WATER RESOURCES RESEARCH, VOL. 30, NO. 2, PAGES 499-507, FEBRUARY 1994

An Eulerian-Lagrangian approach with an adaptively corrected method of characteristics to simulate variably saturated water flow

K. Huang, R. Zhang,[1] and M. T. van Genuchten

U.S. Salinity Laboratory, Agricultural Research Service, USDA, Riverside, California

Abstract. A relatively simple method of characteristics is developed to simulate one-dimensional variably saturated water flow. The method uses the Eulerian-Lagrangian approach to separate the governing flow equation into "convection" and "diffusion" parts, which are solved with the method of characteristics and the conventional finite element method, respectively. The method of characteristics combines a single-step reverse particle tracking technique with a correction strategy to ensure accurate mass balances. The correction process is implemented by weighing the calculated convective contribution to the pressure head at each node with the pressure head values of two upstream nodes, using an adaptive weighing factor A. The value of A is automatically adjusted by considering the global mass balance at each time step. Numerical experiments for ponded infiltration are presented to illustrate the scheme's performance for situations involving highly nonlinear soil hydraulic properties and extremely dry initial conditions. Results indicate that the proposed method is mass-conservative, virtually oscillation-free, and computationally quite efficient. The method is especially effective for simulating highly nonlinear flow scenarios for which traditional finite difference and finite element numerical methods often fail to converge.

Introduction

Water flow in variably saturated media is an important topic in several branches of hydrology, soil science, and agricultural engineering dealing with subsurface flow and chemical transport processes. Many authors have used a variety of analytical techniques to solve the governing equations for infiltration and water flow in soils [e.g., ***Philip, 1957; Parlange, 1972; Broadbridge and White, 1988; Warrick et al., 1991].*** Analytical solutions provide useful tools for studying relatively simple unsaturated flow problems such as infiltration in homogeneous soil profiles. However, they are usually inappropriate for more complicated flow situations because of their reliance on relatively simple initial and boundary conditions and the need to adopt simplified functions for the unsaturated soil hydraulic properties. Such situations generally require the flexibility of a numerical approach.

Accurate implementation of numerical techniques often encounters difficulties when nonlinear variably saturated flow problems are simulated. Numerical problems are usually manifested by a lack of convergence and/or the presence of undesired oscillations in computed results when sharp moisture fronts are present. Such situations arise, for example, when calculating ponded infiltration in initially very dry soils, particularly when the soil hydraulic properties are highly nonlinear. Several methods have been proposed in efforts to reduce the computational effort and/or provide more stable numerical solutions. For example, ***Abriola [1986]*** used hierarchical basis functions to increase the order, and hence accuracy, of the numerical scheme in elements near the wetting front. ***Dane and Mathis [1981]*** presented an adaptive finite difference scheme that allows a fine grid system to move with the wetting front. More recently, ***Gottardi and Venutelli [1992]*** developed a relatively sophisticated moving finite element model which uses fewer nodes as compared to fixed-grid methods, and hence can save CPU time when a relatively large flow domain is involved. However, initial results indicate that this method may yield larger mass balance errors than the conventional finite difference and finite element methods. ***Gamliel and Abriola [1992]*** proposed similar schemes which refine the numerical mesh in areas where the soil water pressure head changes rapidly but expand the mesh in less transient regions.

Numerical simulations based on the Richards' equation for variably saturated water flow generally use the pressure head as the dependent variable. Many studies have observed mass balance problems using this approach, presumably because of nonlinearities in the soil water capacity term. ***Milly [1985]*** proposed a mass-conservative numerical scheme which greatly reduces the mass balance errors. He evaluated the storage coefficient as an average value over each element during each time step. ***Ross*** [1990] concluded that relatively large time steps are possible when the mass-conservative mixed-form of the Richards' equation is used. Relatively large spatial increments were also possible when a hyperbolic sine transform was implemented. ***Hills et al. [1989]*** described a water content based algorithm, which appears to be superior to the traditional pressure-based formulation for simulating unsaturated flow in layered soils. Their scheme, however, could not be applied to conditions

[1]Now at Department of Plant, Soil, and Insect Sciences, University of Wyoming, Laramie.

Paper number 93 WR02881.

also involving saturated flow (i.e., positive pressure heads). ***Celia and Pinder [1986],*** and more recently ***Celia et al.*** [1990], proposed schemes which implement more accurate and mass-conservative solutions of the mixed-form of the Richards' equation. In a different approach, ***Hansen et al. [1992]*** combined the adaptive grid refinement method with an operator splitting technique to solve the two-phase immiscible flow equations.

Most of the studies above used relatively standard finite element (FE) or finite difference (FD) methods to simulate the variably saturated flow process. While the studies have led to improvements, conventional FE and FD methods always suffer to some degree from inherent numerical difficulties, such as numerical oscillations, excessive numerical dispersion, and mass balance errors. Besides the presence of highly nonlinear hydraulic properties, numerical problems are often also caused by the hyperbolic nature of the governing equation when the gravitational term in the Richards' equation is relatively important ***[Sorek and Braester, 1986; Huang, 1988].*** Using this last attribute, ***Sorek and Braester [1986]*** extended the adaptive Eulerian-Lagrangian approach, initially introduced by ***Neuman [1984]*** for solving convection-dispersion transport problems, to unsaturated flow problems. While their method appears to improve convergence, no successful simulation example was reported in the paper. Moreover, their method requires a relatively sophisticated scheme to track the cluster of particles in a nonlinear flow field. After similarly separating the governing flow equation into convection and diffusion parts, ***Huang [1988]*** used a different approach by implementing a single-step reverse particle tracking (SRPT) technique to solve the convection problem. Preliminary results indicated several advantages of this approach, such as increased computational efficiency and fewer numerical oscillations. However, the scheme still exhibited relatively large mass balance errors, as well as some minor numerical oscillations, especially for highly nonlinear soil hydraulic properties.

