Kinetic Theory of Granular Gases

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talk, papers available: http://cnls.lanl.gov/~ebn

Los Alamos NATIONAL LABORATORY

Plan

1. **Basics: collision rules, collision rates**
2. **The Boltzmann equation**
3. **Extreme statistics and the linearized Boltzmann equation**
4. **Forced steady states**
5. **Freely cooling states**
6. **Stationary states and energy cascades**
7. **Hybrid solutions**

Experiments

- **Friction**

  D Blair, A Kudrolli 01
- **Rotation**

  K Feitosa, N Menon 04
- **Driving strength**

  W Losert, J Gollub 98
- **Dimensionality**

  J Urbach & Olafsen 98
- **Boundary**

  J van Zon, H Swinney 04
- **Fluid drag**

  K Kohlstedt, I Aronson, EB 05
- **Long range interactions**

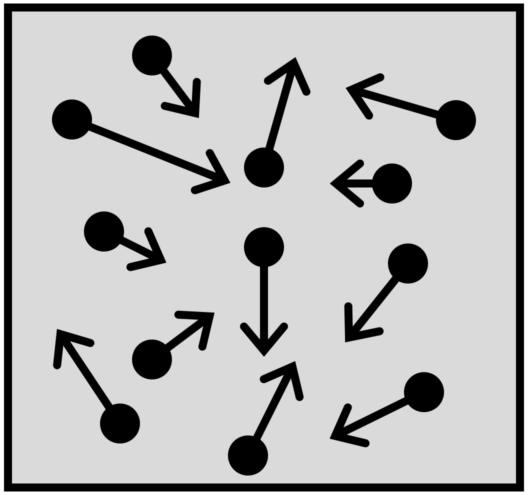
  D Blair, A Kudrolli 01; W Losert 02

  K Kohlstedt, J Olafsen, EB 05
- **Substrate**

  G Baxter, J Olafsen 04

Deviations from equilibrium distribution

Driven Granular gas



- Vigorous driving
- Spatially uniform system
- Particles undergo binary collisions
- Velocities change due to
    1. Collisions: lose energy
    2. Forcing: gain energy
- What is the typical velocity (granular “temperature”)?

$$T = \langle v^2 \rangle$$

- What is the velocity distribution?

$$f(v)$$

Inelastic Collisions (1D)

- **Relative velocity reduced by** $0 < r < 1$

$$v_1 - v_2 = -r(u_1 - u_2)$$

- **Momentum is conserved**

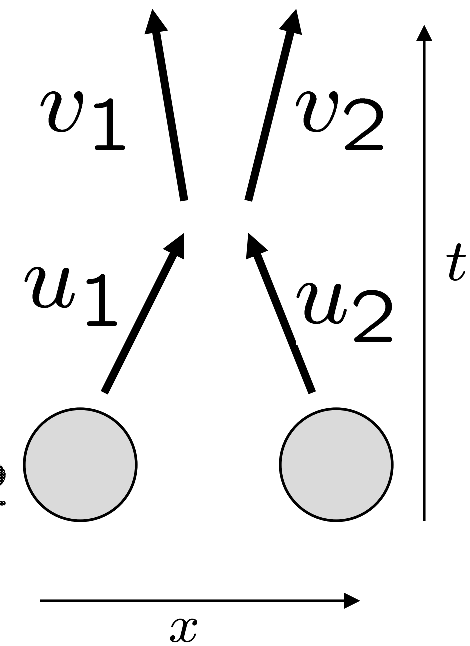
$$v_1 + v_2 = u_1 + u_2$$

- **Energy is dissipated**

$$\Delta E = \frac{1 - r^2}{4}(u_1 - u_2)^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \text{max}) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$



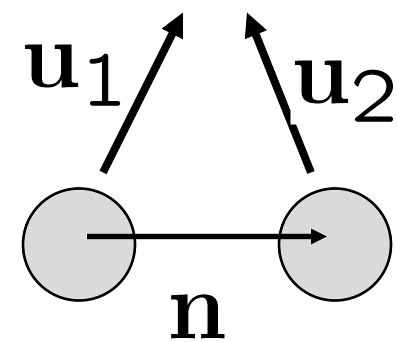
Inelastic Collisions (any D)

- **Normal relative velocity reduced by** $0 < r < 1$

$$(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n} = -r(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}$$

- **Momentum conservation**

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$$



- **Energy loss**

$$\Delta E = \frac{1 - r^2}{4}[(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}]^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \text{max}) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$

Non-Maxwellian velocity distributions

JC Maxwell, Phil Trans Roy. Soc. **157** 49 (1867)

1. Velocity distribution is isotropic

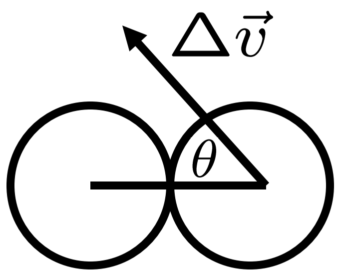
$$f(v_x, v_y, v_z) = f(|v|)$$

2. No correlations between velocity components

$$f(v_x, v_y, v_z) \neq f(v_x) f(v_y) f(v_z)$$

Only possibility is Maxwellian

$$f(v_x, v_y, v_z) \neq C \exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2T}\right)$$



Granular gases: collisions create correlations

The Boltzmann equation (1D)

- **Collision rule (linear)** $r = 1 - 2p, \quad p + q = 1$

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Boltzmann equation (nonlinear and nonlocal)**

$$\frac{\partial f(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

- **Collision rate related to interaction potential**

$$U(r) \sim r^{-\gamma} \qquad \lambda = 1 - 2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

Theory: non-linear, non-local, dissipative

The collision rate

- **Collision rate**

$$K(\mathbf{u}_1, \mathbf{u}_2) = |(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}|^{\lambda}$$

