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THE ISSUE OF TIME EVOLUTION IN QUANTUM GRAVITY

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We discuss the relation between the concept of time and the dynamic structure of quantum gravity. We briefly review the problems of time associated with the standard procedures of gravity quantization. By explicitly utilizing York's analysis of the geometrodynamic degrees of freedom, and imposing the constraints as expectation–value equations, we describe a new procedure of gravity quantization. In particular, this "minimally constrained canonical" quantization procedure leads to a linear Schrödinger equation augmented by the super Hamiltonian and super momentum constraints imposed on expectation-values. This approach supplies a description of time evolution in quantum geometrodynamics free from the standard problems of time associated with canonical approaches. Furthermore, the theory is applicable to the full theory of general relativity, without the need to impose symmetries or reduce dimensionality. Utilizing this method we arrive at an analytic expression for the quantum evolution of the Kasner cosmology, as well as a numerically-generated solution of the quantized Taub model. The key new feature of this approach is that we impose the constraints weakly — instead of the quantum–mechanical wave function satisfying the constraints exactly, it need only satisfy them on average. Thus the constraints only constrain the observables.

1. Introduction

The task of describing time evolution in canonical quantum gravity appears to encounter numerous and seemingly insurmountable obstacles. These difficulties have been reviewed by Kuchař,[1] who has formulated them as a set of "problems of time." He concluded that these problems of time have not been resolved successfully in either the Dirac and square root ADM (Arnowitt, Deser and Misner) approaches to gravity quantization. In this paper we review both these procedures of gravity quantization in light of time evolution in the classical geometrodynamic theory and then present a novel approach to overcome these problems.

The standard ADM and Dirac procedures of canonical gravity quantization are based upon a slicing of 4–dimensional spacetime by spacelike hypersurfaces, and an

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identification of the whole 3–geometry (or even 3–metric) of a spacelike slice as the principal dynamic object of the underlying classical theory. The time evolution is thus understood as the change of the 3–geometry from slice to slice. The proper arena for classical geometrodynamics is superspace (either the space of all spacelike 3–metrics, or, in more sophisticated versions, the space of all 3–geometries).[2] We will call this approach the basic ADM paradigm. The existence of a unique 4–space geometry, coupled with the ADM paradigm, determines the proper variational procedure (all the components of the 3–metric, their conjugate momenta, as well as shift and lapse are varied independently). This generates a system of twelve Hamilton equations and four constraint equations — an over-determined, but not contradictive, system. Furthermore, the existence of a unique spacetime 4–geometry makes the classical theory of gravity in the basic ADM paradigm a fully constrained dynamic theory. The first sign of troubles in the basic ADM paradigm shows up when the Hamilton–Jacobi (hereafter referred to as HJ) equation is derived. A straightforward quantization based on such a HJ theory (Dirac's quantization) leads to the Wheeler–DeWitt equation (hereafter referred to as the WDW equation). The quantum Hamiltonian of this equation does not generate a time translation. None of the equations in Dirac's procedure of quantization contain any information pertinent to time evolution.

A description of time evolution can be introduced into this theory. A key to achieving this can be found through a proper analysis of geometrodynamic degrees of freedom in the classical theory. For general situations, York has shown that only two out of the six functions describing a 3–metric represent the geometrodynamic degrees of freedom; the remaining four, frequently called the embedding parameters, determine the coordinatization of the 3–slices of spacetime and the many–fingered time parameter.[3] If this split of the 3–metric is carried over to the quantum theory, with the embedding parameters treated as functional parameters of the solution of the WDW equation rather than dynamical variables, a notion of functional time arises (particular procedures for introducing a functional time into such a version of quantum gravity are described differently by different authors,[1] but the idea basically remains the same). Such procedures change the principal geometrodynamic object from the 3–metric (or the 3–geometry) to its true dynamically free part. In Dirac quantization the switch from the basic ADM paradigm of time evolution to the York paradigm occurs at the level of solutions to the WDW equation.

The differences between the ADM square root procedure of gravity quantization and Dirac's procedure originate at the classical HJ equations. Up until then they both follow the ADM paradigm, and the theory remains fully constrained. However, the split of the 3–metric into its true dynamic variables and embedding parameters occurs at the introduction of the HJ equation. Constraints are solved with respect to the momenta conjugate to the embedding variables, and the Hamiltonian of the theory is redefined (the so–called square root Hamiltonian is introduced) in such a way that it generates time translation. The resulting theory is quantized, producing first order functional derivative equations, one of which is interpreted as the Schrödinger equation. The functional time evolution is included explicitly in this equation, and only the true dynamic variables are quantized.

We wish to stress here that in both the ADM and Dirac procedures the fundamental dynamic equations are handled according to the basic ADM paradigm of time evolution. In particular, the theory is fully constrained on both classical and quantum levels — all of geometrodynamics is determined by the constraints on any reasonable slicing of the spacetime 4–geometry. The spacetime 4–geometry is assumed to be unique, and the embedding relations of the spacelike slicing of this 4–geometry are necessary to recover the picture of time evolution (cf. Sec. II). Such relations do not depend on a particular geometry of spacetime, but rather depend upon the existence of such a geometry. The classical theory of gravity assumes such existence, and is thus a constrained dynamic theory. It is our thesis that this feature need not be mirrored in quantum gravity, and our observation that the source of the "time" difficulties in both the Dirac and ADM quantization approaches arise because the York split of the dynamical and time parameterization variables occurs too late. Herein we introduce such a split before the dynamical theory is developed, and adhere to the York paradigm of time evolution in the development of the geometrodynamic theory and its quantization. This, we suggest, supplies a more cogent quantization procedure, in addition to bypassing the "problems of time."

A brief review of the techniques of canonical gravity quantization in the light of the ADM and York paradigms of time evolution is given in Sec. II and Sec. III. In Sec. IV we advance a new procedure of gravity quantization (minimally constrained canonical quantization, hereafter referred to as MC^2 quantization) which incorporates York's analysis of the geometrodynamic degrees of freedom. In Sec. V we apply this new theory of quantum geometrodynamics to two simple examples: the axisymmetric Kasner and Taub cosmologies. Sec. VI provides a brief discussion of the structure of this quantization approach, and an exploration of the consequences for time evolution.

2. The Basic ADM Paradigm

Both the Dirac and square root Hamiltonian approaches to canonical quantization make use of the 3+1 split of the spacetime geometry by Arnowitt, Deser and Misner.[4] According to the ADM prescription of classical general relativity, one considers a slicing of the spacetime by a family of spacelike hypersurfaces labeled by a parameter t. This parameter can be thought of as a time coordinate, so that each slice is identified by the relation t = const. The remaining three spatial coordinates x^i determine a coordinatization of each slice. The spacetime metric ${}^{(4)}g_{\mu\nu}$ is decomposed into shift N^i, lapse N and the 3-metric of the slice g_{ik}.

$$\begin{pmatrix} {}^{(4)}g_{00} & {}^{(4)}g_{0k} \\ {}^{(4)}g_{i0} & {}^{(4)}g_{ik} \end{pmatrix} = \begin{pmatrix} (N_s N^s - N^2) & N_k \\ N_i & g_{ik} \end{pmatrix} . \tag{1}$$

The Hilbert action of general relativity for the gravity field

$$I = \int \mathcal{L} d^4x = \int {}^{(4)}R\sqrt{-{}^{(4)}g}\, d^4x, \tag{2}$$

after augmentation by appropriate boundary terms, can be represented as

$$I = \int \left[\pi^{ij} \frac{\partial g_{ij}}{\partial t} - N\mathcal{H}(\pi^{ij}, g_{ij}) - N_i \mathcal{H}^i(\pi^{ij}, g_{ij}) \right] d^4x. \tag{3}$$

In this equation π^{ij}, $\mathcal{H}$, and $\mathcal{H}^i$ are given as follows:[5]

$$\pi^{ij} = \begin{pmatrix} \text{“geometrodynamic} \\ \text{field momentum”} \\ \text{dynamically conjugate to} \\ \text{the “geometrodynamic} \\ \text{field coordinate”} \, g_{ij} \end{pmatrix} = g^{\frac{1}{2}}(g^{ij}\, K - K^{ij}), \tag{4}$$

$$\mathcal{H} = (\text{“super–Hamiltonian”}) = g^{-\frac{1}{2}} \left(\mathrm{Tr}\Pi^2 - \frac{1}{2} \left(\mathrm{Tr}\Pi\right)^2 \right) - g^{\frac{1}{2}} R, \quad \text{and} \tag{5}$$

$$\mathcal{H}^i = (\text{“supermomentum”}) = -2\pi^{ik}{}_{|k}, \tag{6}$$

where K^{ij} is the extrinsic curvature tensor, $K = K^i_i$ is the trace of the extrinsic curvature tensor, Π is the matrix of π^{ij}, and the vertical line represents the covariant derivative with respect to the 3-metric. All the quantities in (4) – (6), including the covariant derivative in (6), are related to the 3–geometry of the slice.