The purpose of this paper is to extend and generalize the Eulerian-Lagrangian solution scheme of ***Huang [1988].*** A numerically robust and computationally efficient method will be presented which combines the modified method of characteristics, SRPT, with the conventional finite element approach. The scheme is further improved by incorporating a self-adaptive weighing procedure which virtually eliminates mass balance errors.

Governing Equations

The governing equation for one-dimensional, vertical water flow in a variably saturated rigid porous medium is given by

$$C\frac{\partial h}{\partial t}=\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}-K\right)-S(z,\ t) \tag{1}$$

where h is the pressure head, C = $d\theta/dh$ is the soil water capacity in which θ is the volumetric water content, K is the hydraulic conductivity, $S(z,t)$ represents a source/sink term (e.g., root water uptake rate), z is soil depth assumed to be positive downward, and t is time. The flow equation is solved subject to the general initial condition

$$h(z,\ 0)\ =\ h_i(z) \tag{2}$$

and either a first- or second-type boundary condition at the soil surface (z = 0), that is,

$$h(0,t)\ =\ \mathrm{h,(t)} \tag{3}$$

or

$$\left(-K\frac{\partial h}{\partial z}+K\right)\Bigg|_{z=0}=q_0(t) \tag{4}$$

where $h,(t)$ and $q_0(t)$ are the prescribed pressure head and net fluid flux, respectively. The auxiliary condition used here for the lower boundary (z = 1) is that of a free draining soil profile as follows:

$$\frac{\partial h}{\partial z}\Big|_{z=l}=0 \tag{5}$$

We further assume that the soil hydraulic properties, $\theta(h)$ and $K(\theta)$, can be described by the parametric functions of ***van Genuchten*** [1980]

$$\theta(h)\ =\ \theta_r+\frac{\theta_s-\theta_r}{(1+|\alpha h|^n)^m} \tag{6}$$

$$K(S_e)=K_sS_e^{1/2}[1-(1-S_e^{1/m})^m]^2 \tag{7}$$

in which S_e is the effective saturation

$$S_e=\frac{\theta-\theta_r}{\theta_s-\theta_r} \tag{8}$$

In these equations, θ_r and θ_s are the residual and saturated water contents, respectively; K_s is the saturated hydraulic conductivity; α and n are shape parameters; and m = 1 – l/n.

Numerical Implementation

As was pointed out by ***Sorek and Braester [1986]*** and ***Huang [1988], flow*** equation (1) has a parabolic-hyperbolic nature similar to the convection-dispersion solute transport equation. Numerical solutions of the flow and transport equations often exhibit numerical oscillations and/or artificial dispersion. These difficulties are most often encountered when the convection term in the transport equation, or equivalently, the gravitational term in the flow equation, dominates ***[Neuman, 1984; Casulli, 1988]. Huang [1988]*** extended the Eulerian-Lagrangian method of ***Neuman [1984]*** to the solution of the variably saturated flow equation by applying a simple, efficient SRPT solution scheme to the gravity term in (1). While successful for some tests, the method suffered from mass balance errors and numerical oscillations when the flow problem was highly nonlinear and/or extremely dry initial soil profiles were considered. We shall show below that the accuracy and efficiency of the method of ***Huang [1988]*** can be improved considerably by implementing an adaptively corrected SRPT method.

Reformulation of the Richards' equation

Following ***Huang [1988],*** (1) can be rewritten in the form

$$C\frac{\partial h}{\partial t}=\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right)-\frac{\partial K}{\partial h}\frac{\partial h}{\partial z}-S(z,\ t) \tag{9}$$

which has a similar structure as the convection-dispersion solute transport equation. Therefore we can follow the Eulerian-Lagrangian approach which has proved to be very effective for solving the convection-dispersion problem [Neuman, 1984]. For this purpose, (9) is first rearranged as follows

$$C\left[\frac{\partial h}{\partial t}+\left(\frac{1}{C}\frac{\partial K}{\partial h}\right)\frac{\partial h}{\partial z}\right]=\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right)-S(z,t) \quad (10)$$

Using the substantial or Lagrangian derivative

$$\frac{Dh}{Dt}=\frac{\partial h}{\partial t}+v^*\frac{\partial h}{\partial z} \quad (11)$$

(10) can be formulated in Lagrangian form as

$$C\frac{Dh}{Dt}=\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right)-S(z,t) \quad (12)$$

in which h no longer represents the pressure head at a particular point in space and time but rather the pressure head of a fluid particle moving along a characteristic path described by the equation

$$dZ/dt = v^* \quad (13)$$

In the above equations, Z represents the characteristic path of a moving particle, and v^* is the velocity of the fluid particle along the characteristic path:

$$v^*=\frac{1}{c}\frac{\partial K}{\partial h}=\frac{\partial K}{\mathbf{ae}} \qquad h<0$$
$$v^*=0 \qquad h\geq 0 \quad (14)$$

We emphasize that v^* is different from the pore water velocity, $v = q/\theta$, in which q is the Darcian fluid flux density. The transformed Richards' equation (12) holds now for a predominantly parabolic type diffusion problem that is well suited for solution using the FE method. Numerical solution of (12) will lead to a symmetric matrix equation.

Equation (12) subject to the initial and boundary conditions can be solved on a fixed grid using the Eulerian-Lagrangian approach by adopting the following FD approximation of the Lagrangian derivative [Neuman, 1984]:

$$\frac{Dh}{Dt}\approx\frac{h^{t+\Delta t}-\bar{h}}{\mathbf{At}} \quad (15)$$

where At is the time increment and $\bar{h}$ is the 'convective component', that is, the contribution of the gravitational term to the pressure head. This contribution can be obtained by implementing an adaptively corrected method of characteristics to be described later.

Assume that the soil profile is discretized into N – 1 elements of variable size, Δz_i (i = 1, 2, ···, $N-1$). The dependent variable h can then be approximated using linear finite elements as follows

$$h\cong\hat{h}=\sum_{i=1}^{N}h_i(t)\phi_i(z) \quad (16)$$

where N is the number of nodes, and $\phi_i(z)$ are chapeau functions which vary linearly between neighboring nodes, being one at node i and zero at all other nodes. If the pressure head h_i^t at time t is known, $h_i^{t+\Delta t}$ at time level t + At can be solved using steps outlined in the next four sections.