- **Collision rate related to interaction potential**

$$U(r) \sim r^{-\gamma} \qquad \lambda = 1 - 2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

- **Balance kinetic and potential energy**

$$v^2 \sim r^{-\gamma} \quad \Rightarrow \quad r \sim v^{-2/\gamma}$$

- **Collisional cross-section**

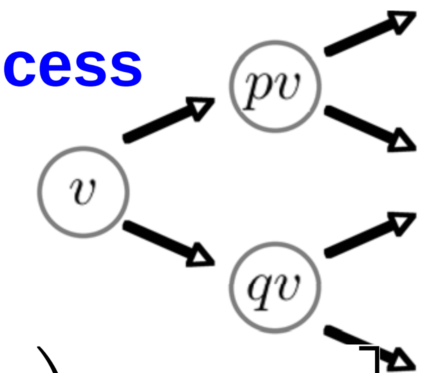
$$\sigma \sim v r^{d-1} \quad \Rightarrow \quad \sigma \sim v^{1-\frac{2}{\gamma}(d-1)}$$

Extreme Statistics (1D)

- **Collision rule: arbitrary velocities**

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Large velocities: linear but nonlocal process**



$$v \xrightarrow{v^\lambda} (pv, qv)$$

- **High-energies: linear equation**

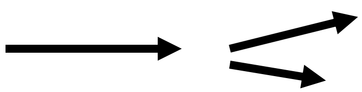
$$\frac{\partial f(v)}{\partial t} = v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

gain gain loss

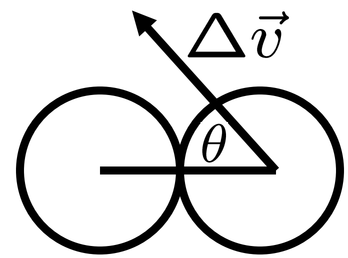
Linear, nonlocal evolution equation

Extreme Statistics (any D)

- **Collision process: large velocities**



$$v \xrightarrow{(v\cos\theta)^\lambda} (\alpha v, \beta v)$$



- **Stretching parameters related to impact angle**

$$\alpha = (1-p)\cos\theta \qquad \beta = \sqrt{1-(1-p^2)\cos^2\theta}$$

- **Energy decreases, velocity magnitude increases**

$$\alpha^2 + \beta^2 \le 1 \qquad \alpha + \beta \ge 1$$

- **Linear equation**

$$\frac{\partial f(v)}{\partial t} = \left\langle (v\cos\theta)^\lambda \left[\frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) - f(v)\right]\right\rangle$$

Forced steady states: overpopulated tails

T van Noije, M Ernst 97

- **Energy injection: thermal forcing (at all scales)**

$$dv/dt = \eta$$

- **Energy dissipation: inelastic collision**

$$v \longrightarrow (pv, qv)$$

- **Steady state equation**

$$0 = D\frac{d^2 f(v)}{d^2 v} + v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