The ADM approach treats all six components of the 3–metric g_{ik} as the gravitational “field coordinates” (with their conjugate momenta π^{ik}), while the lapse N and the shift N^i are treated as Lagrange multipliers (no conjugate momenta for them are present in the expression for action). The expression

$$\mathcal{H}_{ADM}(\pi^{ij}, g_{ij}, N, N_i) = N\mathcal{H}(\pi^{ij}, g_{ij}) + N_i \mathcal{H}^i(\pi^{ij}, g_{ij}) \tag{7}$$

plays the part of the Hamiltonian. Variations with respect to π^{ij} and g_{ij} produce the twelve Hamilton equations

$$\frac{\partial g_{ij}}{\partial t} = \frac{\partial \mathcal{H}_{ADM}}{\partial \pi^{ij}} = 2Ng^{-\frac{1}{2}} \left(\pi_{ij} - \frac{1}{2}\mathrm{Tr}\Pi \right) + N_{i|j} + N_{j|i}, \tag{8}$$

$$\frac{\partial \pi^{ij}}{\partial t} = -\frac{\partial \mathcal{H}_{ADM}}{\partial g_{ij}}. \tag{9}$$

We do not display this last equation in all detail as it is rather lengthy and is not necessary for this paper. In addition, there are three supermomentum constraints

$$\mathcal{H}^i(\pi^{ij}, g_{ij}) = 0, \tag{10}$$

and the super–Hamiltonian constraint

$$\mathcal{H}(\pi^{ij}, g_{ij}) = 0. \tag{11}$$

These four constraint equations are obtained by varying the ADM action with respect to N_i and N, respectively. The equations (8), (9), (10), (11) are not independent. The interplay between the Hamilton equations (8), (9) and the constraints (10), (11) is rather involved. In particular, if the constraints are satisfied on an initial slice then the Hamilton equations guarantee that they are also satisfied on all spacelike slices. On the other hand, if the constraints are satisfied on each slice of all possible spacelike foliations of a given spacetime then necessarily the Hamilton equations are satisfied. This last feature of general relativity has led people to refer to gravitational dynamics as a "fully-constrained theory." While it is clear that this property is essential to the classical theory, it is implicitly based on the assumption of the uniqueness of the spacetime 4–geometry.[6] We claim it is far from clear that a quantum theory must also be fully constrained. In particular, quantum geometrodynamics does not necessarily have an *a priori* notion of a unique 4-geometry. Efforts to impose the constraints fully are, in our opinion, responsible for the notorious "problems of time" in quantum gravity.

Within the ADM picture of gravitational dynamics, the supermomentum constraints (10) express the 3–dimensional diffeomorphism invariance (the freedom of choice of coordinates on the slices)

$$\mathcal{H}^i\left(\frac{\delta S}{\delta g_{ij}}, g_{ij}\right) = 0, \tag{12}$$

and the super–Hamiltonian constraint (11) is interpreted as the HJ equation of the theory[5,7]

$$\mathcal{H}\left(\frac{\delta S}{\delta g_{ij}}, g_{ij}\right) = 0, \tag{13}$$

where $S = S\left[\pi^{ij}, g_{ij}\right]$ is the Hamilton principal functional, and

$$\pi^{ij} = \frac{\delta S}{\delta g_{ij}}. \tag{14}$$

Although this HJ equation involves all six components of the 3–metric, the 3–diffeomorphic invariance as expressed by the supermomentum constraints

$$\left(\frac{\delta S}{\delta g_{ij}}\right)_{|j} = 0 \tag{15}$$

allows one to identify the HJ equation as a description of the evolution of the 3–geometry rather than the 3–metric. Equation (13) has a number of disturbing features, the worst of which being that the super–Hamiltonian $\mathcal{H}$ cannot be interpreted as the generator of the time translation.[8] In addition, unlike the situation in mechanics, the super–Hamiltonian $\mathcal{H}$ in the HJ equation (13) does not coincide with the $\mathcal{H}_{ADM}$ Hamiltonian (7) which generates the dynamic evolution via Hamilton's equations (8), (9). This $\mathcal{H}_{ADM}$ Hamiltonian is related to the Lagrangian $\mathcal{L}$ by the standard relation

$$\mathcal{H}_{ADM} = \pi^{ij} \frac{\partial g_{ij}}{\partial t} - \mathcal{L} \tag{16}$$

while, ordinarily, $\mathcal{H}$ is not.

The quantum theory obtained from equations (14), (13) following Dirac's prescription (cf. Sec. III), inherits many features of the classical theory, and in particular does not admit an interpretation of the quantum Hamiltonian as a generator of time translation.

An attempt to remedy this situation by replacing the super–Hamiltonian $\mathcal{H}$ by a Hamiltonian that does generate time translations, has been undertaken in the ADM square root procedure of quantization.[1] In this procedure the set of six 3–metric parameters is split into two subsets $\{\beta_1, \beta_2\}$ and $\{\alpha_1, \alpha_2, \alpha_3, \Omega\}$. The first of these is treated as the set of true gravitational degrees of freedom, and the second is considered to be the set of embedding variables. The α parameters are often referred to as coordinatization parameters, while Ω is called, depending on context, the slicing parameter, the scale factor, or the many fingered time parameter. Information relevant to the dynamics of the gravity field is carried by the β parameters, while the α and Ω parameters are reserved for a description of time. These ideas are related to York's analysis of gravitational degrees of freedom for generic gravitational fields. In his analysis the β variables describe the conformal part of a 3–geometry, Ω represents the scale factor, and the α parameters are determined by the choice of coordinatization of 3–slices. It is possible that there are other ways to perform such a split. In the two examples within this paper we utilize the York split; however, is important to realize that the theory is not limited to this prescription.

Although there are many different ways to specify coordinatization and slicing parameters, not all of them allow a resulting time that can be related to measurements. A standard procedure to introduce a meaningful time uses lapse and shift functions. In this approach coordinatization and slicing conditions are given arbitrarily only on an initial slice. After this, the coordinatization and slicing conditions are propagated by specifying shift N^i and lapse N functions. In order to investigate the implications of such a description for quantization procedures, it is useful to rewrite the action principle and Hamilton's equations in terms of β, α, and Ω.

The expression (3) for the action can be written symbolically in the form (for the sake of notational simplicity we omit indices on β and α parameters)

$$I = \int \left[\pi^\beta \dot{\beta} + \pi^\alpha \dot{\alpha} + \pi^\Omega \dot{\Omega} - N\mathcal{H}(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega) - N_i \mathcal{H}^i(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega)\right] d^4x, \tag{17}$$

where

$$\pi^\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}}, \qquad \pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\alpha}}, \qquad \pi^\Omega = \frac{\partial \mathcal{L}}{\partial \dot{\Omega}}, \tag{18}$$

are the momenta conjugate to β, α, and Ω respectively. Expression (7) takes the form

$$\mathcal{H}_{ADM}(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega, N, N_i) = N\mathcal{H}(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega) + N_i\mathcal{H}^i(\pi^\beta, \pi^\alpha, \pi^\Omega, \beta, \alpha, \Omega). \tag{19}$$

The Hamilton equations (8), (9) become

$$\dot{\beta} = \frac{\partial \mathcal{H}_{ADM}}{\partial \pi^\beta}, \tag{20}$$

$$\dot{\alpha} = \frac{\partial \mathcal{H}_{ADM}}{\partial \pi^\alpha}, \tag{21}$$

$$\dot{\Omega} = \frac{\partial \mathcal{H}_{ADM}}{\partial \pi^\Omega}, \tag{22}$$

and

$$\dot{\pi}^\beta = -\frac{\partial \mathcal{H}_{ADM}}{\partial \beta}, \tag{23}$$

$$\dot{\pi}^\alpha = -\frac{\partial \mathcal{H}_{ADM}}{\partial \alpha}, \tag{24}$$

$$\dot{\pi}^\Omega = -\frac{\partial \mathcal{H}_{ADM}}{\partial \Omega}. \tag{25}$$

The first three equations are merely kinematic relations between β, α, Ω and their conjugate momenta, and can be introduced in any spacetime, not just a solution of the Einstein equations. The role played by the relations (21), (22) is to provide a parameterization (by shift and lapse) of slicing and coordinatization conditions from the initial hypersurface to the whole spacetime. This parameterization is intertwined with the dynamics of the problem.

Parameterization of slicing and coordinatization conditions can be achieved by different means. However, any reasonable set of slicing and coordinatization conditions can be translated into the language of shift and lapse, and such a translation provides an interpretation of the conditions in terms of measurements.

In what has been done thus far, all the variables β, α, Ω have been treated in a symmetric fashion, in adherence to the basic ADM paradigm of time evolution. This symmetry is apparently violated only in the HJ theory. Namely, the classical super–momentum and super–Hamiltonian constraints (10), (11) are solved on a classical level with respect to the momenta conjugate to the embedding variables[5,9]

$$\pi^\beta = \frac{\delta S}{\delta \beta}, \qquad \pi^\alpha = \frac{\delta S}{\delta \alpha}, \qquad \pi^\Omega = \frac{\delta S}{\delta \Omega} \tag{26}$$

leading to the following four equations:

$$\frac{\delta S}{\delta \alpha} = -h_\alpha\left(\frac{\delta S}{\delta \beta}, \beta, \alpha, \Omega\right), \tag{27}$$

$$\frac{\delta S}{\delta \Omega} = -h_\Omega\left(\frac{\delta S}{\delta \beta}, \beta, \alpha, \Omega\right). \tag{28}$$

h_Ω contains a square root operation in it (we assume from now on that York's variables are used), which is why it is called the square root Hamiltonian. The square root Hamiltonian h_Ω in the classical theory, just as was the case with the super-Hamiltonian $\mathcal{H}$, is not related to the Lagrangian $\mathcal{L}$ as it would be in a standard dynamic theory (cf. equation (16)).