Single-step reverse particle tracking (SRPT)

The convective component $\bar{h}$ can be obtained using the SRPT technique described by ***Neuman*** [1984] and ***Galeati et al.*** [1992]. The method considers each node i to be a fictitious particle which moves from location $\bar{Z}_i$ at time t along its characteristic path to location z_i, the coordinate of node i, at time $t + At$. The location of the fictitious moving particle i at time t can be tracked backward according to (13), that is,

$$\bar{Z}_i=z_i-\int_t^{t+\Delta t}v^*\,dt \quad (17)$$

Since the velocity v^* is a function of z and t, an iterative procedure is required to accomplish the above integration. Once location $\bar{Z}_i$ is computed, the corresponding convective component of node i can be obtained by interpolation between nodal values as follows:

$$\bar{h}_i=\sum_{j=1}^{N}h_j^t\phi_j(\bar{Z}_i) \quad (18)$$

Adaptive correction

Formally, the computed convective component can be directly substituted into (15), after which a conventional FE or FD method can be used to solve (12) for $h_i^{t+\Delta t}$. However, similar to numerical solutions for convection-dominated solute transport [Neuman, 1984], the SRPT technique tends to produce numerical dispersion near the wetting front when water flow is dominated by the gravity component $\partial K/\partial z$ in (1). In addition, numerical oscillations and mass balance errors may appear when the hydraulic properties are highly nonlinear. The following adaptive correction procedure was developed to overcome these numerical difficulties.

As before, let $\bar{h}_i$ be the convective component obtained by the SRPT. A modified convective component $\bar{h}_i^*$ may be calculated at each time step by weighing $\bar{h}_i$ with the pressure heads of the first two nodes upstream of node i, that is,

$$\bar{h}_i^*=\frac{1}{2}([(1-\lambda)h_{i-2}^t+\lambda h_{i-1}^t]+\bar{h}_i) \quad (19)$$

where $0 \leq A \leq 1$ is a correction factor. The value of A is automatically adjusted during each time step, according to the global mass balance error δ, defined as

$$\delta=\frac{(W_q+W_i-W_{\text{out}})-W_{\text{stor}}}{W_{\text{stor}}}\times 100\% \quad (20)$$

where W_q, W_i, W_{out}, and W_{stor} are the volumes of water infiltrated, initially present, drained, and currently stored in the soil profile respectively.

When the mass balance error at time t is negative and its absolute value is larger than the error at $t - At$, **A is** increased at time $t + At$ by some factor (e.g., 0.05). When

the mass balance error at time t is positive and greater than the error at $t - At$, A is decreased. However, A is kept unchanged when the absolute mass balance error remains the same or decreases. This approach assumes that the state variable h at a node is affected primarily by the pressure heads of the first two upstream nodes.

Galerkin finite element representation

We used the Galerkin finite element method to solve the transformed flow equation (12). The method is similar to the scheme described in detail by *van Genuchten [1982]* and *Kool and van Genuchten [1991].* Substituting (16) into the Lagrangian governing equation (12), performing the Galerkin orthogonalization, and applying the L_2 mass-lumping scheme used by *van Genuchten [1982]* and *Milly [1985],* leads to the matrix equation

$$[A]\{h\} + [B]\frac{D\{h\}}{Dt} = \{F\} \tag{21}$$

where

$$[A_{ij}] = \int_0^l K \frac{d\phi_i}{dz}\frac{d\phi_j}{dz}\,dz \tag{22a}$$

$$[B_{ij}] = C_i \int_0^l \phi_i\,dz \tag{22b}$$

$$\{F_i\} = (-q + K)\phi_i\Big|_0^l - \int_0^l S\phi_i\,dz \tag{22c}$$

in which the subscripts i and j represent nodal indices. The first term on the right-hand side of (22c) is not needed for prescribed pressure head boundary conditions. Analogous to the pressure head, the nonlinear coefficients K and C are interpolated linearly across each element as follows:

$$K \cong \sum_{i=1}^{N} K_i\phi_i(z) \tag{23a}$$

$$C \cong \sum_{i=1}^{N} C_i\phi_i(z) \tag{23b}$$

where K_i and C_i are nodal values of the respective coefficients. A similar interpolation was also used for the root uptake term S.

According to the Eulerian-Lagrangian approach, the nodal Lagrangian derivative in (21) may be discretized by means of backward finite differences using the approximation

$$\frac{Dh_i}{Dt} \simeq \frac{h_i^{t+\Delta t} - \bar{h}_i^*}{At} \tag{24}$$

where the modified convective component $\bar{h}_i^*$ is calculated using (19). This approximation is consistent with the notion that node i is regarded as a particle reaching z_i at t_{k+1} *[Neuman,* 1984]. Substituting (24) into (21) leads to the final matrix equation

$$\left([a] + \frac{1}{\Delta t}[B]\right)\{h\}^{t+\Delta t} = \frac{1}{\Delta t}[B]\{\bar{h}^*\} + \{F\} \tag{25}$$

Time-stepping procedure and evaluation of nonlinear coefficients

Since the hydraulic conductivity K and the soil water capacity C in (1) are nonlinear functions of the pressure head h, an iterative procedure is necessary at every time step to obtain the correct solution. We used the Picard iteration method with under-relaxation *[van Genuchten, 1982; Kool and van Genuchten, 19911.* Convergence was assumed when the following condition was satisfied:

$$|\hat{h}_i^{t+\Delta t} - h_i^{t+\Delta t}| \le \delta_1|h_i^{t+\Delta t}| + \delta_2 \tag{26}$$

where $\hat{h}^{t+\Delta t}$ and $h^{t+\Delta t}$ are, respectively, the projected and computed values of the pressure head at the ith node at the new time level, and δ_1 and δ_2 are relative and absolute convergence criteria. The value of At was automatically updated for each time step depending on how quickly the solution converged at the previous time step. For this purpose we used the adaptive time-stepping scheme implemented previously by *Kool and van Genuchten [1991].*

