae. These are also stored in memory. In order to retain 7 ordinates after a step-size is doubled, it is necessary to store in the memory 13 past ordinates. This is done for each body.

The criteria for halving and doubling follow from the difference tables. Since the tables are not stored, the differences are computed as follows

$$\nabla^6 F_o = F_o - 6F_{-1} + 15F_{-2} - 20F_{-3} + 15F_{-4} - 6F_{-5} + F_{-6}$$

$$\nabla^4 F_o = F_o - 4F_{-1} + 6F_{-2} - 4F_{-3} + F_{-4} ,$$

where $F_o = D^2 X_o$, D being the step-size.

Then, if $|\nabla^6 F_o| + |\nabla^6 G_o| + |\nabla^6 H_o| > \epsilon_H$,

with $G_o = D^2 Y_o$, $H_o = D^2 Z_o$,

the interval is halved.

Similarly, if

$|\nabla^4 F_o| + |\nabla^4 G_o| + |\nabla^4 H_o| < \epsilon_D$, the interval is doubled.

It has been found that $\epsilon_H = 100\ \epsilon_D$ is about the best ratio. The quantity ϵ_H determines the truncation error which may be seen to be $\sim 0.003\ \epsilon_H$.

Output

The program stores the output on magnetic tape in binary form at equal time intervals of specified length. The output quantities are the time, the coordinates, the velocities, and the step-sizes of all bodies. These quantities must usually be interpolated, since there is no guarantee that the integration will iterate on the exact time desired. The output tapes may then be used for any analysis since the complete system is fully determined by these output quantities. In other words the state of motion manifold is known.

The program will also print out - in addition to the storing on the tape - the state of the system at certain times. This is done whenever the smallest step-size used by a body changes. Such a procedure is useful in indicating the more active parts of the system's evolution, as for example, close encounters. At these print outs, the values of the known integrals are listed as checks. Other quantities include the time, the virial factor, and optionally, the complete set of coordinates, velocities, step-sizes. Also, the machine time may be printed.

This shows the time the program spends computing certain stages of the evolution of the system.

The Force function

The force function has been written as a separate subroutine, enabling the substitution of different potentials, such as a galactic potential, a tidal force, etc. These may be added to the standard force subroutine which is the mutual gravitational attraction of n mass points. The program is so designed that with k of the n bodies being iterated, the number of mutual distances computed is the lesser of the two quantities n k or n (n-1)/2.

Other programs

Subsidiary programs have been written to analyse the output from the magnetic tapes. These include

1) A program to plot any number of computed quantities versus time. Such quantities, for example, may be the variation of total energy, the potential energy, the kinetic energy, the virial factor, the moment of inertia, the departure from the original point in phase space, etc.

2) A program to list the osculating elements of a Keplerian orbit. Also the program lists the elements corresponding to a harmonic oscillator for every body around the center of mass.

3) A program to locate and show the formation, evolution, and disruption of any bodies in mutual osculating elliptic motion (i.e., binaries).

4) A program to list population desities in spherical shells of specified thickness at each time.

Part 2. Regularization of the problem of n bodies

A generalization of the method of regularization of collision orbits of the problem of two bodies was introduced. The independent variable occuring in the dynamical system was transformed by a modification of Sundmann's transformation and the equations of motion were formulated using a new time-variable. This transformation regularized the equations of motion for double collisions or for the case of close approaches. The dependent variables were not transformed. It was found by numerical experiments that for a relatively small number of bodies ($3 \leq n \leq 10$) this process to eliminate the singularities was rather efficient. The total computer-time required was shorter using the more complicated regularized equations than employing the original equations of motion. When the number of participating bodies was above say 10 the original (not regularized) system proved to be more efficient.

The method of regularization, consequently, is kept in readiness - even for a large number of participating bodies - if unusually close approaches would occur in the system.

It is highly probable that a combined transformation of the position coordinates and the time would increase the accuracy and at the same time would reduce the computer time. Such generalization of the transformations customary in the restricted problem of three bodies, was not found up to this time. A publication is in preparation on this subject.