Although a departure from the basic ADM paradigm of time evolution has occurred, its influence on the theoretical structure of classical geometrodynamics is hardly noticeable. An appropriate solution of equations (27), (28) is a functional $S = S[\beta, \alpha, \Omega]$ with all the variables treated in the same way. No meaningful parameterization of embedding parameters (and, consequently, no meaningful time) has been introduced thus far. It is possible to introduce such a parameterization using shift and lapse, for example by resorting to equations (21), (22). Such a procedure implies transition from the basic ADM paradigm of time evolution to York's paradigm. This transition occurs at the same stage in both the original and square root Hamiltonian formulations of classical theory. The situation is quite different for the quantum theory.

3. Canonical Quantum Gravity

Dirac's procedure of canonical quantization utilizes the standard prescription

$$\pi^{ij} = \frac{\delta S}{\delta g_{ij}} \longrightarrow \widehat{\pi}_{ij} = \frac{\hbar}{i}\frac{\delta}{\delta g_{ij}}. \tag{29}$$

Applied to equations (12), (13), this leads to four functional differential equations based solely on the constraint equations of the classical theory

$$\widehat{\mathcal{H}}^i \left(\widehat{\pi}^{ij}, g_{ij}\right) \Psi = 0, \tag{30}$$

$$\widehat{\mathcal{H}} \left(\widehat{\pi}^{ij}, g_{ij}\right) \Psi = 0. \tag{31}$$

The three equations (30) are often interpreted as a requirement that the state functional is a functional of the 3–geometry $\Psi = \Psi[^{(3)}\mathcal{G}]$ rather than of the 3–metric. Equation (31) is the WDW equation[5,9] which is considered to be a proper wave equation for quantum gravity. This equation is reminiscent more of a Klein–Gordon equation than a Schrödinger equation. As we have already mentioned, the state functional Ψ in this equation is considered to be a functional of the underlying 3–geometry $\Psi = \Psi[^{(3)}\mathcal{G}]$. The commutation relations are imposed on all the components of the 3–metric, with proper care taken to ensure that they are diffeomorphically invariant. The WDW equation, just as its classical HJ counterpart, does not admit an interpretation of the Hamiltonian as a generator of time translations.

The ADM square root Hamiltonian procedure of quantization utilizes the split of the 3–metric into its true dynamic variables β, and its embedding parameters α, Ω. The standard prescription for quantization,

$$\frac{\delta S}{\delta \alpha} \longrightarrow i\hbar\frac{\delta}{\delta \alpha}, \qquad \frac{\delta S}{\delta \Omega} \longrightarrow i\hbar\frac{\delta}{\delta \Omega}, \qquad \frac{\delta S}{\delta \beta} \longrightarrow \widehat{\pi}^\beta = \frac{\hbar}{i}\frac{\delta}{\delta \beta}, \tag{32}$$

is applied to equations (27), (28) and leads to the four quantum equations,

$$i\hbar \frac{\delta \Psi}{\delta \alpha} = \widehat{h}_{\alpha} \left(\widehat{\pi}^{\beta}, \beta, \alpha, \Omega \right) \Psi, \tag{33}$$

$$i\hbar \frac{\delta \Psi}{\delta \Omega} = \widehat{h}_{\Omega} \left(\widehat{\pi}^{\beta}, \beta, \alpha, \Omega \right) \Psi, \tag{34}$$

where $\Psi = \Psi[\beta, \alpha, \Omega]$. The choice of parameters α and Ω is frequently made in such a way that the three equations (33) are interpreted as a requirement of the diffeomorphism invariance, while equation (34) is considered to be the proper Schrödinger equation. In this case parameters α can be interpreted as coordinatization parameters while the function Ω can be thought of as the many-fingered time parameter with h_{Ω} treated as the Hamiltonian.[10,11]

Both the ADM square root and Dirac quantization procedures admit, in principle, a time evolution prescription to be introduced in them, although this should be done on different stages. In Dirac's quantization procedure the state functional Ψ is a functional of either the slice 3-metric or the slice 3-geometry, which can be written in terms of β, α, Ω functions as

$$\Psi = \Psi[{}^{(3)}g] = \Psi[\beta, \alpha, \Omega] \quad \text{or} \quad \Psi = \Psi[{}^{(3)}\mathcal{G}] = \Psi[\beta, \Omega], \tag{35}$$

where α, Ω and β are treated on equal footing. In particular, a computation of any quantity similar to an expectation function, if it were possible, ordinarily would involve functional integration over either all six functional parameters α, Ω, β or the three functional parameters β and Ω. Along these lines, we wish to remind the reader that in Dirac's quantization approach the commutation relations are imposed on all the components of the 3-metric (which include the embedding parameters and their conjugate momenta), and in quantum mechanics the evaluation of expectations involves integration over all the variables participating in the commutation relations. Such a calculation, however, would ordinarily exclude a possibility for one to introduce a functional time and a time evolution in the theory. A way to avoid such an unpleasant situation is to split the set of variables in two subsets, one of which (β) is to be considered as a set of true functional arguments of the state functional (and to be integrated over during computing the expectations), while the others (α, Ω) are to be considered as a set of functional parameters (one per slice) and to be interpreted as a representation of functional time.[12] This split should be introduced only after the solution for the state functional is obtained and prior to the evaluation of expectation values. Such a procedure is equivalent to treating the embedding variables as quantum in one part of the theory yet as classical in another. In addition, it still does not yield a meaningful parameterization of time.

Such a parameterization by shift and lapse can be achieved in a way similar to that of the classical theory via invoking equations (21), (22). One should notice, however, that the right hand sides of these equations depend on both α, Ω and the true dynamic variables β that remain quantized and, consequently, are given by distributions even after the quantum nature of embedding variables is suppressed.

The situation can be remedied if their expectation values (better to be called the modified or reduced expectation values as functional integration over true dynamic variables only is involved) over the solution WDW equation are substituted in (21), (22). No contradictions are introduced this way (which follows from the simple count of equations and unknowns). The relations (21), (22) can be thought of as definitions of N^i, N in quantum gravity. In practice, this procedure can become rather complicated. In particular, the parameterization contains functions (the expectation functions of dynamic variables as well as of their conjugate momenta) that can be completely determined only after the WDW equation has been solved. This means that parameterization of coordinatization functions and the many-fingered time scale parameter by the shift and lapse results in an implicit procedure in which even the parameterization itself remains undetermined until the final solution of the entire problem has been obtained. Nevertheless, it seems to be unavoidable if one is to interpret the solutions of the quantum gravity equations in terms of observations. Appropriate care should be taken as equation (20) does not ordinarily have to be satisfied anymore even for the expectations. Consequently, the expectation values of the momenta are not related to the extrinsic curvature of $t = \mathrm{const}$ slices of the spacetime which is determined by the expectation values of the 3-metric, shift, and lapse. The spacetime itself is not ordinarily a solution of Einstein equations.

In the square root Hamiltonian quantization all basic equations are derived within the framework of the basic ADM paradigm of time evolution only up to the HJ theory. The split of variables occurs before quantization. The quantum theory assumes from the very start that there is only one function Ω per slice. The state functional Ψ when restricted to one slice becomes essentially a functional of β only, which can be properly written as

$$\Psi = \Psi[\beta; \alpha, \Omega] \quad \text{or} \quad \Psi = \Psi[\beta; \Omega]. \tag{36}$$

The computation of expectation functions (or any other momenta) now assumes functional integration over the two parameters β only, with the result depending on Ω as a parameter (of course, it will also depend on three other functional parameters α, which we call coordinatization parameters). This functional parameter (Ω) plays the role of the many-fingered time parameter. In other words, in the square root Hamiltonian approach the switch from the basic ADM paradigm of evolution to York's paradigm occurs at the final stage of formulating classical geometrodynamics. The quantum theory is formulated entirely within York's paradigm of time evolution. However the structure of all equations and the structure of Hamiltonian (the square root Hamiltonian) is determined by the basic ADM paradigm, which leads subsequently to numerous difficulties, including the notorious "problems of time" (cf. below). As in Dirac's procedure of quantization, the quantum equations (33), (34) themselves do not supply a meaningful parameterization of time. Such a parameterization by shift and lapse can be introduced in the ADM square root Hamiltonian procedure of quantization just as in Dirac's procedure via invoking equations (21), (22). The procedure is exactly the same except the embedding variables are classical to start with, and the expectations are calculated via functional integration over the space of true dynamic variables with the state functional being the solution of the Schrödinger equation (instead of the WDW equation).

The main conclusion of our exposition of the time evolution description in quantum gravity is that it appears to us that any meaningful description of time evolution can be achieved only within York's paradigm while the structure of all basic equations in both Dirac's and ADM square root Hamiltonian procedures of canonical quantum gravity is determined by the basic ADM paradigm of time evolution. In a formal description (cf. above) this leads to numerous disturbing features making quantum gravity a special case different from any other quantum dynamic theory.