Convergence and accuracy of the flow solution are strongly influenced by how the nonlinear hydraulic conductivity K and the soil water capacity C are evaluated in the numerical scheme. Different ways of evaluating the hydraulic conductivity, and especially the soil water capacity, can have pronounced effects on the mass balance error, particularly when steep moisture fronts are present. For such situations, alternative solution schemes using water content as the dependent variable have been proposed [e.g., *Hills et al., 1989].* The advantage of O-based schemes is that they are inherently mass conservative. However, as pointed out by *Milly [1985],* a mass-conservative solution may also be obtained with the head-based formulation if the capacity term is evaluated in an appropriate manner. Mass conservation may be achieved by defining the soil water capacity so that at the element level the following equality holds:

$$\int_e C_e(h^{t+\Delta t} - h')\,de = \int_e (\theta^{t+\Delta t} - \theta^t)\,de \tag{27}$$

in which C_e is the effective element soil water capacity given by (23b) and where integration is performed on an element by element basis. According to the L_2 mass-lumping scheme *[Milly, 1985]* the resulting expressions for the nodal values of C at time t are given by

$$C_i^t = \frac{\theta_i^{t+\Delta t} - \theta_i^t}{h_i^{t+\Delta t} - h_i^t} \tag{28}$$

When changes in the nodal pressure heads are very small, the denominator of (28) approaches zero, in which case the soil water capacity is best computed analytically as the derivative of the soil water retention curve evaluated at the half-time level $h_i^{t+1/2\Delta t} = 1/2(h^t + h_i^{t+\Delta t})$, that is,

$$C_i = \left.\frac{d\theta(h)}{dh}\right|_{h^{t+1/2\Delta t}} \tag{29}$$

In our simulations, nodal values of hydraulic conductivity were always evaluated at the half-time level using (7), that is, $K_i = K(h_i^{t+1/2\Delta t})$.

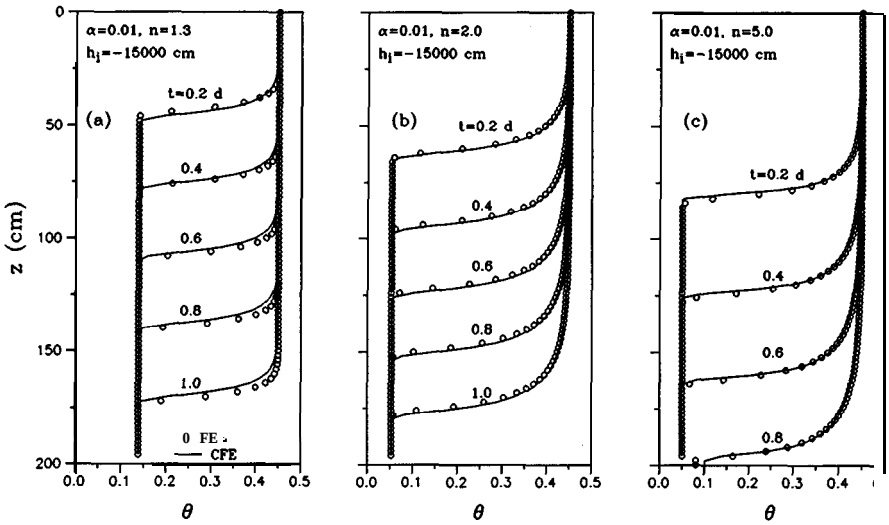

Figure 1. Calculated water content (θ) profiles versus depth (z) for ponded infiltration in soils with α = 0.01 cm^{-1} and (a) n = 1.3, (b) n = 2.0, and (c) n = 5.0. Solid lines represent characteristic finite element (CFE) solutions; circles give finite elements (FE) results.

Numerical Verification

The proposed characteristic finite element scheme (CFE) was tested by simulating ponded infiltration into a 200-cm-deep soil profile. Results obtained with the CFE method will be compared with solutions of the mass-conservative finite element (FE) method of Kool and van Genuchten [1991]. To facilitate comparisons between CFE and FE, all simulations used L_2 mass lumping, employed the same adaptive time-stepping scheme, and implemented the same spatial discretization, using uniform spatial increments Az of 2 cm. The simulations were carried out on an IBM 486/33 personal computer. CPU time was used to compare the computational efficiency of the different methods, while the numerical accuracy was evaluated in terms of the mass balance error δ given by (20). Simulations were carried out for different hydraulic parameters and different initial pressure heads h_i. The residual and saturated water contents in (6) were assumed to be 0.05 and 0.45, respectively, in all simulations. Results will be presented for ponded infiltration assuming a zero pressure head (h_0 = 0) at the soil surface.

Figure 1 compares water content profiles calculated using the CFE method (solid lines) with those obtained using standard FE (open circles) assuming h_i = − 15,000 cm, α = 0.01 cm-', K_s = 50 cm d^{-1}, and three values of n; that is, n = 1.3, typical of a relatively fine-textured soil, n = 2 for a medium- to coarse-textured medium, and n = 5 for a coarse-textured soil *[Camel and Parrish, 1988; Yates et al., 1992].* The CFE and FE simulation results are in close agreement except for those in Figure la where the FE solution appears to be slightly ahead of the CFE solution. Both methods yielded mass balance errors of less than 0.05% in this example. For n = 1.3, the CFE simulation required slightly less CPU time (13.0 min) than the FE scheme (15.4 min). The improved efficiency of the CFE method is more apparent when n = 5 (Figure lc), for which the CFE and FE schemes used 9.1 and 22.6 min CPU time, respectively. By comparison the CFE and FE schemes for n = 2 (Figure lb) required 9.8 and 9.2 min, respectively.