- **Stretched exponentials**

$$f(v) \sim \exp\left(-v^{1+\lambda/2}\right)$$

Nonequilibrium velocity distributions

A **Mechanically vibrated beads**

F Rouyer & N Menon 00

B **Electrostatically driven powders**

I Aronson & J Olafsen 05

◆ **Gaussian core**

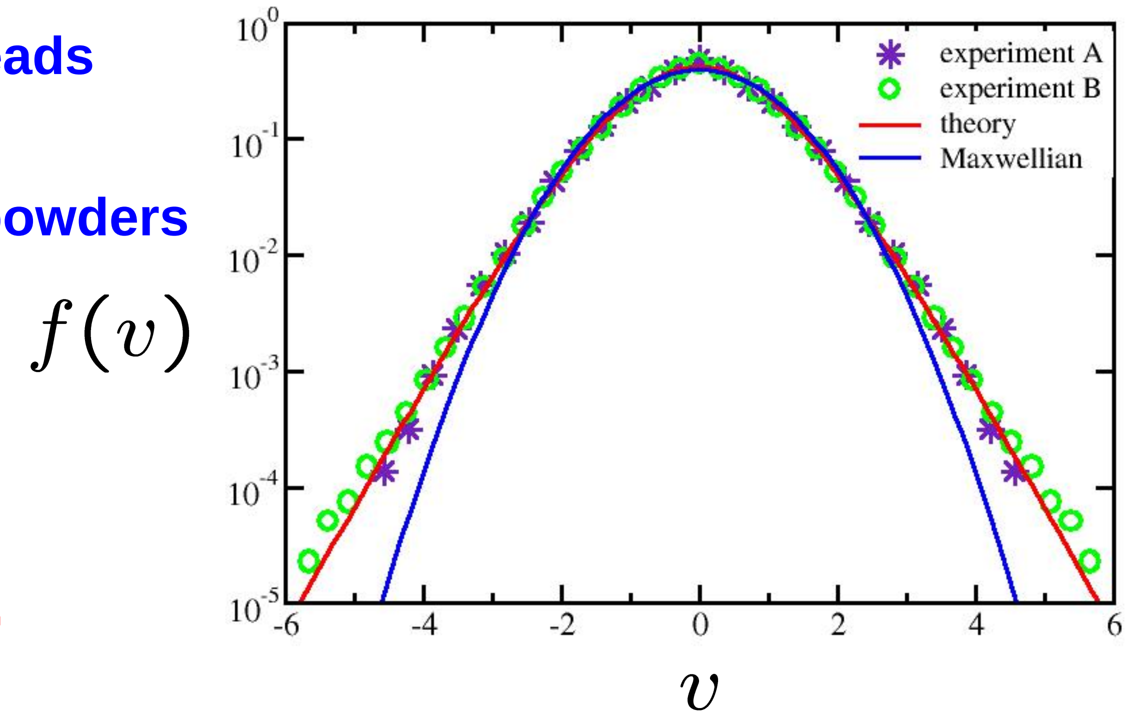
◆ **Overpopulated tail**

$$f(v) \sim \exp\left(-|v|^{\delta}\right)$$

$$1 \le \delta \le 3/2$$

◆ **Kurtosis**

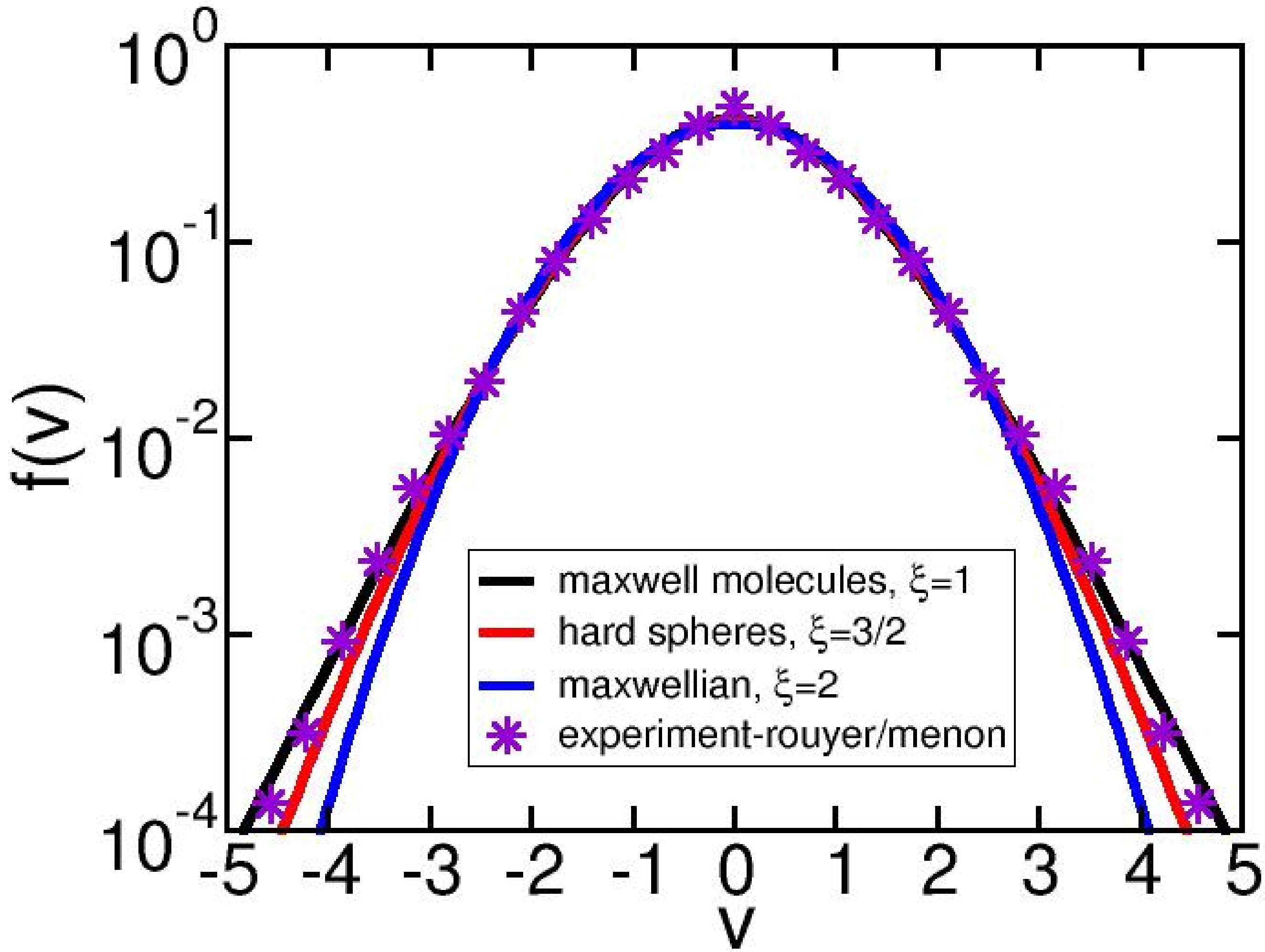
$$\kappa = \begin{cases} 3.55 & \text{theory} \\ 3.6 & \text{experiment} \end{cases}$$



Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls

Comparing kinetic theories



Freely cooling states: temperature decay

Haff, JFM 1982

- **Energy loss** $\Delta T \sim (\Delta v)^2$
- **Collision rate** $\Delta t \sim 1/(\Delta v)^\lambda$
- **Energy balance equation**

$$\frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}$$

- **Temperature decays, system comes to rest**

$$T \sim t^{-2/\lambda} \quad \Rightarrow \quad f(v) \to \delta(v)$$

System comes to rest

Temperature decay: dimensional analysis

- **Collision rate**

$$K \sim (\Delta v)^{\lambda} \sim v^{\lambda}$$

- **Collision rate inversely proportional to time**

$$K \sim t^{-1} \qquad \Rightarrow \qquad v \sim t^{-1/\lambda}$$

Freely cooling states: similarity solutions

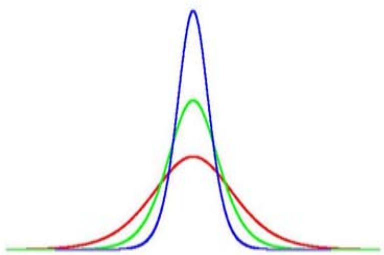
Esipov, Poeschel 97

- **Linearized equation**

$$\frac{\partial f(v)}{\partial t} = v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

- **Similarity solution**

$$f(v) \to t^{1/\lambda}\Phi(vt^{1/\lambda})$$



- **Steady state equation**

$$\frac{1}{\lambda}[\Phi(z)+z\frac{d}{dz}\Phi(z)] = z^\lambda \left[\frac{1}{p^{1+\lambda}}\Phi\left(\frac{z}{p}\right) + \frac{1}{q^{1+\lambda}}\Phi\left(\frac{z}{q}\right) - \Phi(z)\right]$$

- **Stretched exponentials (overpopulation)**

$$\Phi(z) \sim \exp\left(-z^\lambda\right)$$

An exact solution

- **One-dimensional Maxwell molecules**
- **Fourier transform obeys a closed equation** $F(k) = \int dv\, e^{ikv} f(v)$

$$F(k) = F(pk)F(qk)$$

- **Exponential solution**

$$F(k) = \exp(-v_0|k|)$$

- **Lorentzian velocity distribution**

$$f(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2}$$

Nontrivial stationary states do exist!