3.1.

It is popular, in discussing the difficulties related to the description of time evolution, to refer to the "problems of time" formulated by Kuchař.[1] We describe them here very briefly as, in our opinion, the nature of these problems can be made quite transparent from the point of view of two paradigms of time evolution. We we will return to this question (cf. Sec. VI) after we develop the formal machinery to handle it (Sec. IV, V). At this point we will only briefly describe them following Kuchař.

The first of the time problems is formulated as a dependence of the final state $\Psi_{\Sigma_{FIN}}$ at a given $\Psi_{\Sigma_{IN}}$ on the foliation that connects the spacelike hypersurfaces Σ_{IN} and Σ_{FIN}. This problem is called the problem of functional evolution.

The second problem is observed by Kuchař in the context of the square root Hamiltonian quantization. It stems from the fact that the split of the metric variables into the true dynamic variables β as opposed to the coordinatization parameters α and the slicing parameter Ω is not unique. Different splits might lead to different and possibly inequivalent quantum theories. Kuchař calls this the multiple-choice problem. In our opinion, this difficulty is not restricted to the square root Hamiltonian quantization. It might show up in Dirac's procedure at the final stage when shift and lapse parameterization is introduced.

The third difficulty is the so called Hilbert space problem and is particular to the Dirac quantization procedure. In the square root Hamiltonian approach, the Schrödinger equation automatically determines an inner product and a possibility to form, at given values of α and Ω, meaningful observables. That is, observables for which expectation values can be evaluated in a way similar to that of standard quantum mechanics (modulo some technical difficulties caused by the functional nature of variables). Nothing similar is granted by the WDW equation of the Dirac approach. The WDW equation is a second order functional differential equation for which the problem of supplying the space of its solution with the structure of a Hilbert space encounters numerous difficulties. Roughly speaking, the equation does not admit a one–Universe interpretation.

We wish to add to these three problems one more (also described by Kuchař), namely the spectral-analysis problem that shows up if the square root Hamiltonian is involved. It is well known that, if the operator expression under the square root is not positively definite, the spectral analysis procedure yields, as the result, a Hamiltonian that is not self–adjoint. This implies that the Schrödinger equation with such a Hamiltonian does not produce unitary evolution.

As we will argue later on in this paper the interpretational problems of time (the functional evolution problem and the multiple choice problem) appear to be simply ill posed while the technical problems (the Hilbert space problem and the spectral analysis problem) are the results of assumptions concerning the nature of quantum geometrodynamics that are classical in their nature.

The equations of quantum gravity, be it equations (30), (31) of Dirac's quantization or equations (33), (34) of the square root Hamiltonian quantization, do not provide a natural split of the variables into the true dynamic variables and the embedding parameters. This is quite understandable in light of the basic ADM paradigm of time evolution. This would not cause any problems in the classical theory because the two paradigms are equivalent formally. Both the true dynamic and embedding variables are treated the same way in the classical theory. Until a meaningful definition of time is introduced into these two quantization procedures, the same is true in the quantum theory. However, after time is introduced, the true dynamic variables are quantized while embedding variables are not. Even in Dirac's procedure the embedding variables are "dequantized" when a shift and lapse parameterization is introduced. The quantum and classical variables are treated quite differently (commutation relations, functional integration in computing expectation values, and so on). Different splits correspond to different physical systems. The split reflects, for any particular gravitational system, our understanding of the system's dynamics. In a sense, it is similar to the situation described by N. Bohr regarding the split of any system into its quantum and classical parts. Such a split depends not only on the object that we are considering but, also, on the questions that we are asking. It is clear that different splits represent physically different systems which would not happen if both parts were classical.

In quantum gauge field theory the unified treatment of the gauge variables with the dynamic variables was solved by Faddeev and Popov (F-P).[13] We wish to emphasize here that this F-P prescription appears to be incompatible with quantum gravity. The nature of time in general relativity does not allow one to model gravitational phenomena as purely quantum. We merely wish to emphasize here that our proposed circumvention of the "problems of time" in canonical quantum gravity will make use of a different paradigm — a paradigm in which the constraints are handled as expectation-vale equations. Therefore, the F-P procedure does not apply in our approach.

In the next section we advance a theoretical framework entirely within York's paradigm of time evolution in both the classical and quantum theories. This approach does not involve a transition from one paradigm to another. The resulting theory sheds new light on the issue of time evolution in quantum gravity.

4. Minimally Constrained Canonical Quantization (MC^2 Quantization)

We propose here an alternative approach to the quantization of gravity in which the theoretical framework is developed entirely within York's paradigm of time evolution. We split the 3–metric in its dynamic part described by true dynamic variables β and embedding parameters α, Ω. The true dynamic variables form what we call a dynamic superspace while the embedding variables essentially parameterize time (and coordinatization). The proper arena for geometrodynamics in this approach is the dynamic superspace instead of the superspace of 3–metrics or 3–geometries.

In its spirit, our approach is motivated by York's analysis of the gravitational degrees of freedom.[3,13,14] Historically, this development was initially motivated by Wheeler's semi–intuitive remark that the 3–geometry of a spacelike hypersurface has encoded within it the two gravitational degrees of freedom as well as its temporal location within spacetime. It is this notion that the 3–geometry is a carrier of information on time that has been referred to as "Wheeler's *many–fingered time*."[4,9,10] It was J. York who first made this thesis precise. He forwarded what has now become almost the canonical split of the 3–geometry into its underlying conformal equivalence class (its *shape* representing the two dynamic degrees of freedom of the gravitational field coordinate per space point) and the conformal scale factor (its *scale* representing Wheeler's many-fingered time). Only the conformal 3–geometry is truly dynamic in the sense that it can be specified freely as the initial data. The scale factor is non-dynamic and essentially specifies Wheeler's many-fingered time. The results of York have demonstrated that the true dynamic part of the gravitational field is not the 3-geometry but only its conformal part, and that the proper configuration space or "arena for geometrodynamics" should be the underlying conformal superspace (the space of all conformal 3-geometries) rather than Wheeler's superspace (the space of all 3-geometries). The conformal scale factor and three other functional parameters of the 3–metric (responsible for coordinate conditions) thus become external parameters. According to York, the true dynamic variables β are associated with conformal 3-geometry, the many fingered time variable Ω with the 3-geometry scale factor, and the remaining three variables α of the 3–metric with a coordinatization of a spacelike 3-surface. The many-fingered time variable and coordinatization variables are assumed to be fixed initially by some conditions, and then subsequently controlled by the shift and lapse. In this description the dynamic superspace is the space of conformal classes.

For the rest of this section (and for most of the paper) a particular choice of the split of 3–geometry in true dynamic variables and embedding variables is not important. The only important point is that the arena for geometrodynamics is the true dynamic superspace. In other words, we suggest that one should interpret geometrodynamics as an evolution of the true dynamic part of 3–geometry β in an external field determined by embedding variables Ω, α. Such an approach calls for a reformulation of geometrodynamics on the classical level. We start from the standard Lagrangian $\mathcal{L}$ (written in terms of the 3–metric, shift and lapse) and the associated action (with appropriate boundary terms, as needed, to remove the second time derivatives terms) and we introduce the momenta conjugate to the true dynamic variables

$$\pi_\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}}. \tag{37}$$

We then use these π_β's to form the dynamic Hamiltonian $\mathcal{H}_{DYN}$ of our approach,

$$\mathcal{H}_{DYN} = \pi_\beta \dot{\beta} - \mathcal{L}. \tag{38}$$

The new Hamiltonian $\mathcal{H}_{DYN}$ is distinctly different from $\mathcal{H}_{ADM}$ and its arguments do not coincide with those of $\mathcal{H}_{ADM}$, namely

$$\mathcal{H}_{DYN} = \mathcal{H}_{DYN}(\beta, \pi_\beta; \Omega, \alpha). \tag{39}$$

The variables preceding the semicolon are the coordinates and momenta of the gravitational true degrees of freedom, i.e. of the true geometrodynamics, while the ones following the semicolon are treated as describing an external field. The variation of β and π_β leads to the equations of geometrodynamics, i.e. to two pairs of Hamilton equations,

$$\dot{\beta} = \frac{\partial \mathcal{H}_{DYN}}{\partial \pi_\beta}, \tag{40}$$

$$\dot{\pi}_\beta = -\frac{\partial \mathcal{H}_{DYN}}{\partial \beta}, \tag{41}$$

and, subsequently, to the HJ equation

$$\frac{\delta S}{\delta t} = -\mathcal{H}_{DYN}\left(\beta, \frac{\delta S}{\delta \beta}; \Omega, \alpha\right). \tag{42}$$

Here S is a functional of β and, in addition, a function of time t,

$$S = S[\beta; t). \tag{43}$$

and $\frac{\delta S}{\delta t}$ is defined by

$$\frac{\partial S}{\partial t} = \int \frac{\delta S}{\delta t}\, d^3x. \tag{44}$$

Neither the Hamilton equations (40, 41) nor the HJ equation (42) are capable of providing, in and by themselves, any predictions, as their solutions depend on the functional parameters Ω and α which are not yet known. One can complete the system of equations by adding to the Hamilton equations, or to the HJ equation, the standard constraint equations of general relativity. These constraints should be satisfied when the solution for β, π_β of (40,41) or (42) (with appropriate initial data) is substituted in them. We use symbols $[\beta]_s$, $[\pi_\beta]_s$ for such a solution.

$$\begin{aligned} \mathcal{H}^i\left([\beta]_s, [\pi_\beta]_s, \Omega, \alpha\right) &= 0 \\ \mathcal{H}\left([\beta]_s, [\pi_\beta]_s, \Omega, \alpha\right) &= 0 \end{aligned} \tag{45}$$

We wish to emphasize that these four constraint equations cannot be derived from a variational principle in our theory.

