Nonlinear Rayleigh-Taylor growth in converging geometry

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June 2004

This work was performed under the auspices of the U.S. Department of Energy by the University of California Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.

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UCRL-POST-204342

Abstract

The early nonlinear phase of Rayleigh-Taylor (RT) growth is typically described in terms of the classic model of Layzer [1955] in which bubbles of light fluid rise into the heavy fluid at a constant rate determined by the bubble radius and the gravitational acceleration. However, this model is strictly valid only for planar interfaces and hence ignores any effects which might be introduced by the spherically converging interfaces of interest in inertial confinement fusion. The work of G. I. Bell [1951] and M. S. Plesset [1954] introduced the effects of spherical convergence on RT growth but only for the linear regime. Here, a generalization of the Layzer nonlinear bubble rise rate is given for a spherically converging flow of the type studied by Kidder [1974]. A simple formula for the bubble amplitude is found showing that, while the bubble initially rises with a constant velocity similar to the Layzer result, during the late phase of the implosion, an acceleration of the bubble rise rate occurs. The bubble rise rate is verified by comparison with full, 2-D hydrodynamics simulations.

Motivations

- Understanding the nonlinear phase of RT growth is important for IFE where optimized capsule designs may “push the limits” of hydrodynamic instability.
- Layzer’s model [1955] simply & quite accurately describes nonlinear RT bubble growth but is strictly valid only in planar geometry.
- Is there an analogous model for a spherically converging system more relevant to IFE targets?
- Finding an analogous analytical model could provide a rigorous & relevant nonlinear test problem to validate hydro codes.
- Model might also reveal interesting scaling properties of nonlinear bubble growth.
- Problem has not been solved, but appears solvable…

Outline

- Review of Layzer's nonlinear RT bubble model
- 1st attempts at adapting Layer's model to converging geometry
- Review of Kidder's self-similar spherical implosion — from an RT perspective
- A model of nonlinear RT growth in a self-similar implosion
- Comparison with HYDRA simulations

Nonlinear bubble growth in planar geometry: Davies & Taylor [1950], Layzer [1955]

Classic description of nonlinear RT growth likens nonlinear state to large bubbles of light fluid advancing into denser fluid within cylindrical tube at constant velocity u.

Moving to frame of bubble where flow is static, simultaneously solve

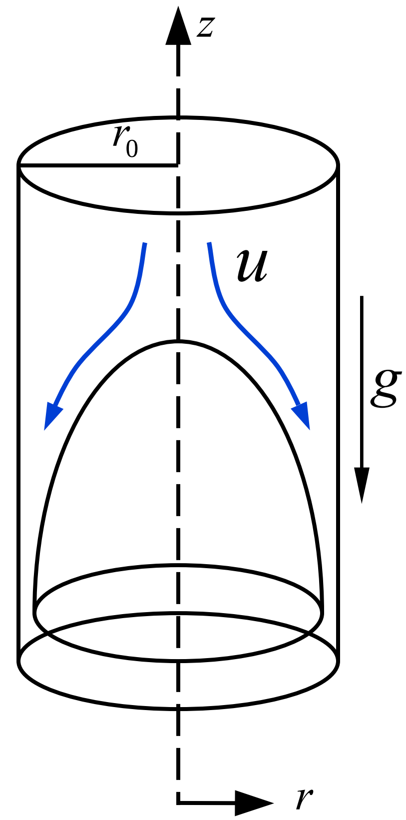
$$\nabla^2\phi = 0 \quad \text{and} \quad F(t) = \phi_t - \frac{1}{2}|\nabla\phi|^2 - \frac{p}{\rho} - gz\Big|_{S=0}$$

with boundary conditions $v_r(r = r_0) = 0$ and $v_z(z = +\infty) = -u$ and where eqn. for the bubble surface $S = Z(r) - z$ is