Are there nontrivial steady states?

- **Stationary Boltzmann equation**

$$0=\iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

<u>Naive answer: NO!</u>

- **According to the energy balance equation**

$$\frac{dT}{dt} = -\Gamma$$

- **Dissipation rate is positive**

$$\Gamma > 0$$

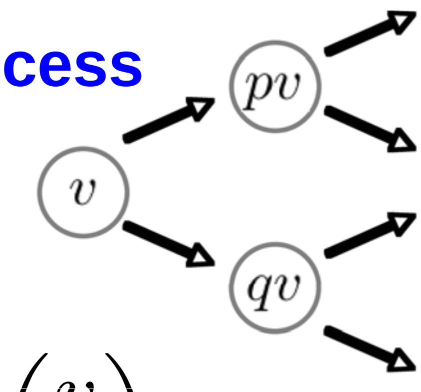
Cascade Dynamics (1D)

- **Collision rule: arbitrary velocities**

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Large velocities: linear but nonlocal process**

$$v \to (pv, qv)$$



- **High-energies: linear equation**

$$f(v) = \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)$$

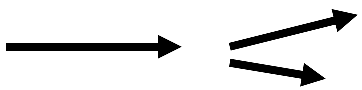
loss gain gain

- **Power-law tail**

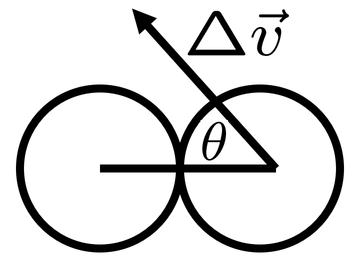
$$f(v) \sim v^{-2-\lambda}$$

Cascade Dynamics (any D)

- **Collision process: large velocities**



$$v \to (\alpha v, \beta v)$$



- **Stretching parameters related to impact angle**

$$\alpha = (1-p)\cos\theta \qquad \beta = \sqrt{1-(1-p^2)\cos^2\theta}$$

- **Energy decreases, velocity magnitude increases**

$$\alpha^2 + \beta^2 \le 1 \qquad \alpha + \beta \ge 1$$

- **Steady state equation**

$$f(v) = \left\langle \frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) \right\rangle$$