One should be careful when comparing equations (40), (41) with standard equations (20), (23) of Sec. II. In computing $\frac{\partial}{\partial\beta}$ and $\frac{\partial}{\partial\pi_\beta}$, the set of independent variables $\{\beta, \pi_\beta, \alpha, \Omega\}$ for (40), (41) differs from the set of independent variables $\{\beta, \pi_\beta, \alpha, \pi_\alpha, \Omega, \pi_\Omega\}$ for (20), (23), and so does the meaning of the partial derivatives in these two sets of equations. A proper comparison of equations (40), (41) with standard equations (20), (23) is rather trivial in the sense that it does not involve anything but the Hamiltonian dynamics. It shows that these two sets of equations simply coincide. Thus equations (40), (41), (45) or (42), (45) are equivalent to the standard equations of classical geometrodynamics.

For the purpose of quantization, we start from our HJ equation (42) which describes, effectively, what we refer to as the *true geometrodynamics* (*conformal geometrodynamics* in the original York formulation). Our HJ equation elolves the true dynamic variables, where this evolution is parameterized by the "external" field represented by embedding variables, such as, for instance, (1) the scale parameter and (2) the coordinatization variables. Using the HJ equation (42) we may transition to the corresponding Schrödinger equation

$$i\hbar\frac{\delta\Psi}{\delta t} = \widehat{\mathcal{H}}_{DYN}\left(\beta, \widehat{\pi}_\beta; \Omega, \alpha\right)\Psi \tag{46}$$

where $\widehat{\pi}_\beta = \frac{\hbar}{i}\frac{\delta}{\delta\beta}$. The Schrödinger equation (46) treats embedding variables as external classical fields and quantizes only the true dynamic variables, β. The state functional Ψ in this equation is a functional of β and a function of t.

$$\Psi = \Psi[\beta; t) \tag{47}$$

This Schrödinger equation (with specific initial data) can be solved (cf., for instance examples of the Bianchi 1A and Taub cosmological models in Sec. V). The resulting solution Ψ_s of this Schrödinger equation is not capable of providing any definite predictions as it depends on four functional parameters Ω, α which remain at this stage undetermined. All expectations, such as, for instance the expectation values of β

$$<\beta>_s = \langle\Psi_s|\beta|\Psi_s\rangle = \int \Psi_s^*\beta\Psi_s\,\mathcal{D}\beta \tag{48}$$

or of $\widehat{\pi}_\beta$

$$<\pi_\beta>_s = \langle\Psi_s|\widehat{\pi}_\beta|\Psi_s\rangle = \int \Psi_s^*\widehat{\pi}_\beta\Psi_s\,\mathcal{D}\beta \tag{49}$$

also depend on these functional parameters. To specify these functions we resort to the constraint equations. As in case of classical geometrodynamics the constraints should be imposed on the solution of the dynamic equations (Schrödinger equation) with appropriate initial data. In this way the constraints, determine the unique values of Ω and α. It is possible that there are several ways to couple the constraints to the quantization of the true dynamic variables, β. We propose here that the four constraints be imposed only on the expectation values of the conformal dynamics

$$\begin{aligned} \mathcal{H}^i\left(<\beta>_s, <\pi_\beta>_s, \Omega, \alpha\right) = 0 \\ \mathcal{H}\left(<\beta>_s, <\pi_\beta>_s, \Omega, \alpha\right) = 0 \end{aligned} \tag{50}$$

i.e. only on measurable quantities. In so doing, we explicitly avoid the interpretational conundrums associated with the problems of time, and we form "classical" gravitational time (including a clock) driven by the quantized geometrodynamic system — i.e. *quantum-driven many-fingered time.*

This procedure of quantization utilizes explicitly York's paradigm of time evolution and introduces a meaningful time parameterization utilizing the shift and lapse. The super-momentum and super-Hamiltonian constraints are not satisfied as strictly as in the Dirac and ADM square root approaches. Our point of view is that the question of slicing independence of evolution is not a well posed question in quantum gravity and should be recovered only in the classical limit. It is clear that this weakening of the constraints, so that they hold only on the expectation values over the solution of the initial-value problem of the Schrödinger equation, makes the theory less restrictive and enlarges the set of possible solutions.

The way evolution occurs can be described as follows. Initial data at $t = t_0$ consists of the initial state functional $\Psi = \Psi_0$, and the initial values (functions) of embedding variables. In addition, shift and lapse are supposed to be given. Equations (48), (49) yield the expectation values (functions) of true dynamic variables and their conjugate momenta. The results are substituted in constraints (50). After this, the constraints are solved with respect to the time derivatives of embedding parameters. A step forward in time (say, with an increment Δt) is performed by integration of obtained expressions to evolve embedding variables and by integration of the Schrödinger equation (46) to evolve the state functional. This concludes one step forward in time. The next step is performed by repeating the same operations in the same order.

The goal of this paper is limited to a clarification of our basic thesis regarding the quantization of gravity — minimally constrained canonical quantum gravity.[15] We believe that the best way to achieve this goal is to consider a couple of simple illustrative examples. These examples also stress some technical points. In particular they demonstrate the way constraints (50) participate in describing the time evolution.

5. MC^2 Quantization: The Kasner and Taub Cosmologies

5.1. MC^2

The Bianchi 1A cosmological model is commonly referred to as the axisymmetric Kasner model.[12] Its metric is determined by two parameters, the scale factor Ω and the anisotropy parameter β (we choose $N^i = 0$ and $N = 1$ values of shift and lapse)

$$ds^2 = -dt^2 + \mathrm{e}^{-2\Omega}\left(\mathrm{e}^{2\beta}dx^2 + \mathrm{e}^{2\beta}dy^2 + \mathrm{e}^{-4\beta}dz^2\right). \tag{51}$$

As this cosmology is homogeneous the two functions Ω and β are functions of the time parameter t only. The scalar 4–curvature can be expressed in terms of these two functions to yield the Hilbert action and, after subtracting the boundary term, the cosmological action,

$$I_C = I_H + \frac{3V}{8\pi}\dot{\Omega}\mathrm{e}^{-3\Omega}|_{t_0}^{t_f} = \frac{3V}{8\pi}\int\limits_{t_0}^{t_f}\left(\dot{\beta}^2 - \dot{\Omega}^2\right)\mathrm{e}^{-3\Omega}dt, \tag{52}$$

where $V = \iiint dxdydz$ is the spatial volume element.

We treat the scale factor $\Omega(t)$ as the many-fingered time parameter and the anisotropy $\beta(t)$ as the dynamic degree of freedom. The momentum conjugate to β is

$$p_\beta = \frac{\partial L}{\partial \dot{\beta}} = \frac{3V}{4\pi}\mathrm{e}^{-3\Omega}\dot{\beta}. \tag{53}$$

As is usually the case with homogeneous cosmologies, we work here with the momentum p_β rather than with the density π_β. The Hamiltonian of the system in our approach can be expressed in terms of the momentum conjugate to β and the Lagrangian.

$$\begin{aligned} H_{DYN} &= p_\beta\dot{\beta} - L \\ &= \frac{4\pi}{3V}\mathrm{e}^{3\Omega}p_\beta^2 - \frac{3V}{8\pi}\left(\frac{4\pi}{3V}\right)^2\mathrm{e}^{3\Omega}p_\beta^2 + \frac{3V}{8\pi}\dot{\Omega}^2\mathrm{e}^{-3\Omega} \\ &= \frac{2\pi}{3V}\mathrm{e}^{3\Omega}p_\beta^2 + \frac{3V}{8\pi}\dot{\Omega}^2\mathrm{e}^{-3\Omega}. \end{aligned} \tag{54}$$

Only β, our single true degree of freedom, is used as a dynamic variable. In the classical theory this Hamiltonian can be used to produce either one pair of Hamilton equations or the equivalent HJ equation. In either case, the dynamical picture derived in this way is incomplete. To complete it we need to impose, in addition, the super-Hamiltonian constraint.

$$p_\beta^2 = \left(\frac{3V}{4\pi}\right)^2\mathrm{e}^{-6\Omega}\dot{\Omega}^2. \tag{55}$$

Using the HJ equation,

$$\frac{\partial S}{\partial t} = -H_{DYN}\left(\frac{\partial S}{\partial \beta}, \Omega(t), \dot{\Omega}(t)\right), \tag{56}$$

together with the expression (54) for the Hamiltonian H, we obtain the Schrödinger equation for the axisymmetric Kasner model:

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{2\pi\hbar^2}{3V}\mathrm{e}^{3\Omega}\frac{\partial^2\Psi}{\partial\beta^2} + \frac{3V}{8\pi}\dot{\Omega}^2\mathrm{e}^{-3\Omega}\Psi. \tag{57}$$

The constant $\hbar$ in this equation should be understood as the square of Planck's length scale, rather than the standard Planck constant. We wish to stress here that the scale factor Ω in the Schrödinger equation is thus far an unknown function of time. To complete the dynamical picture we impose, in addition to equation (57), the super-Hamiltonian constraint:

$$< p_\beta >_s^2 = \left(\frac{4\pi}{3V}\right)^2 \mathrm{e}^{-6\Omega}\dot{\Omega}^2. \tag{58}$$

Here $< p_\beta >_s$ is the expectation value of the momentum operator $\widehat{p}_\beta = \frac{\hbar}{i}\frac{\partial}{\partial\beta}$,

$$< p_\beta >_s = \langle\Psi_s|\widehat{p}_\beta|\Psi_s\rangle = \int\limits_{-\infty}^{\infty} \Psi_s^*(\beta,t)\widehat{p}_\beta\Psi_s(\beta,t)d\beta, \tag{59}$$

with Ψ_s the solution of the Schrödinger equation with specified initial data. The system of equations (57), (58) provide us with a complete quantum dynamic picture of the axisymmetric Kasner model and, when augmented by appropriate initial and boundary conditions, can be solved analytically.

Treating the scale factor Ω as a function of time generating an external potential, and separating variables, one can easily find the general solution to the Schrödinger equation (57) (cf. ref. 15)

$$\Psi(\beta,t) = \int\limits_{-\infty}^{\infty} A_k \mathrm{e}^{\frac{i}{\hbar}k\beta}\mathrm{e}^{-\frac{i}{\hbar}[fk^2+g]}dk. \tag{60}$$

where $f = f(t)$ and $g = g(t)$ are shorthand notations for two functions of time determined by yet unknown function of $\Omega(t)$ via expressions

$$f = f(t) = \frac{2\pi}{3V}\int\limits_{t_0}^{t} \mathrm{e}^{3\Omega}dt, \qquad g = g(t) = \frac{3V}{8\pi}\int\limits_{t_0}^{t} \dot{\Omega}^2\mathrm{e}^{-3\Omega}dt. \tag{61}$$

To specify a particular problem one has to furnish appropriate initial data,

$$\Psi(\beta,t)|_{t_0} = \Psi_s(\beta,t_0) = \int\limits_{-\infty}^{\infty} A_k\mathrm{e}^{\frac{i}{\hbar}k\beta}dk. \tag{62}$$

In this section we consider the simplest example comparable with the quantum mechanics of a particle, namely a Gaussian wave packet centered initially at a value k_0 of k (we will further discuss k_0 later):

$$A_k = C\mathrm{e}^{-a(k-k_0)^2}, \tag{63}$$

where the constant a determines the initial width of the wave packet in momentum space, and C is a normalization constant. This leads to an initial wave function:

$$\Psi(\beta,t_0) = C\int\limits_{-\infty}^{\infty} \mathrm{e}^{-a(k-k_0)^2}\mathrm{e}^{\frac{i}{\hbar}k\beta}dk = C\sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{i}{\hbar}\beta k_0}\mathrm{e}^{-\frac{\beta^2}{4a\hbar^2}}. \tag{64}$$

The value of the normalization constant C is determined by the condition $\langle\Psi(\beta,t_0)|\Psi(\beta,t_0)\rangle = 1$, leading to

$$C^2 = \frac{\sqrt{a}}{\hbar\pi^{\frac{3}{2}}\sqrt{2}}. \tag{65}$$

Using expression (63) for A_k, we can write down the solution $\Psi_s(\beta,t)$ for the wave packet.

$$\Psi_s(\beta,t) = C\mathrm{e}^{-\frac{i}{\hbar}g}\int_{-\infty}^{\infty}\mathrm{e}^{-a(k-k_0)^2}\mathrm{e}^{\frac{i}{\hbar}\beta k}\mathrm{e}^{\frac{i}{\hbar}fk^2}dk. \tag{66}$$

The integral on the right hand side of (66) can be evaluated. The final expression for the solution describing a Gaussian wave packet of Kasner universes allows one to evaluate the expectation values of dynamic variables. In particular, for the expectation $< p_\beta >_s$ of the momentum $\widehat{p}_\beta = \frac{\hbar}{i}\frac{\partial}{\partial\beta}$ one obtains

$$< p_\beta >_s = \langle\Psi_s|\widehat{p}_\beta|\Psi_s\rangle = k_0. \tag{67}$$

In other words the expectation value of the momentum $< p_\beta >_s$ does not change with time. It is determined by the k–center of the packet at $t = t_0$ (for the detailed calculations, cf. ref. 15).

It is clear that the solution of the Schrödinger equation given by (66) and describing the wave packet cannot provide any definite predictions as it contains the two functions of time $f(t)$ and $g(t)$ which are themselves related to the as yet undetermined scale factor Ω. To find $\Omega(t)$ we need to (1) use the expectation value $< p_\beta >_s$ computed above, (2) substitute this expectation value into the constraint (58) and (3) solve the resulting equation with respect to $\dot\Omega$. Substitution of (67) in (58) yields

$$k_0^2 = \left(\frac{3V}{4\pi}\right)^2\mathrm{e}^{-6\Omega}\dot\Omega^2. \tag{68}$$

This equation is identical to its classical counterpart. Once the solution of this equation is substituted in (66), the geometrodynamic problem (57), (58) for the wave packet (63) is solved completely. To summarize, the many-fingered time of quantum geometrodynamics in the case of a Gaussian wave packet over axisymmetric Kasner spacetimes coincides with its classical counterpart if the expectation value of the momentum of the packet is identified with the (conserved) value of the momentum of the classical solution.

The expectation value for the anisotropy parameter β, where β is the only quantum dynamic variable in this model, is given by:

$$< \beta >_s = \langle\Psi_s|\beta|\Psi_s\rangle = 2k_0 f(t). \tag{69}$$

Thus "the center" of the wave packet evolves as the classical Kasner universe with momentum value k_0. The spread of the wave packet in time is given by the variance in β.

$$< (\beta - < \beta >_s)^2 >_s = \langle\Psi_s|\left(\beta - < \beta >_s\right)^2|\Psi_s\rangle = \frac{\hbar^2a^2+f^2}{a} \tag{70}$$

The wave packet spreads in time, similar to the case of a quantum-mechanical free particle. This is not surprising, as the Bianchi I cosmology is the free–particle analogue in quantum cosmology.

5.2. MC^2

In this subsection we apply our MC^2 quantization approach to the Taub cosmology and numerically solve the Schrödinger equation, coupled though it is to the expectation-valued Hamiltonian constraint equation.[12]

The Taub cosmology is an axisymmetric homogeneous cosmology parameterized by a scale factor $\Omega(t)$, and an anisotropy parameter $\beta(t)$. The line element may be expressed as

$$ds^2 = -dt^2 + a_\circ^2 e^{2\Omega} \left(\mathrm{e}^{2\beta}\right)_{ij} \sigma^i \sigma^j , \tag{71}$$

where $(\beta) = \mathrm{diag}(\beta, \beta, -2\beta)$, and the one forms σ^i are: $\sigma^1 = \cos\psi d\theta + \sin\psi d\phi$, $\sigma^2 = \sin\psi d\theta - \cos\psi d\phi$, and $\sigma^3 = d\psi + \cos\theta d\phi$. The choice for shift and lapse is $N = 1$, $N^i = 0$. The scalar 4-curvature is expressed in terms of Ω and β and yields the action,

$$I_c = \frac{3\pi a_\circ^3}{4} \int \left\{ (\dot\beta^2 - \dot\Omega^2) + \frac{1}{6}\,{}^{(3)}R \right\} \mathrm{e}^{3\Omega}\, dt. \tag{72}$$

Here

$${}^{(3)}R = \frac{\mathrm{e}^{-2\beta}}{2a_\circ^2 \mathrm{e}^{2\Omega}} (4 - \mathrm{e}^{-6\beta}) \tag{73}$$

represents the scalar 3-curvature.