$$0 = S_t + \mathbf{v}\cdot\nabla S.$$

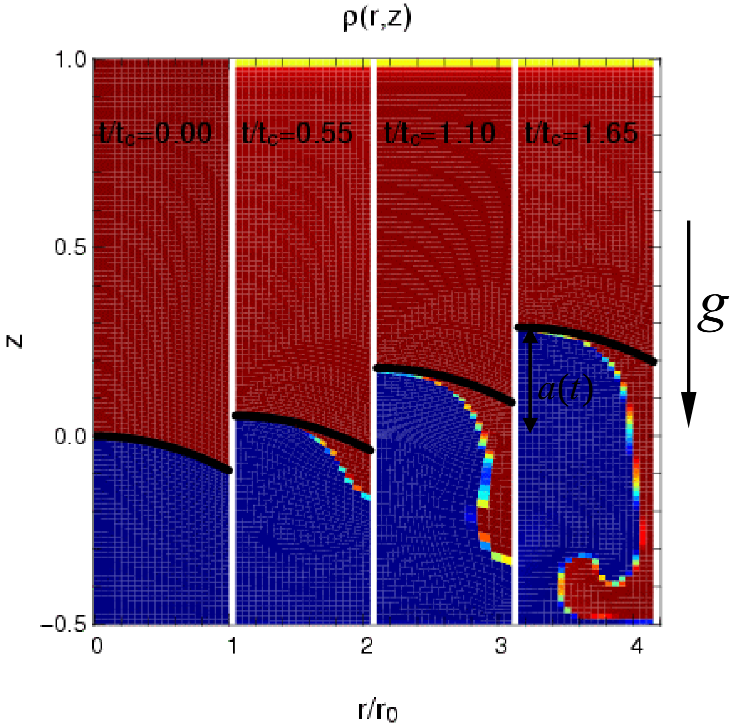
Using $\phi \approx A(t)e^{-k_0 z/r_0} J_0(k_0 r/r_0)$, self-consistent solution near bubble apex gives

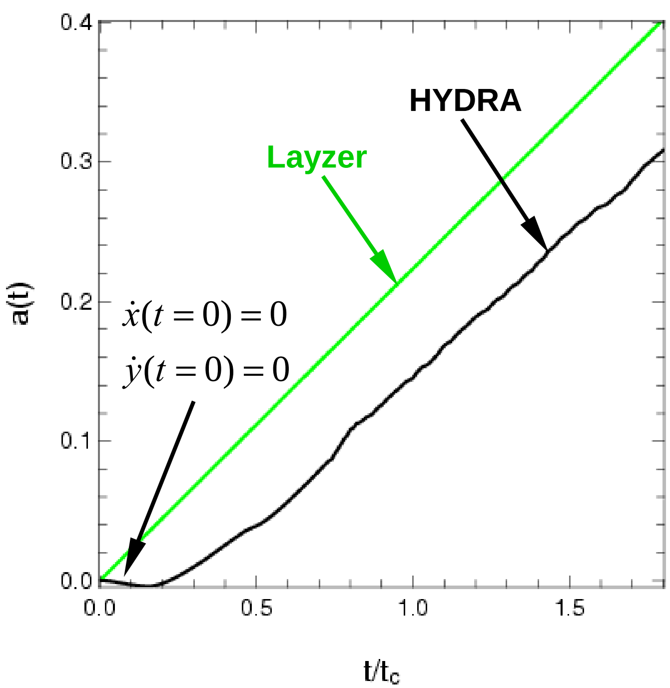
$$u \approx \sqrt{g r_0 / k_0} \approx 0.511\sqrt{g r_0} \quad ; \quad J_1(k_0) = 0.$$



Layzer model agrees remarkably well with hydrodynamics simulations

Measuring bubble growth from ALE hydrodynamics simulations with HYDRA (with uniform gravity g) shows approximately linear bubble growth in time.





Symmetry along z and constancy of acceleration make it "easy" for flow to "attract" to Layzer solution. Symmetry does not apply in converging case…

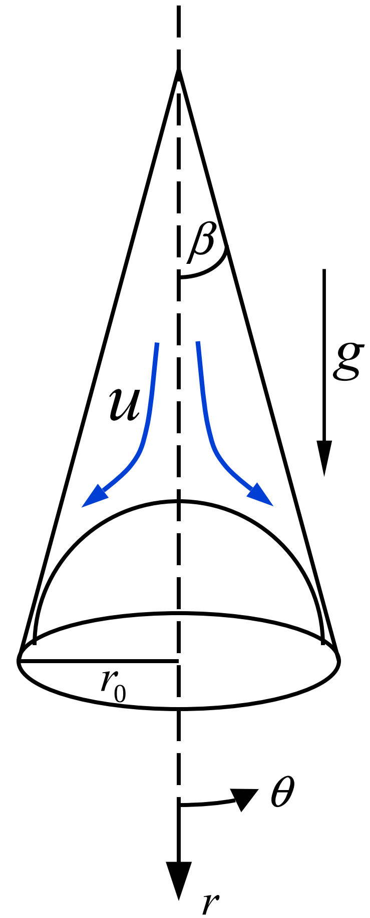
Nonlinear growth in converging geometry: convert cylindrical boundaries to conical

Replace bubble rising in cylinder with bubble rising in cone to capture lowest order convergence effect:

I. **no longer expect constant bubble rise velocity**

II. **cannot choose convenient coordinates fixed in bubble frame where flow is time independent**

III. **frame of accelerating interface no longer equivalent to uniform gravitational field**



Now seek self-consistent, time-dependent solution near bubble apex to

$$\nabla^2\phi = 0 \quad \text{and} \quad F(t) = \phi_t - \frac{1}{2}\left|\nabla\phi\right|^2 - \frac{p}{\rho} - gz\Big|_{S=0}$$

with $\upsilon_\theta(\theta = \beta) = 0$ **and** $\left|\mathbf{v}\right|(r \to 0) < \infty$.