Power-laws are generic

- **Velocity distributions always has power-law tail**

$$f(v) \sim v^{-\sigma}$$

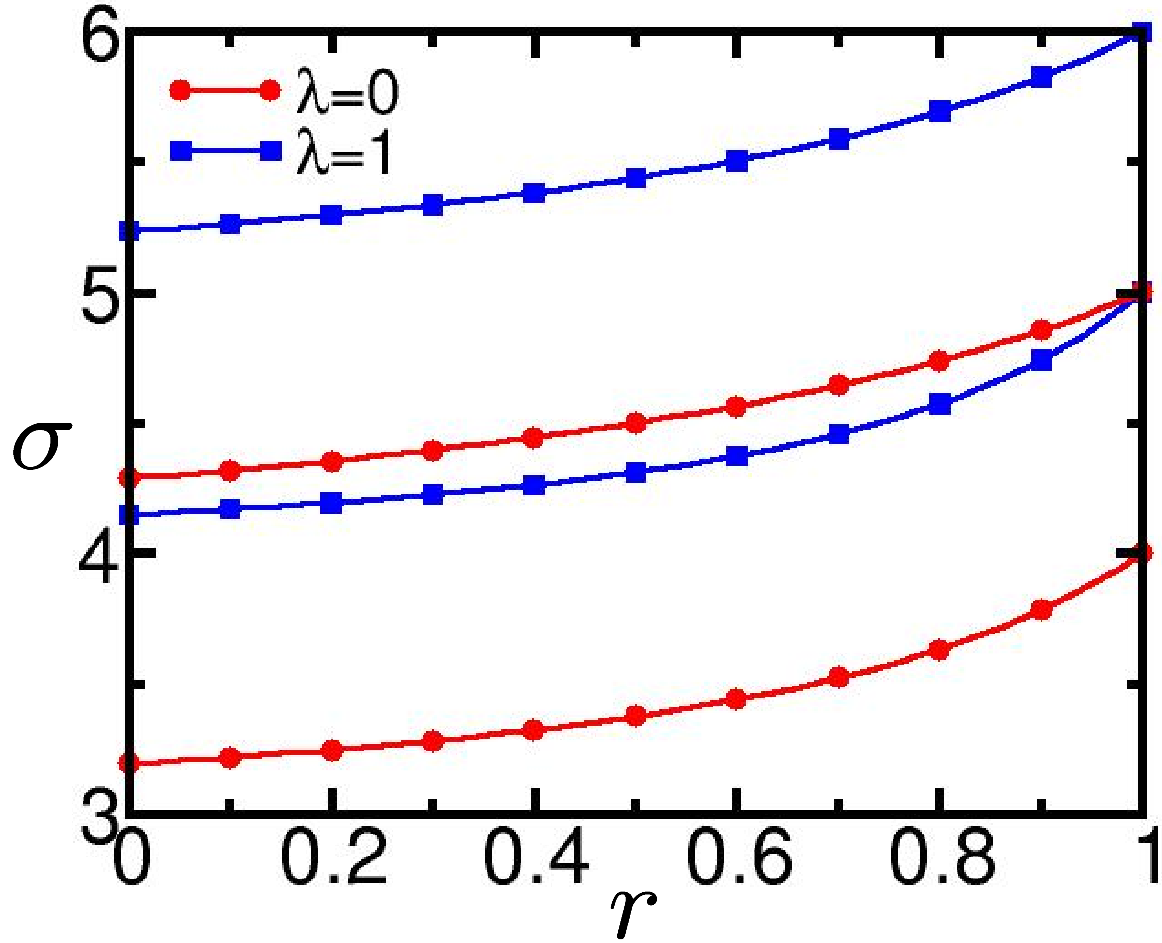
- **Exponent varies with parameters**

$$\frac{1 - {}_2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}$$

- **Tight bounds** $1 \leq \sigma - d - \lambda \leq 2$
- **Elastic limit is singular** $\sigma \to d+2+\lambda$

Dissipation rate always divergent
Energy finite or infinite

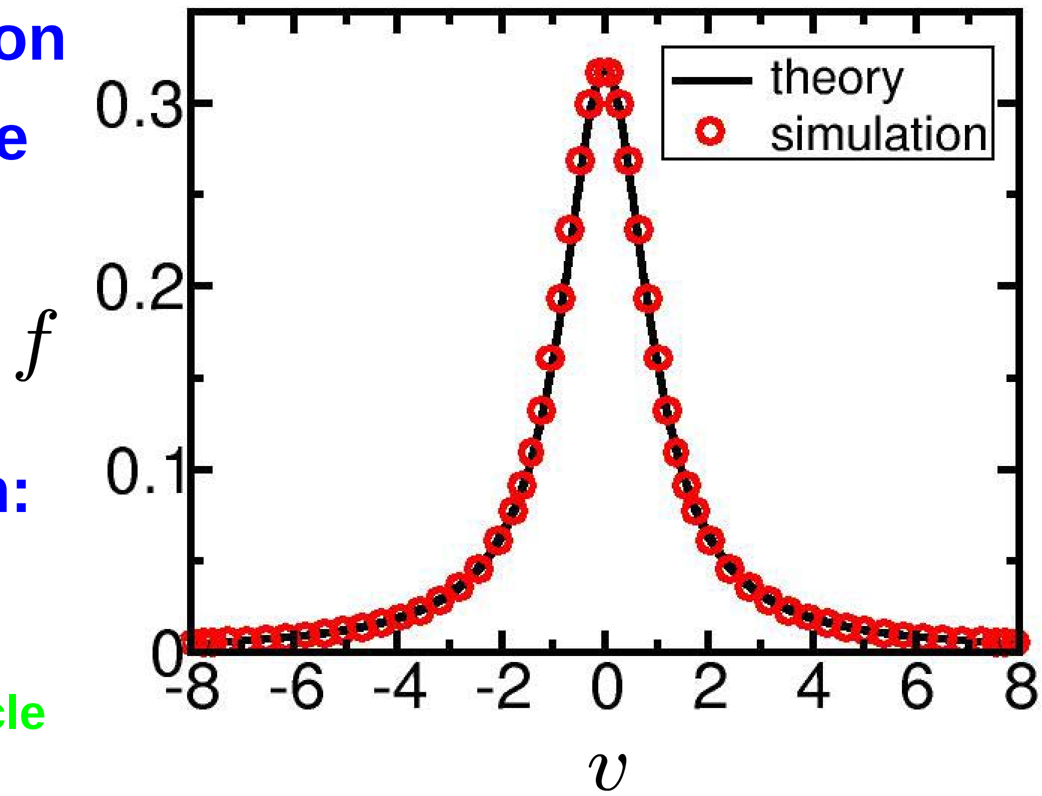
The characteristic exponent σ



σ varies with spatial dimension, collision rules

Monte Carlo Simulations

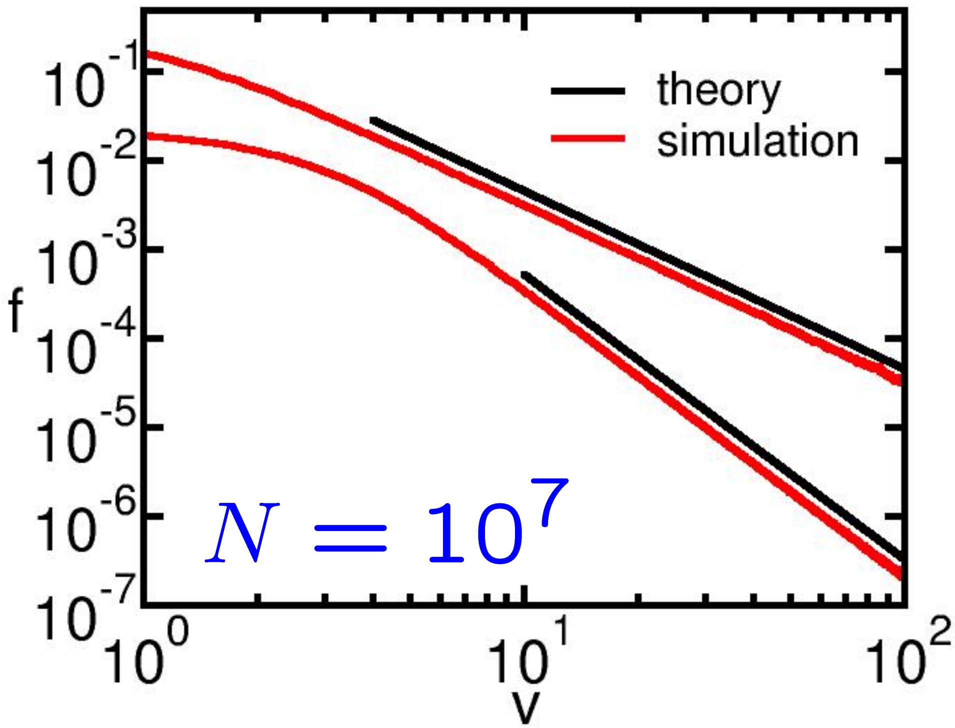
- Compact initial distribution
- Inject energy at very large velocity scales only
- Maintain constant total energy
- "Lottery" implementation:
  - Keep track of total energy dissipated, E_T
  - With small rate, boost a particle by E_T