We treat the scale factor $\Omega(t)$ as the many-fingered time parameter and the anisotropy $\beta(t)$ as the dynamic degree of freedom.[2,10] The momentum conjugate to β is obtained from from the Lagrangian L.

$$p_\beta = \frac{\partial L}{\partial \dot\beta} = \frac{3}{2}\pi a_\circ^3 \mathrm{e}^{3\Omega} \dot\beta = m\dot\beta \tag{74}$$

The dynamical Hamiltonian for this cosmology can be expressed in terms of this momentum and the Lagrangian,

$$H_{DYN} = p_\beta \dot\beta - L = \frac{1}{2m} p_\beta^2 + \underbrace{\frac{m}{2}\left(\dot\Omega^2 - \frac{1}{6}{}^{(3)}R\right)}_{V} . \tag{75}$$

In the classical theory H_{DYN} can be used to construct either the HJ equation or the two Hamilton equations; however, to complete the dynamics we must in addition impose the super-Hamiltonian constraint,

$$p_\beta^2 = m^2 \left(\dot\Omega^2 + \frac{1}{6}{}^{(3)}R\right) . \tag{76}$$

Using this dynamical Hamiltonian (75), the corresponding HJ equation and the standard quantization prescription we obtain the Schrödinger equation for the Taub cosmology.

$$-i\hbar\frac{\partial\Psi(\beta,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(\beta,t)}{\partial\beta^2}+V\Psi(\beta,t) \tag{77}$$

The scale factor Ω in V and m in this equation should be treated as an unknown function of time, t. To complete the quantization we impose, in addition, (76) as an expectation-value equation over β (where $<\bullet>_s=\int\Psi_s^*\bullet\Psi_s d\beta$ and Ψ_s is the solution of of (77) augmented by appropriate initial data, cf. (79)).

$$\dot{\Omega}^2=\Big(\frac{<p_\beta>_s}{m}\Big)^2+\frac{1}{6}<{}^{(3)}R>_s \tag{78}$$

This system of equations (77) and (78) provides us with a complete quantum-dynamic picture of the axisymmetric Taub model, and when augmented by the appropriate boundary conditions can be solved numerically. The equations (77) and (78) are coupled and can be integrated according to the prescription provided in the end of Sec. .

We display in Figs. 1-2 a solution of the these two coupled equations for an initially Gaussian wave packet.

$$\Psi_0=C\mathrm{e}^{i(\beta-\beta_\circ)p_\circ/\hbar}\mathrm{e}^{-(\beta-\beta_\circ)^2/\Delta\beta^2} \tag{79}$$

with $C^2=\sqrt{2/\pi}/\Delta\beta$, $p_\circ=-100$, $\hbar=1$, $\beta_\circ=0$, $\Delta\beta=0.1$ and $\Omega(t=0)=1$. The solution was obtained using a 2nd-order split operator unitary integrator which preserves the norm exactly (up to round off errors).[16] The width of the initial Gaussian wave packet (79) is determined by $\Delta\beta$ and is related to the parameter a (used for the same purpose in previous subsection) by the simple relation $\Delta\beta=2\hbar\sqrt{a}$.

We have introduced in this section an application of our MC^2 approach to the quantization of gravity (this approach need not be restricted to quantum cosmology). When the Taub cosmology was quantized our theory generated a quantum picture based on York's paradigm of time evolution. This MC^2 quantization appears to be free from the conceptual difficulties related with time usually associated with both the Dirac and ADM quantization procedures. Only the dynamical part, the underlying anisotropy (β) has been quantized. The Hamiltonian participating in the quantization ($\mathcal{H}_{DYN}$) is not a square root Hamiltonian. This absence of a square root Hamiltonian is a generic feature of our MC^2 quantization procedure.

We have demonstrated here using the Taub cosmology that the concept of time may very well be inextricably intertwined and woven to the initial conditions as well as to the quantum dynamics over the space of all conformal 3-geometries.

Both examples of this section illustrate one important technical detail. They show that the constraint equation is used in the description of time evolution for evaluating the time derivative of the embedding variable (scale factor). It is easy to see that this feature will remain unchanged in a generic setting. The constraint equations, with expectation values of the true dynamic variables, generate the time evolution of the embedding variables. Although the equations are of the same nature

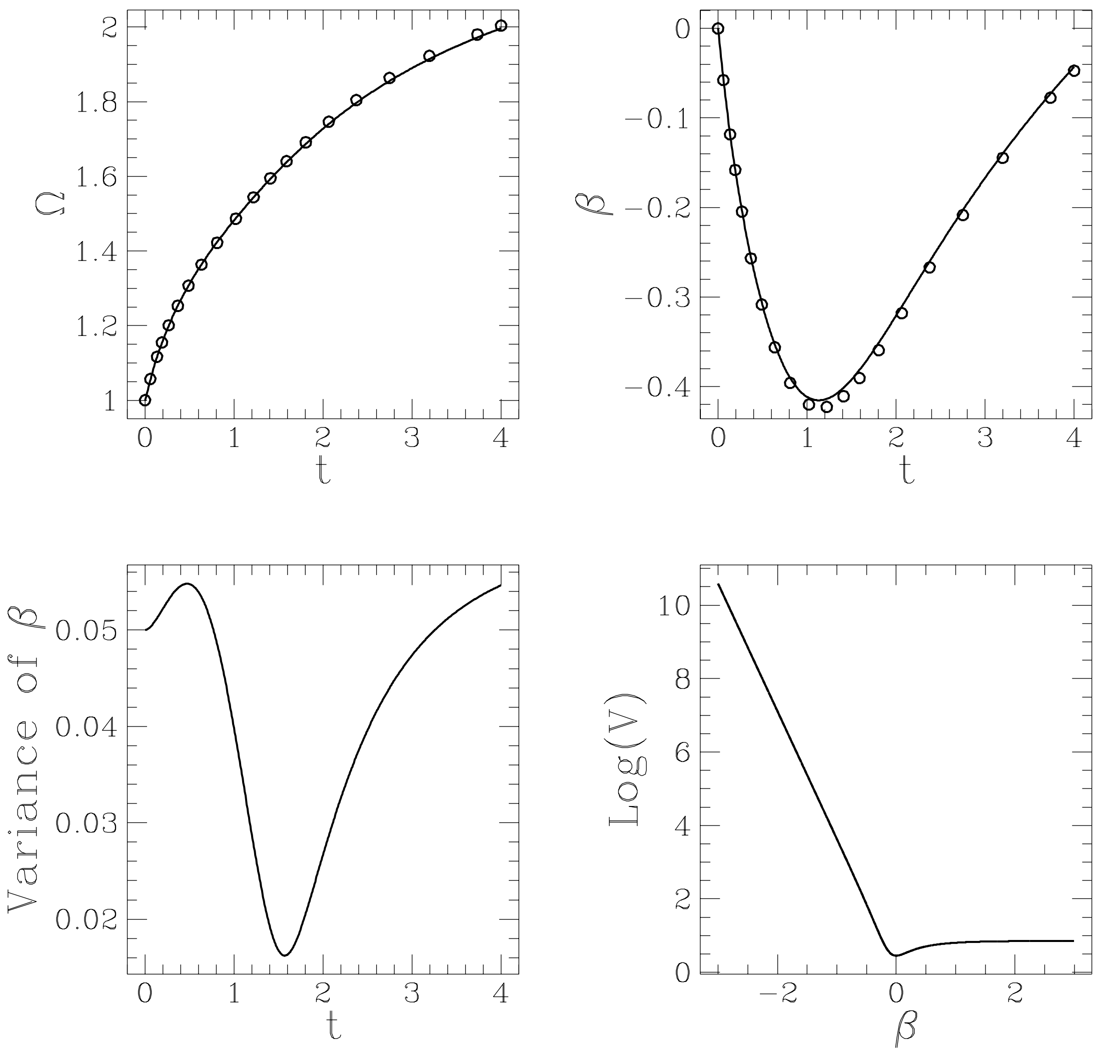

Fig. 1. *The quantum geometrodynamics of the Taub cosmology. The solid lines in each of the four graphs represent the quantum solution, while the small circular dots in the upper two graphs represent the classical trajectory. The scale factor's (Ω) dependence on time (t) is shown in the upper left, while the expectation of the anisotropy (< β >) is displayed in the upper right. The lower left shows the variance in* Ψ_s *throughout its evolution, and the lower right graph is a snapshot of the potential (V) at the end of the simulation.*