Figure 2 shows pressure head distributions corresponding to the water content profiles in Figure 1. It is evident from Figures 1 and 2 that the pressure head wetting fronts are much steeper than those for the water content. All CFE simulations were found to be stable (no oscillations were observed). The FE solutions, on the other hand, exhibited some minor oscillations when n = 1.3 in the near-saturated zones between the wetting fronts and the soil surface. The FE scheme for n = 1.3 also produced slightly positive pressure head values (0.5 ∼3.0 cm) in this zone. These results, not further noticeable in Figure 2a, are inconsistent with the simulated ponded infiltration problem which invoked a zero pressure head boundary condition at the soil surface. The simulated positive pressure heads probably also caused the slightly steeper wetting fronts in Figures la and 2a for the FE scheme as compared to the CFE method.

The FE results in Figures 1 and 2 were obtained with the L_2 mass-lumping scheme of *Milly [1985].* Implementing the mass-conservative mixed-form solution of the Richards' equation *[Celia et al., 1990]* did not significantly improve the results for n = 1.3. This scheme also exhibited oscillatory solutions and, like the standard FE method, produced positive head values between the wetting fronts and the soil surface. By comparison, the CFE never produced oscillations in computed pressure head distributions. The numerical oscillations and positive pressure values in the FE results were found to increase at increasingly lower n values, and likely were caused by the shape of the conductivity function near saturation. As shown elsewhere *[van Genuchten and* Nielsen, 1985; *van Genuchten et al.,* 1991], relatively small n values (e.g., 1.0 < n < 1.3) in (6) and (7) will lead to extremely nonlinear *K(h)* functions near saturation, sometimes with important repercussions for accurately simulating variably saturated flow *[Vogel et al., 1992].* Adoption of alternative *K(h)* functions, such as those by *Brooks and Corey [1964],* or modifications of (6) and (7) as proposed by *Vogel and Cislerova [1988],* may significantly decrease the nonlinearities in the soil hydraulic functions and lead to more stable flow (but dramatically different) simulation results *[Vogel et al.,* 1992].

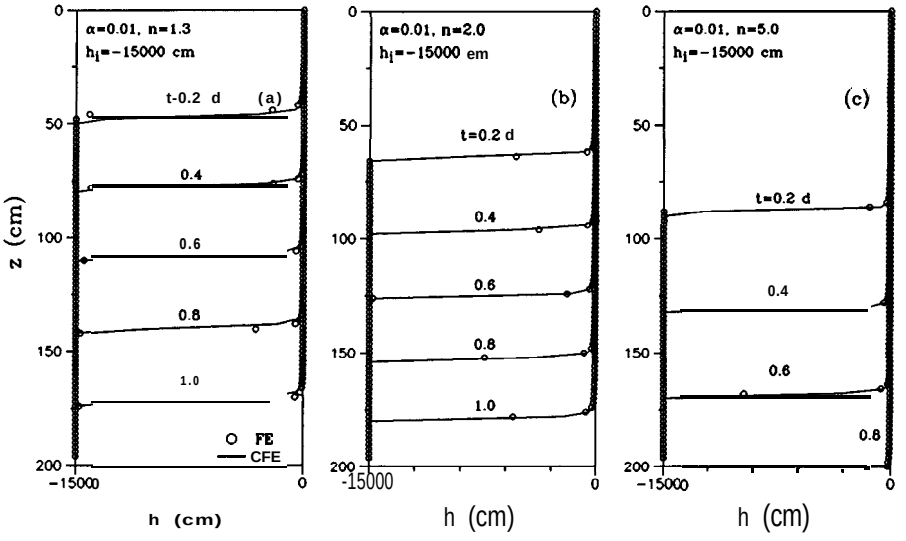

Figure 2. Calculated pressure head (h) distributions versus depth (z), corresponding to the results in Figure 1. (a) n = 1.3; (b) n = 2.0; and (c) n = 5.0.

Figure 3 examines the effect of the parameter α on numerical simulations assuming n = 2, K_s = 50 cm d^{-1}, an initial pressure head h_i of − 10,000 cm, and three values of the parameter α: 0.1, 0.01, and 0.001 cm^{-1}. The CFE solutions show relatively good agreement with the FE results for all three values of α. The mass balance error for both methods was always less than 0.02%. The results in Figure 3 are indicative of the fact that simulations with higher α values (Figure 3a) generally produce steeper wetting fronts than simulations with smaller α values (Figure 3c), with the CFE scheme requiring somewhat less computer time at the higher α as compared to the FE method. The CPU times for the CFE and FE schemes were, respectively, 3.3 and 7.8 mins for α = 0.1, 8.3 and 7.4 min for α = 0.01, and 4.5 and 4.2 min for α = 0.001.

Next, we examined the performance of the CFE scheme for infiltration in soils having relatively large n values. Figure 4 demonstrates the capability of the CFE method to solve the flow problem with an extremely high value of 10 for II. Soils of this type are characterized by very narrow pore and particle size distributions (usually coarse-textured soils) and typically have very nonlinear θ (h) as well as ***K(h)*** hydraulic functions [van ***Genuchten and Nielsen,*** 1985]. The simulations again pertained to ponded infiltration into a soil profile having an initial pressure head h_i of -10,000 cm and a saturated hydraulic conductivity K_s of 500 cm d^{-1} and assuming α = 0.1 cm-'. The CFE computed water content and pressure head distributions are shown in Figures 4a and 4b, respectively. CPU time for this example was 6.0 min, while the mass balance error was 0.53%. The standard finite element (FE) method assuming the same finite element grid system as used for CFE (constant Az of 2 cm) failed to converge for this infiltration problem. The mixed-form method of ***Celia et al. [1990]*** also failed to converge when the standard 2 cm for the grid spacing Az was used. Smaller nodal spacings also did not lead to convergent FE solutions.