Excellent agreement between theory and simulation

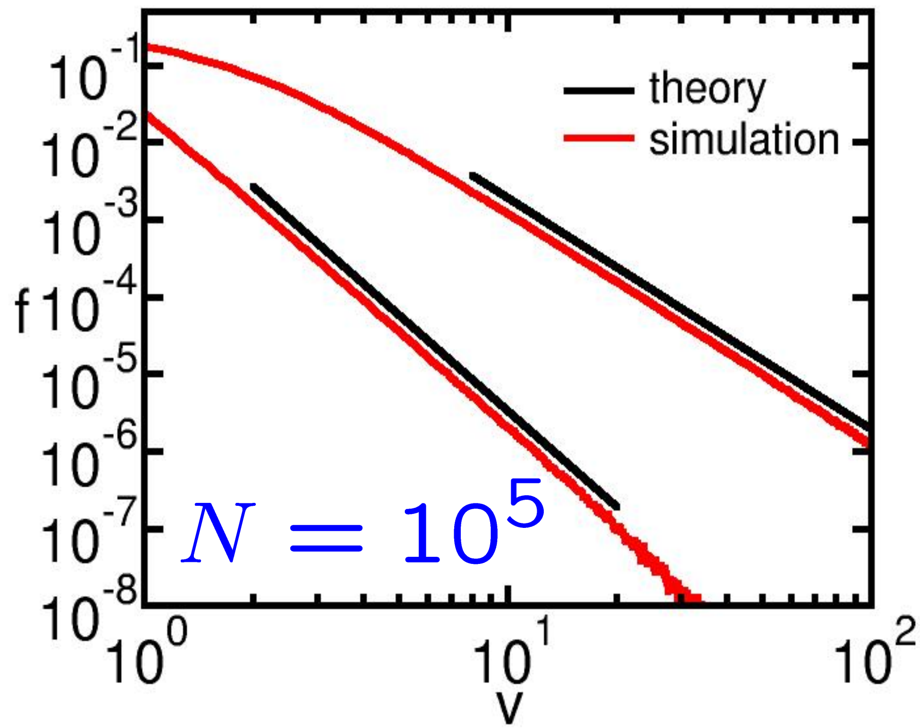
Further confirmation

Maxwell molecules (1D, 2D)



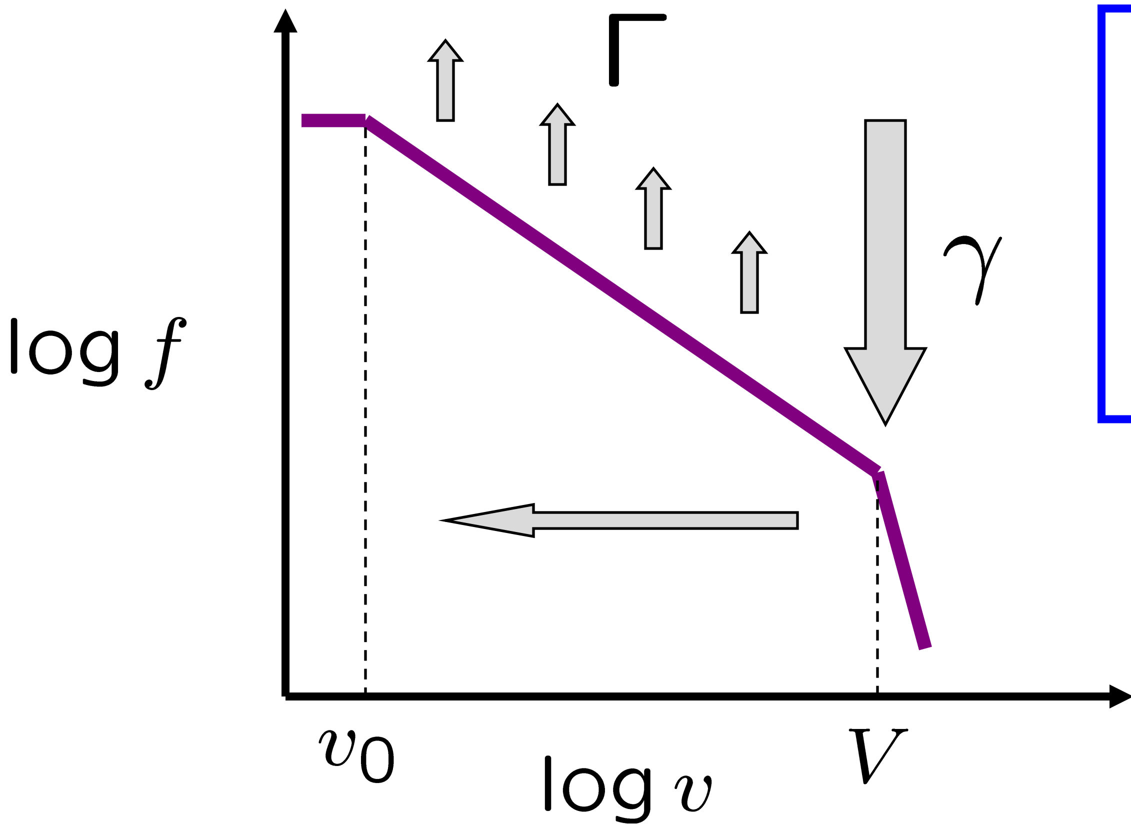
d	theory	simulation
1	2	1.995
2	3.19520	3.19