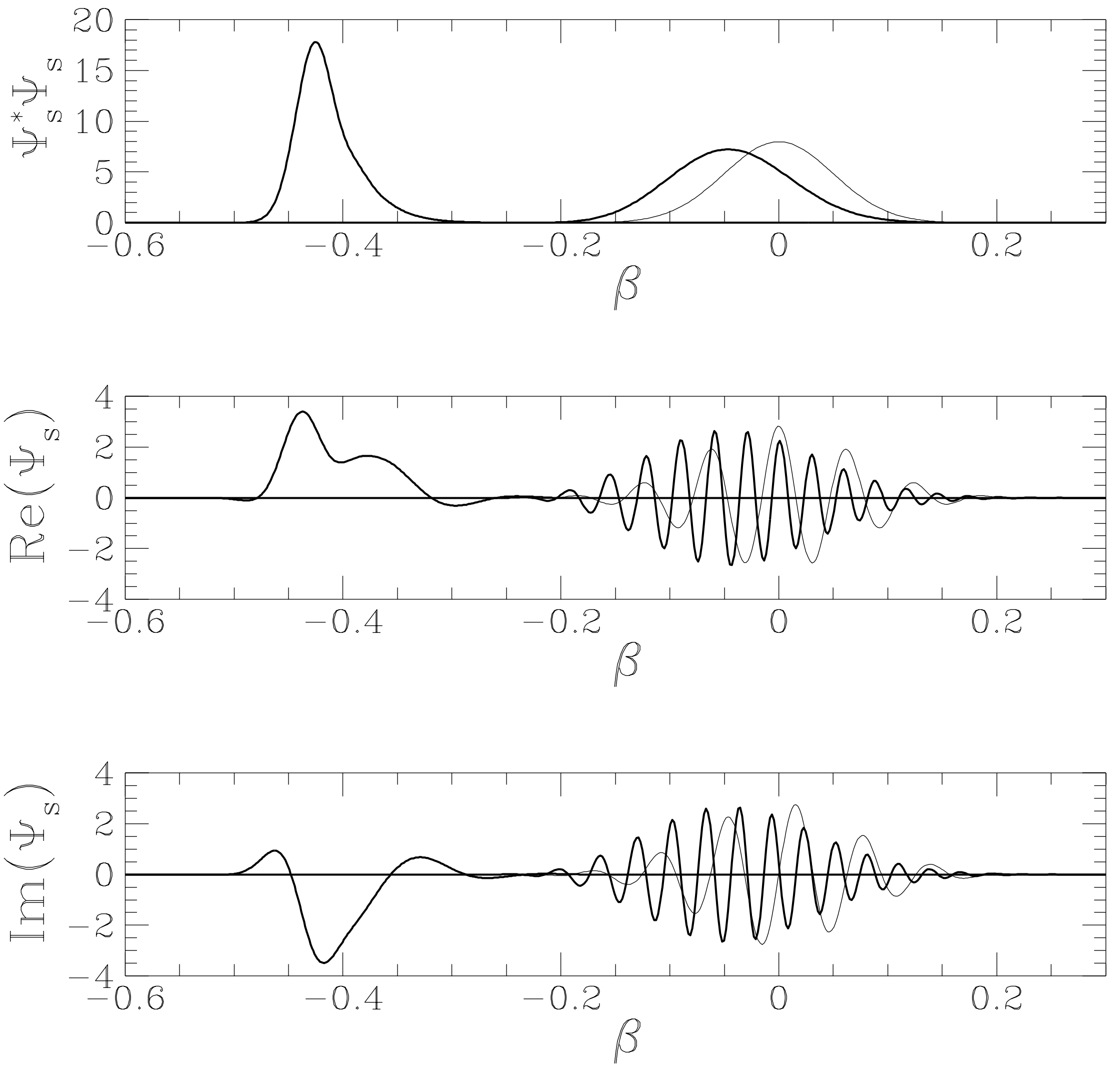

Fig. 2. *The Taub wave function. A snapshot of the wave function at three distinct times throughout the evolution is represented on each of the three graphs. The first row plots* $\Psi_s^*\Psi_s$*, while the second and third are the real and imaginary parts of* Ψ_s*, respectively. The right-most wave function (plotted in a slightly narrower line width) shows the initial* ($t = 0$) *Gaussian wave function. The left-most and higher-peaked curve represents the wave function near the bounce at* $t = 1$*, while the remaining curve displays the right-going wave function well after the bounce at* $t = 4$*. This reflected wave function is spreading in time and therefore appears, and will continue to appear, lower in amplitude than the initial wave function.*

as the ones used by York in solving the initial value problem, the flow of data is essentially in opposite direction. The resulting problem is significantly simpler than the initial value problem.

6. Discussion

In this paper we have introduced a novel quantization scheme which, while sharing many of the motivations of the standard approaches, differs in one significant way — a weakening of the constraints. This MC^2 procedure of gravity quantization, advanced in Sec. IV and illustrated by quantum cosmological examples in Sec. V, is developed entirely within the framework of York's paradigm of time evolution.

In particular, the split of the 3–metric into its true dynamic part (functional variables β) and four embedding variables (α, Ω) is introduced prior to the very first step of deriving the classical theory. This split amounts to a representation of a quantum gravitational system as two interacting subsystems. The first subsystem is described by two true dynamic variables and represents the principal object of geomerodynamics, while the second subsystem described by four embedding variables reflects our understanding of the nature of time. Only the true dynamic variables are considered as functional arguments of the action functional (to be varied in variational principles), while the embedding variables are considered as mere functional parameters.

The dynamic theory in this approach, in contrast to the one described in Sec. II, has all the features of the simplest dynamic theories. It utilizes the same Hamiltonian ($\mathcal{H}_{DYN}$) in the Hamilton equations and in the HJ equation. This Hamiltonian generates time translation and is related to the Lagrangian by the standard relation (16). When the variational principle on the true dynamic superspace is applied, it produces the system of four Hamilton equations (40), (41) (or the HJ equation (42)) which is incomplete and, consequently, incapable of providing any meaningful predictions. Any solution of this system contains, as parameters, four embedding variables, i. e. four unknown functions. When this system of equations is augmented by the four standard constraints of general relativity it provides one with both predictions and a meaningful time controlled by the shift and lapse (45). The resulting classical theory is equivalent to the standard Hamiltonian formulation of general relativity, except the system of equations (40), (41), (45) is not over determined. The classical theory remains fully constrained even if not manifestly.

The situation is quite different in the MC^2 quantum theory. Only the first subsystem (described by true dynamic variables) is quantized. The time evolution of this subsystem is described by the Schrödinger equation. This equation is a functional differential equation linear on the space of functionals of the true dynamic variables. The second subsystem (described by the embedding variables) is not quantized. The two subsystems are coupled. We consider the second subsystem, which we can referr to as "time fibers," as an external classical field within the quantum dynamics of the first subsystem. On the other hand, this external field is generated by the first subsystem according to the constraint equations.

During the quantum evolution the MC^2 procedure creates a spacetime geometry determined by the shift, lapse, embedding variables, as well as the expectations of true dynamic variables. Generally speaking, the momenta (including the expectations of the momenta conjugate to true dynamic variables) are not related to the geometry of the $t =$ const slice (expressed in terms of the extrinsic curvature) in a standard fashion, and the spacetime is not "classical," i. e. it cannot be considered as a product of a classical evolution of a 3–geometry as the standard equations of time evolution are not utilized in building this spacetime. However there is a "background" spacetime geometry which includes a "background" causality. The light cones of this causality are determined by quantum geometrodynamic evolution.

Within the MC^2 quantization procedure, any two different splits of the gravitational dynamics in two subsystems (dynamical and embedding) will ordinarily correspond to different physical systems. In other words, our quantum procedure yields a finer resolution of gravitational systems than the classical analogue. A similar phenomenon occurs naturally in any generic mechanical system. However, in mechanical systems one usually has some control over the coupling of the subsystems (an ability to switch on and off an interaction between them) which provides a "natural split". There appears to be no such mechanism for splitting common to all gravitational systems. In quantum cosmology a particular split of variables into its quantum and embedding parts essentially reflects our view concerning the nature of time in the Universe. Whether this view is correct can be decided only on the basis of observations. From this point of view the functional evolution problem and the multiple choice problem described in Sec. III are simply ill posed. It is our thesis that the origin of such problems within the standard ADM paradigm is the transition of the classical slicing theorem to the quantum. This slicing theorem requires the existence of a unique spacetime geometry — a geometry which has no foundation in quantum gravity. The MC^2 quantization procedure discards the slicing theorem and recovers it only on the classical level. As a result the functional evolution and the multiple choice problems of time are circumvented. Furthermore, there is no Hilbert space problem within the MC^2 quantization approach because the basic dynamic equation is a Schrödinger equation. There is no spectral analysis problem because the Hamiltonian of the theory is not a square root Hamiltonian.

The MC^2 quantization procedure appears to be quite natural for the theory in which information about time is carried by functions that cannot be decoupled from true dynamic variables. The only relatively arbitrary step in this procedure was the manner in which we imposed the constraints on the expectations of the true dynamic variables. Our particular approach was motivated by the correspondence principle so as to recover standard classical geometrodynamics in the classical limit. It is possible that there are other prescriptions. However, any other approach should be "loose" enough to avoid a situation in which a measurement of the classical variables would provide too much information about quantum state determined by quantum variables.

One feature of time has already emerged from our proposal. By elevating the true dynamic superspace as the arena for quantum dynamics we were led naturally to weaken general covariance at the quantum level (not classical). Thus in our theory the concept of time appears to be inextricably intertwined with the initial conditions, as well as with the quantum dynamics over the true dynamic superspace (i.e. the space of all conformal 3-geometries in York's particular representation).

It is in this sense that we are addressing the issue of time in canonical quantum gravity. In so doing we tackle one of the most fundamental questions in this field: "What is the domain space for the quantum-mechanical wave functional for quantum gravity?"[9] While it appears that we have circumvented the "problems of time", we are faced with a more formidable interpretational dilemma: What does our wave functional mean, and what is in principle measurable? To answer this question in its entirety is beyond the scope of this paper. Nonetheless, we are animated by the words of Wheeler[18] on Bohr and Rosenfeld "The theory itself, however, here as always, defines in and by itself, in its own natural way, the procedures-in-principle for measuring all those quantities that are in principle measurable."[19] Then perhaps, by circumventing the problem of time, while at the same time preserving, as much as possible, the original motivation of quantum gravity, we have for the first time a mathematically consistent theory that may 'in principle' reveal the true quantum dynamics of spacetime.

We are left with a consistent mathematical framework that sidesteps the usual problems of time, but introduces formidable interpretational challenges regarding the enriched spectrum of allowable quantum-mechanical dynamics. This we view as a worthwhile challenge — a problem "soluble in principle" in light of Bohr and Rosenfeld.[19]

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