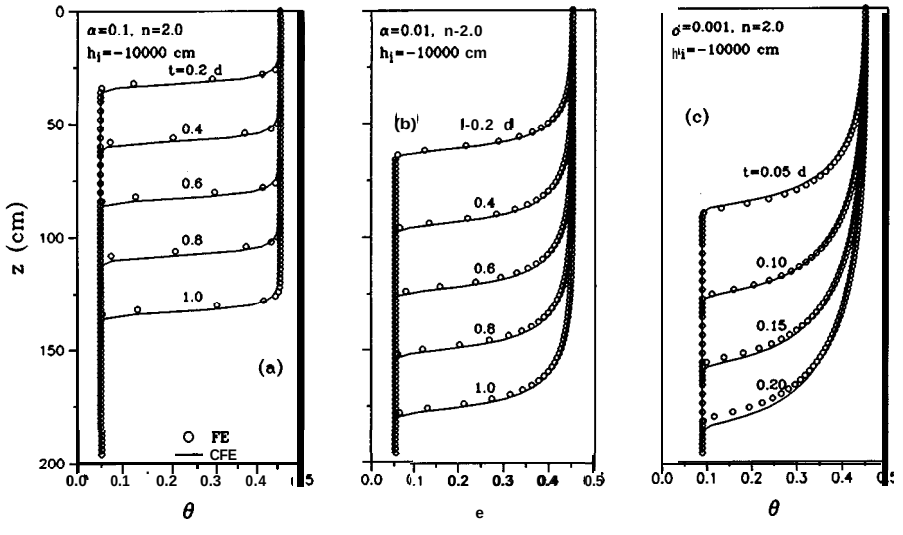

Figure 3. Simulated water content (0) distributions versus depth (z) for ponded infiltration in soils with n = 2.0 and (a) α = 0.1, (b) α = 0.01, and (c) α = 0.001 cm^{-1}.

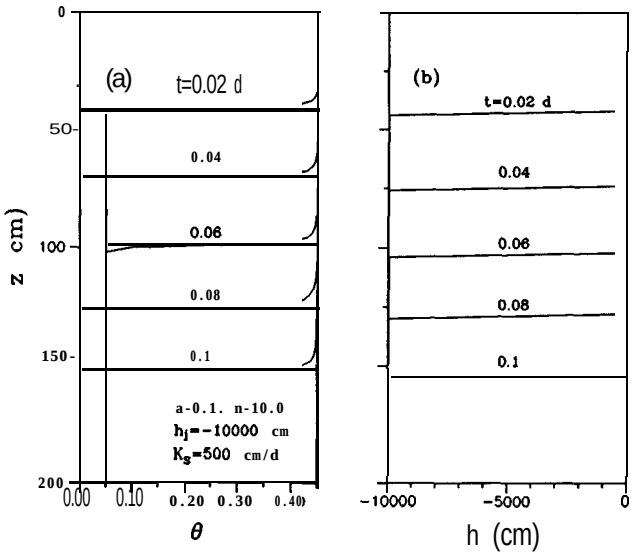

Figure 4. Calculated (a) water content, θ, and (b) pressure head, ***h,*** distributions versus depth, z, obtained with the CFE method for ponded infiltration in a very coarse-textured (n = 10) soil profile.

Accurate and stable simulations of infiltration into very dry soils are often difficult to obtain with standard numerical methods [Zaidel and Russo, 1992] because of the development of steep wetting fronts. Figure 5 shows one set of results obtained with the CFE and FE schemes for ponded infiltration into a soil having a very low initial pressure head h_i of -50,000 cm. The simulation concerned a medium- to coarse-textured soil assuming $\alpha = 0.01$ cm $^{-1}$, $n = 2$, and K_s ***= 50*** cm d-i. Simulation of the O.l-day infiltration event required 10.7 and 12.0 min for the FE and CFE methods, respectively. Notice the close agreement between the two methods in Figure 5.

All calculations thus far were obtained with 2-cm elements. We also examined the effects of different spatial increments on the results, again for ponded infiltration. Figure 6 compares calculated water content distributions assuming element sizes Az of 1, 2, and 4 cm. The corresponding CPU times were 7.5, 2.6, and 0.98 min, and the introduced mass balance errors were 0.0397%, 0.055%, and 0.0125% for Az = 1, 2, and 4 cm, respectively. The results indicate that the solutions with Az = 1 cm and Δz = 2 cm are very close, while some relatively minor deviations appear near the wetting front when Az = 4 cm. This shows that the CFE method can produce accurate solutions using relatively large spatial increments, thus further improving the computational efficiency as compared to the FE method.

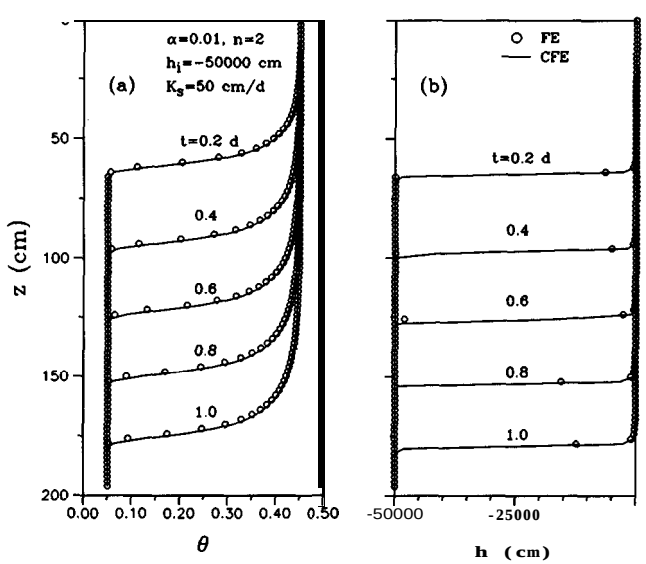

Figure 5. Calculated (a) water content (0) and (b) pressure head ***(h)*** distributions versus depth, z, for ponded infiltration into an initially very dry (h_i ***= -50,000*** cm) soil profile.