Hard spheres (1D, 2D)



d	theory	simulation
1	3	2.994
2	4.14922	4.15

Injection, cascade, dissipation



Experimental realization?
Energetic particle "shot" into static medium

Energy balance

$$\Gamma \sim \gamma V^2$$

- ❖**Energy is injected at large velocity scales**
- ❖**Energy cascades from large velocities to small velocities**
- ❖**Energy dissipated at small velocity scales**

Energy balance

- **Energy injection rate** γ
- **Energy injection scale** V
- **Typical velocity scale** v_0
- **Balance between energy injection and dissipation**

$$\gamma \sim V^{\lambda}(V/v_0)^{d-\sigma}$$

- **For "lottery" injection: injection scale diverges with injection rate**

$$V \sim \begin{cases} \gamma^{-1/(2-\lambda)} & \sigma < d+2 \\ \gamma^{-1/(\sigma-d-\lambda)} & \sigma > d+2 \end{cases}$$

with Ben Machta (Brown)

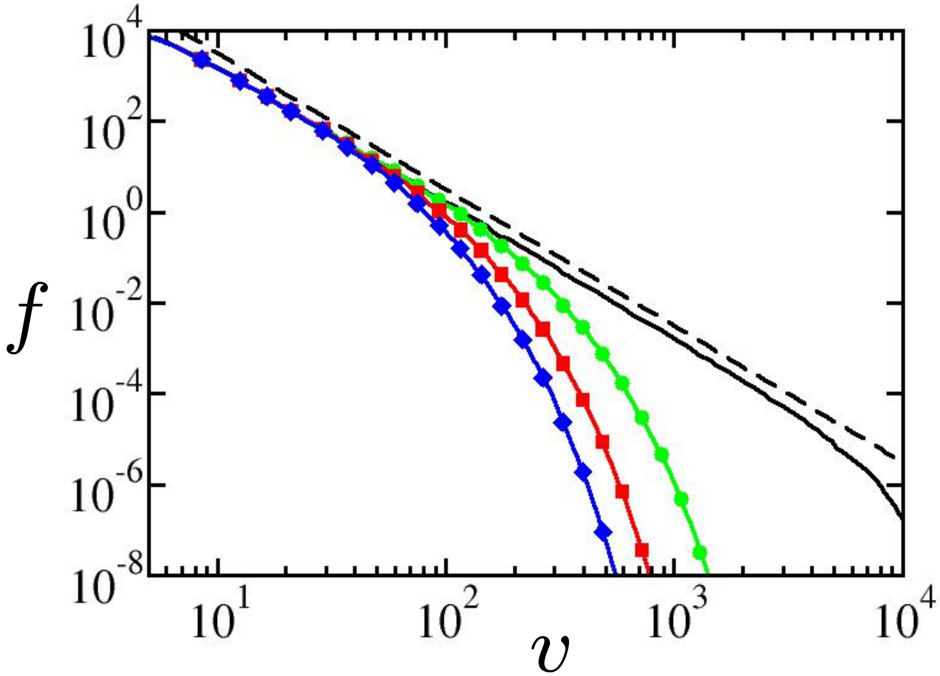
Self-similar collapse

- **Self-similar distribution**

$$f(v,t) \simeq v^{-\sigma}\Phi\left(\frac{v}{V(t)}\right)$$

- **Cutoff velocity decays**

$$V(t) \sim t^{-1/\lambda}$$

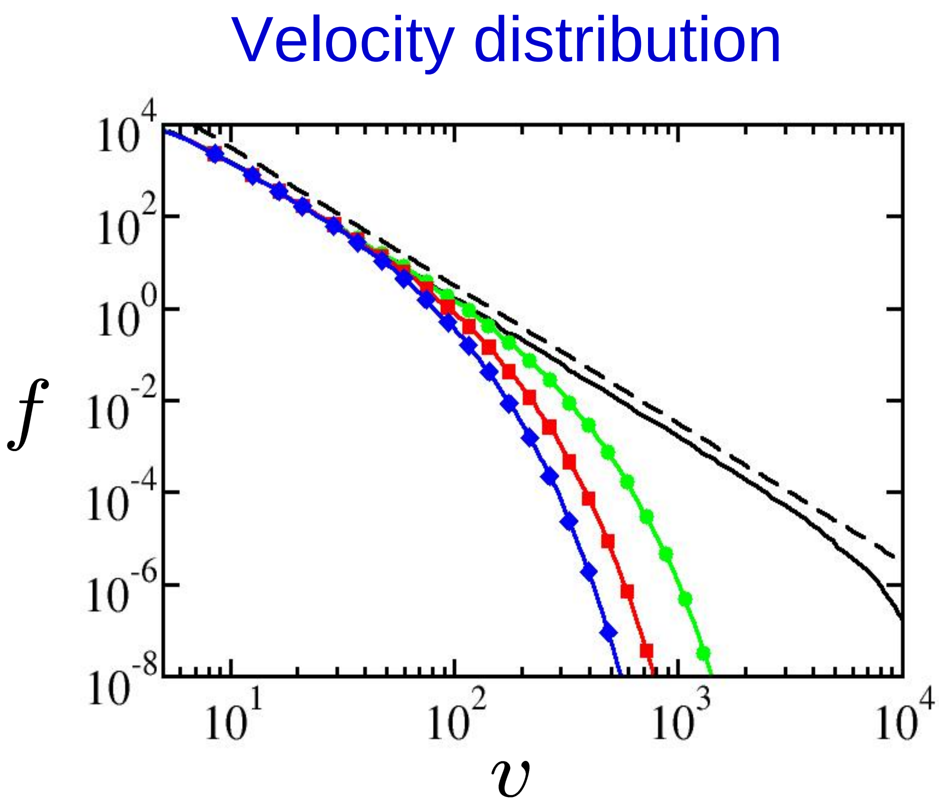


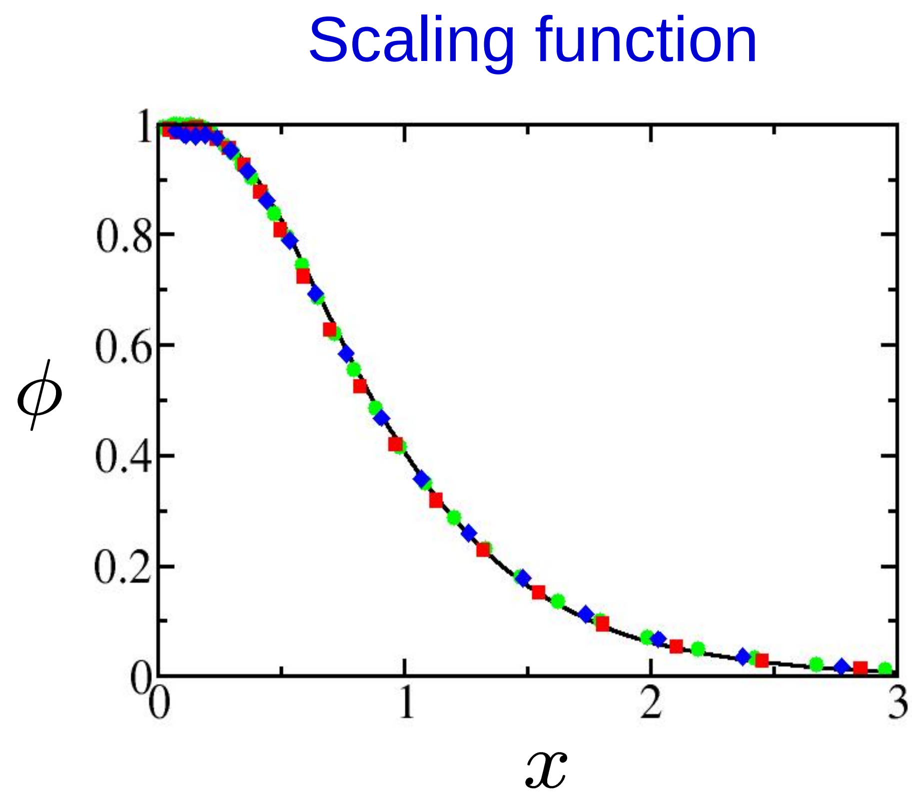
- **Scaling function**