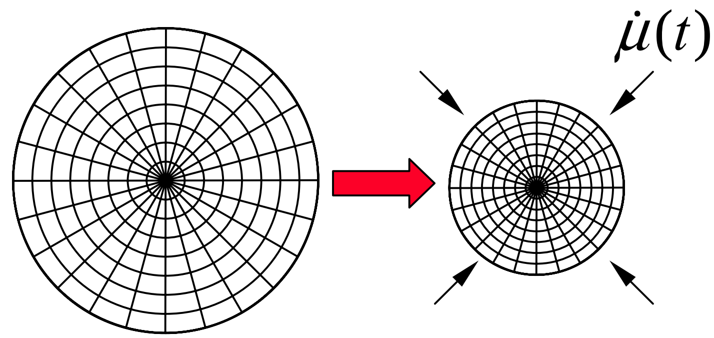
Using $\phi \approx A(t) r^\nu P_\nu(\cos\theta)$, **find that bubble *decelerates* while rising:**

$$u \to -(g/\nu)\left\{\sqrt{\nu a_0 / g} - t\right\}, \quad \nu \to \infty$$

Nonlinear growth in converging geometry: adopt a contracting coordinate system

In place of a bubble rising in a cone under gravity, consider flow satisfying conical boundary conditions in a coordinate system accelerating spherically inward:

$$\left.\begin{aligned} r' &= \mu(t) r \\ \theta' &= \theta \\ t' &= t \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} r &= r'/\mu(t') \\ \theta &= \theta' \\ t &= t' \end{aligned}\right.$$



Velocity potential, Bernoulli's eqn., and surface eqn. transform to

$$\phi' = \mu^2 \phi - \frac{(r')^2}{2} f + \Phi(t') \qquad \text{where} \qquad f = \frac{\dot{\mu}}{\mu}$$

$$F(t') = \phi'_{t'} + \frac{1}{2}\left|\nabla' \phi'\right|^2 - \frac{\gamma}{\gamma - 1}\frac{p'}{\rho'} - 2 f \phi' + \frac{(r')^2}{2}\left(\dot{f} - f^2\right)\Bigg|_{S'=0}$$

$$0 = S'_{t'} + \mathbf{v}' \cdot \nabla S'$$

But once a $\mu(t)$ is chosen, p', ρ', and $F(t')$ can no longer be independently specified as before. Consider the 1-D problem…

Kidder's [1974] self-similar (homogeneous) spherical implosion

Assume a uniform (1-D), isentropic, compressing flow ($h(t) \equiv 1/\mu(t)$)

$$\phi = -\frac{r^2}{2}\frac{\dot{h}}{h} \quad \Rightarrow \quad \rho = \exp\int dt\, \nabla^2 \phi = \rho_0(r_0) h^{-3}(t)$$

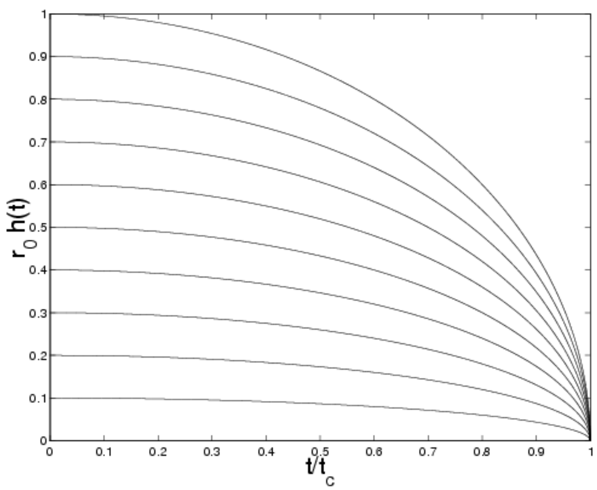
$$p = p_0(r_0)\left[\frac{\rho}{\rho_0(r_0)}\right]^{\gamma} \quad \Rightarrow \quad \frac{p}{\rho} = \frac{p_0(r_0)}{\rho_0(r_0)} h^{3(1-\gamma)}$$

$$F(t) = \phi_t - \frac{1}{2}\phi_r^2 - \frac{\gamma}{\gamma-1}\frac{p}{\rho}\bigg|_{S=0} = -\frac{r^2}{2}\frac{\ddot{h}}{h} - \frac{\gamma}{\gamma-1}\frac{p_0(r_0)}{\rho_0(r_0)} h^{3(1-\gamma)}\bigg|_{r=R_0}$$