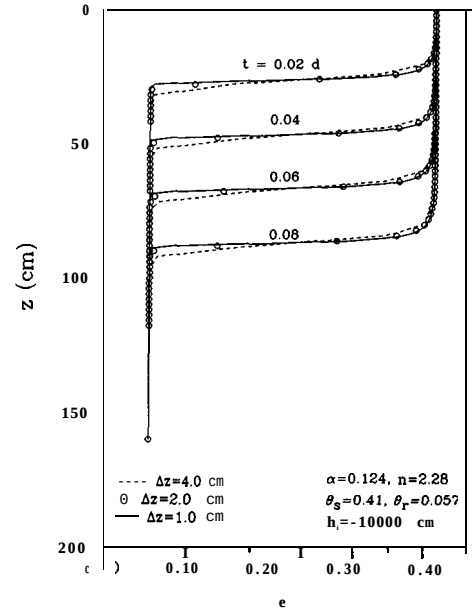

Figure 6. Effect of spatial discretization, Az, on simulated water content (θ) distributions versus depth (z) using the CFE method.

The above results, as well as those of many other simulations, indicate that the proposed CFE method is computationally somewhat ***more*** efficient than the standard FE for situations leading to relatively steep pressure head wetting fronts. Such situations most often arise for soils having very large or very small n values, and/or relatively large α values. However, the FE method is equally or slightly more efficient than the CFE scheme for more moderate nonlinear infiltration problems involving intermediate n and relatively small α values. This feature results from the fact that the CFE method requires additional computations for solving (17) that are associated with the gravity term in the flow equation. Intermediate values of n in (6) and (7) do not lead to strongly nonlinear soil hydraulic properties and steep wetting fronts, and hence there appears little advantage to employing the particle tracking method. These conclusions are consistent with studies by ***van Genuchten and Nielsen [1985]*** and ***van Genuchten et al.*** [1991] who show that the unsaturated hydraulic properties, especially ***K(h),*** become very nonlinear for soils having relatively small (e.g., 1 .O $< n \leq$ 1.3) or large ($n \geq 3$) values of ***n.*** Numerical simulations of infiltration in such soils using standard FE or FD methods are generally the most difficult, especially for initially very

dry soil profiles. The most extreme cases arise when the gravity term of (1) is relatively dominant, in which case the flow equation acquires hyperbolic properties which are best treated with the proposed CFE method.

The results in this paper pertain to ponded infiltration. We also carried out several simulations for constant flux infiltration at the soil surface. The proposed CFE method was found to work efficiently and accurately and found to be oscillation-free for most numerical tests. However, some minor oscillations were observed when the applied surface boundary flux q_0 was close to K_s for infiltration into fine-textured soils having extremely small n values (such as $n = 1.05$).

Summary and Conclusions

The rigorous splitting approach based on the Eulerian-Lagrangian concept, initially introduced for solving convection-dispersion type solute transport problems, was extended to variably saturated water flow. The approach permits one to separate the original flow problem into "convection" and "diffusion" parts, which are then solved with the method of characteristics and the Galerkin finite element method, respectively. A self-adaptive correction method was developed to modify the "convection contribution" obtained by the single-step reverse particle tracking technique. The correction procedure consists of weighting the convection contribution with the pressure heads of two upstream nodes. The weighting factor, $0 \leq \lambda \leq 1$, was continually updated in the program so that the global mass balance error could be minimized.

Numerical simulations of ponded infiltration for a wide range of soils showed that the Eulerian-Lagrangian approach works well for variably saturated flow problems described with the Richards' equation, including situations involving very dry initial conditions. While maintaining the simplicity of the single-step reverse particle tracking (SRPT) method, the proposed adaptive correction technique significantly improved the mass balance accuracy of previous characteristic finite element (CFE) methods. The proposed CFE method yielded results which were also virtually free of oscillations. The only situations where some minor oscillations were observed were for flux infiltration into soils with very small n values when the infiltration rate q_0 approached the value of the saturated hydraulic conductivity, K_s. The CFE scheme appears especially attractive for highly nonlinear flow problems, such as for fine-textured soils having small n values ($n < 1.3$) and coarse-textured soils having large values of n and α, particularly when very dry initial soil profiles are present. Such nonlinear problems are very difficult to simulate with most standard finite difference or finite element methods.

For simulating water flow in soils having intermediate n values ($1.5 < n < 3.0$) the CFE method required approximately the same or slightly more CPU time than the standard finite element (FE) method, mostly because the CFE requires additional calculations associated with the particle tracking scheme. However, the CFE method is more efficient and numerically far more robust than the.standard FE method for problems having highly nonlinear hydraulic properties and/or very dry initial soil profiles. As such, we believe that the CFE method may become an attractive alternative to existing methods for simulating variably saturated water flow.

References

Abriola, L. M., Finite element solution of the unsaturated flow equation using hierarchic basis functions, in Proceedings of the *6th International Conference on Finite Elements in Water Resources,* edited by A. Sa Costa, A. Melo Baptista, W. G. Gray, C. A. Brebbia, and G. F. Pinder, pp. 125-133, Springer-Verlag, New York, 1986.

Broadbridge, P., and I. White, Constant rate rainfall infiltration: A versatile nonlinear model, 1, Analytical solution, *Water Resour. Res., 24, 145-154, 1988.*

Brooks, R. H., and A. T. Corey, Hydraulic properties of porous media, *Hydrol. Pap. 3, 27* pp., Colo. State Univ., Fort Collins, 1964.

Carsel, R. F., and R. S. Parrish, Developing joint probability distributions of soil water retention characteristics, *Water Resour. Res., 24, 7.55-769, 1988.*

Casulli, V., Eulerian-Lagrangian methods for hyperbolic and convection dominated parabolic problems, in *Modeling Surface and Subsurface Flow, Developments in Water Science Series,* vol. *35,* edited by M. A. Celia, L.-A. Ferrand, C. A. Brebbia, W. G. Gray, and G. F. Pinder. pp. 183-191. Elsevier. New York. 1988.

Celia, M. A., and G. F. Pinder, An alternating-direction collocation solution for the unsaturated flow equation, in *Proceedings of the 6th International Conference on Finite Elements in Water Resources,* edited by A. Sa Costa, A. Melo Baptista, W. G. Gray, C. A. Brebbia, and G. F. Pinder, pp. 395-410, Springer-Verlag, New York, 1986.