Part 3 Review of status of research in the gravitational n-body problem.

The following is an alphabetized list of scientists who are or have been involved in research on the gravitational n-body problem. The list also contains their affiliations and a few remarks.

Aarseth, S.J. at Cambridge University has conducted extensive experimental studies using a slight modification of the Newtonian gravitational field. We intend to repeat his work in order to establish the effects of his modifications of the force field on the microscopic and macroscopic results.

King, I. at the University of California, Berkeley, suggested to us to superimpose the galactic field on the n-body gravitational field because of the theoretically expected possibility of significant effects.

Lecar, M. at the Smithsonian Astrophysical Institute and at Harvard University has developed a computer program generating random initial conditions for a gravitational n-body program. He also experimented with a one-dimensional model and was closely connected with the development of our program.

Miller, R. at the University of Chicago described certain fascinating numerical results according to which an exponential instability is exhibited in the phase space regarding the microscopic development of the gravitational n-body problem. Whether this instability is of numerical or dynamical origin is of utmost importance and one of our goals is to answer this question in the future.

Prendergast, K. at Columbia University established the probability of binary formation in the gravitational n-body problem. It is our intention to investigate the same question experimentally with our computer program.

Sherman, J. University of California, Radiation Laboratory integrated an impressive number of bodies under their mutual gravitational attraction. His program is less sophisticated than ours and the development of his cluster of over 300 bodies often comes to a slow-down and to a stop when close approaches demand such short time-steps that the integration does not proceed any more. This is the case when our regularized program would step in to resolve the difficulty.

Thuring, B. of Karlsruhe performed integrations of dynamical systems simulating two "colliding" clusters. His clusters consisted of close binaries and his findings show that these binaries are preserved and only slightly deformed during the "collision". We intend to extend this work to the problem of collision of clusters consisting of many stars.

van Albada, T.S., Kapteyn Lab., Groningen, Holland found a hierarchy of binaries by numerical integration. It seems to be necessary to repeat these calculations with several sets initial conditions before the discovery of different types of binaries can be confirmed.

von Hoerner, S. integrated the n-body problem and obtained tabulated results for binary formation. He also showed a tendency for large increases of density near the center of mass of the system. Such mass-concentrations are of great cosmological interest.

Wielen, R. of the University of Heidelberg has successfully integrated one-hundred gravitating bodies for extended time; up to 24 hours computer-time, corresponding to a few "periods" of the dynamical system. He has not studied in detail and consequently did not observe binary-formation which process is of great interest to us.

Worrall, G., University of Cambridge performed integrations using three bodies. The difficulties of extrapolating such results to say one hundred bodies may be considerable. During our work we have obtained some interesting results regarding three bodies of different masses. These results show an alternating close approach between one of the bodies and the two others. We also found a solution close to a periodic motion of new and non-trivial variety which we intend to study further.

Part 4. Future work and personnel.

The research conducted under the sponsorship of this grant is performed by the principal investigator, Dr. V. Szebehely and by Mr. M. Standish. It is expected that Mr. P. Nacozy will join this undertaking in the coming months.

Inasmuch as the computer program has been developed and perfected during the first phase of the project we will now undertake its actual utilization to investigate the gravitational n-body problem. Special attention will be given to binary formations, close approaches, escapes, density development, variations of the moments of inertia of the system, etc.

Further studies will also be performed regarding periodic solutions of the problem of three bodies. Another major effort will be directed toward the question of accuracy, reversibility, preservation of the integration constants of the system and to the possible establishment of additional integrals above the ten known.

It is expected that a continuation of the project after 1 Feb. 1967 will be applied for another year. This will allow the organizing of a meeting with possibly all participants involved in n-body integration. The meeting will have the sponsorship of the International Astronomical Union and it will be conducted prior to this Union's General Assembly in August 1967.

Since the development of the computer program has been accomplished the continuation of this project will require also a significant amount of funds for computer-time.