$$\Phi(x) = \sum_{n=1}^{\infty} A_n \exp\left[-(2^n x)^{\lambda}\right] \qquad A_n = \prod_{\substack{k=1\\k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

Hybrid between steady-state and time dependent state

Numerical confirmation





A third family of solutions exists

Conclusions

- New class of nonequilibrium stationary states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism

Outlook

- **Spatially extended systems**
- **Spatial structures**
- **Polydisperses granular media**
- **Experimental realization**

E. Ben-Naim and J. Machta, Phys. Rev. Lett. **94**, 138001 (2005)

E. Ben-Naim, B. Machta, and J. Machta, cond-mat/0504187

Deviation from Maxwell-Boltzmann

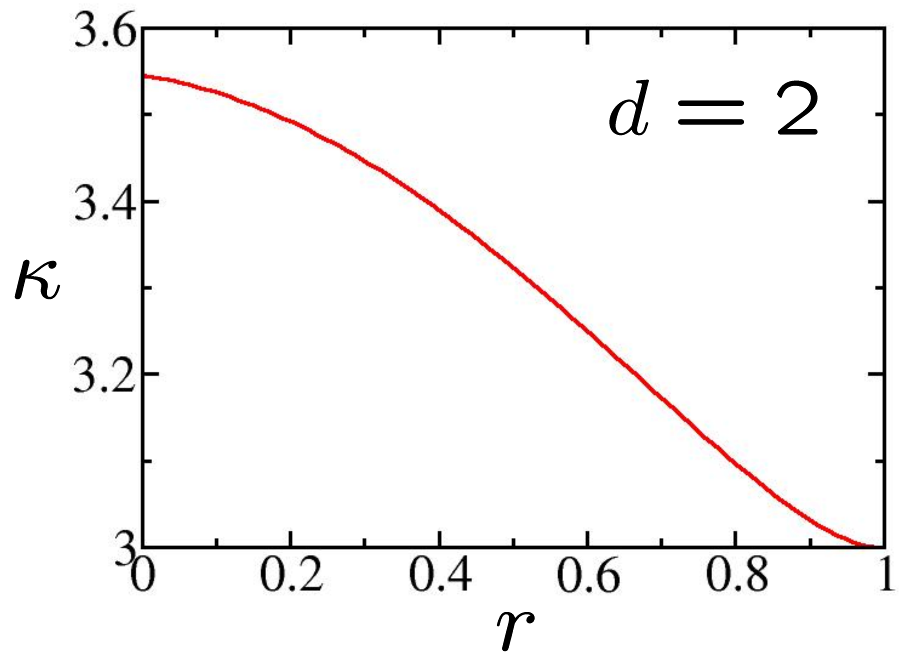
- **Kurtosis κ**

$$\kappa = \langle v^4 \rangle / \langle v^2 \rangle^2$$

$$\kappa = 3 + \frac{18(1-r)^2(1+r)}{33 - 25r + 3r^2 - 3r^3}$$

- **Restitution coefficient r**

$$\Delta E \propto \left(1 - r^2\right) (\Delta v)^2$$



1. **Velocity distribution independent of driving strength**
2. **Stronger dissipation yields stronger deviation**

Exact solution of Maxwell's kinetic theory: thermal forcing balances dissipation

EB, Krapivsky 02