Celia, M. A., E. T. Bouloutas, and R. L. Zorba, A general mass-conservative numerical solution for the unsaturated flow equation, *Water Resour. Res., 26, 1483-1496, 1990.*

Dane, J. H., and F. H. Mathis, An adaptive finite difference scheme for the one-dimensional water flow equation, *Soil Sci. Soc.* Am. J., 45, 1048-1054, 1981.

Galeati, G., G. Gambolati, and S. P. Neuman, Coupled and partially coupled Eulerian-Lagrangian model of freshwater-seawater mixing, *Water Resour. Res., 28, 149-165, 1992.*

Gamliel, A., and L. M. Abriola, A one-dimensional moving grid solution for the coupled non-linear equations governing multiphase flow in porous media, 1, Model development, *Znt. J. Numer. Methods Fluids, 14, 25-45, 1992.*

Gottardi, G., and M. Venutelli, Moving finite element model for one-dimensional infiltration in unsaturated soil, *Water Resow. Res., 28, 3259-3267, 1992.*

Hansen, R., M. S. Espedal, and 0. Nygaard, An operator splitting technique for two-phase immiscible flow dominated by gravity and capillary forces, in *Computational Methods in Water Resources IX, vol.* 1, *Numerical Methods in Water Resources,* edited by T. F. Russell, R. E. Ewing, C. A. Brebbia, W. G. Gray, and G. F. Pinder. DD. 59-67. Elsevier Science. New York. 1992.

Hills, R. G., I. Porro, D. B. Hudson, and P. J. Wierenga, Modeling one-dimensional infiltration into very dry soils, 1, Model development and evaluation, *Water* Resour. *Res.,* 25, 1259-1269, 1989.

Huang, K., Solution of saturated-unsaturated flow by finite element or finite difference methods combined with characteristic technique, in *Proceedings of the 7th Computational Methods in Water Resources, vol. 2,* edited by M. A. Celia, L. A. Ferrand, C. A. Brebbia, W. G. Gray, and G. F. Pinder, pp. 241-246, Elsevier, New York, 1988.

Kool, J. B., and M. T. van Genuchten, HYDRUS: One-dimensional variably saturated flow and transport model, including hysteresis and root water uptake, *Res. Rep. 124,* U.S. Salinity Lab., Agric. Res. Serv., U.S. Dep. of Agric., Riverside, Calif., 1991.

Milly, P. C. D., A mass-conservative procedure for time-stepping in models of unsaturated flow, *Adv. Water Resour., 8, 32-36, 1985.*

Neuman, S. P., Adaptive Eulerian-Lagrangian finite element method for advection-dispersion, *Znt. J. Numer. Methods* Eng., 20, 321-337, 1984.

Parlange, J., Theory of water movement in soils, 6, Effect of water depth over soil, *Soil Sci., 113, 308-312, 1972.*

Philip, J. R., The theory of infiltration, 1, The infiltration equation and its solution, *Soil Sci.,* 83, 345-357, 1957.

Ross, P. J., Efficient numerical methods for infiltration using Richards' equation, *Water Resour. Res., 26, 279-290, 1990.*

Sorek, S., and C. Braester, An adaptive Eulerian-Lagrangian approach for the numerical simulation of unsaturated flow, in *Proceedings of the 6th International Conference on Finite Ele-*

ments in Water Resources, edited by A. Sa Costa, A. Melo Baptista, W. G. Gray, C. A. Brebbia, and G. Pinder, pp. 87-100, Springer Verlag, New York, 1986.

van Genuchten, M. T., A closed-form equation for predicting the hydraulic conductivity of unsaturated soils, ***Soil Sci. Soc. Am. J., 44, 892-898, 1980.***

van Genuchten, M. T., A comparison of numerical solutions of the one-dimensional unsaturated-saturated flow and transport equations, ***Adv. Water Resow., 5, 47-55, 1982.***

van Genuchten, M. T., and D. R. Nielsen, On describing and predicting the hydraulic properties of unsaturated soils, ***Ann. Geophys., 3,*** 615-628, 1985.

van Genuchten, M. T., F. J. Leij, and S. R. Yates, The RETC code for quantifying the hydraulic properties of unsaturated soils, ***EPA/600/2-91/065, 85*** pp., R. S. Kerr Environ. Res. Lab., U.S. Environ. Prot. Agency, Ada, Okla., 1991.

Vogel, T. N., and M. Cislerova, On the reliability of unsaturated hydraulic conductivity calculated from the moisture retention curve, ***Transp. Porous Media, 3,*** l-5, 1988.

Vogel, T. N., M. T. van Genuchten, R. Zhang, and M. Cislerova, Effect of the description of soil hydraulic properties near saturation on numerical simulation of soil water movement, ***Agron. Abstr., 84, 230, 1992.***

Warrick, A. W., A. Islas, and D. 0. Lomen, An analytical solution to Richards' equation for time-varying infiltration, ***Water Resow. Res., 27, 763-766, 1991.***

Yates, S. R., M. T. van Genuchten, and F. J. Leij, Analysis of predicted hydraulic conductivities using RETC, in ***Proceedings of the International Workshop on Indirect Methods for Estimating the Hydraulic Properties of Unsaturated Soils,*** edited by M. Th. van Genuchten, F. J. Leij and L. J. Lund, pp. ***273-283,*** University of California, Riverside, Calif., 1992.

Zaidel. J.. and D. Russo. Estimation of finite difference interblock conductivities for simulation of infiltration into initially dry soils, ***Water Resour. Res., 28, 2285-2295, 1992.***

K. Huang, M. T. van Genuchten, U.S. Salinity Laboratory, 4500 Glenwood Drive, Riverside, CA 92501.

R. Zhang, Department of Plant, Soil, and Insect Sciences, University of Wyoming, P.O. Box 3354, Laramie, WY 82071-3354.

(Received March 11, 1993; revised August 30, 1993;
accepted October 6, 1993.)