Conditions at $r = 0$ determine $F(t)$ in terms of $h(t)$ leading to

$$-h^{3\gamma-2}\ddot{h} = \text{const.} \equiv 1/t_c^2$$

$$\Rightarrow \quad h(t) = \sqrt{1-(t/t_c)^2} \quad ; \quad \gamma = 5/3$$



Requires special (physical?) density & pressure profiles & histories.

Nonlinear RT growth during a Kidder-type implosion

Consider nonlinearly perturbed Kidder-type implosion:

$$\phi = -\frac{r^2}{2}\frac{\dot{h}}{h} + A(t) r^{\nu} P_{\nu}(\cos\theta) \quad \Rightarrow \quad \rho = \exp\int dt\, \nabla^2 \phi = \rho_0(r_0) h^{-3}(t)$$

$$p = p_0(r_0)\left[\frac{\rho}{\rho_0(r_0)}\right]^{\gamma} \quad \Rightarrow \quad \frac{p}{\rho} = \frac{p_0(r_0)}{\rho_0(r_0)} h^{3(1-\gamma)}.$$

Incorporating enthalpy $\gamma p/(\gamma-1)\rho$ **into** $F(t)$ **and evaluating at** $r=0$ **to determine** $F(t)$**, Bernoulli's eqn. is**

$$(\mathrm{const.}) h^{1-3\gamma}(t) = \phi_t + \frac{1}{2}\left[\phi_r^2 + \frac{\phi_\theta^2}{r^2}\right]\Bigg|_{S=0}$$

$$= \dot{A} r^{\nu} P_{\nu} - \frac{1}{2}\left[\left(\nu A r^{\nu-1} P_{\nu}\right)^2 + \left(A r^{\nu-1} \sin\theta\, P_{\nu}'\right)^2\right] +$$

$$+ 2\frac{\dot{h}}{h} A r^{\nu} P_{\nu} - \frac{r^2}{2}\frac{\ddot{h}}{h}\Bigg|_{S=0}.$$

Bubble shape is "self-similar" with flow

Expanding the bubble shape $S = R(\theta,t) - r$ **as** $R(\theta,t) \approx a(t) + b(t)\theta^2$ **near the apex and inserting the chosen flow** ϕ **into the surface eqn. gives**

$$0 = S_t + \mathbf{v}\cdot\nabla S$$

$$\approx \dot{a} + \nu A a^{\nu-1} + \theta^2\Big\{\dot{b} + \nu(\nu-1)Aa^{\nu-2}b +$$

$$+\nu(\nu+1)Aa^{\nu-2}b - \frac{\nu^2(\nu+1)}{4}Aa^{\nu-1}\Big\} + O(\theta^3).$$

Separating the orders in θ**,** $b(t)$ **and** $A(t)$ **can be eliminated in favor of the bubble amplitude** $a(t)$ **:**

$$O(\theta^0): \quad 0 = \dot{a} + \nu A a^{\nu-1} \qquad \Rightarrow \quad A = -\frac{1}{\nu}a^{1-\nu}\dot{a}$$

$$O(\theta^2): \quad 0 = \dot{b} + 2\nu^2 A a^{\nu-2} b - \frac{\nu^2(\nu+1)}{4}Aa^{\nu-1} \quad \Rightarrow \quad b = \frac{\nu}{4}\frac{\nu+1}{2\nu-1}a$$

***i.e.*, a constant curvature (self-similar) bubble**

$$R(\theta,t) \approx a(t)\left\{1 + \frac{\nu}{4}\frac{\nu+1}{2\nu-1}\theta^2\right\} + O(\theta^4).$$

RT Bubble amplitude and implosion scale factor are coupled

Substituting into Bernoulli's eqn. and likewise expanding to $O(\theta^2)$ gives two eqns. to determine $h(t)$ and $a(t)$ simultaneously

$$O(\theta^0): \quad (\text{const.})h^{1-3\gamma} = a\ddot{a} + \left(1 - \frac{\nu}{2}\right)\dot{a}^2 + 2\frac{\dot{h}}{h}a\dot{a} + \frac{\nu}{2}\frac{\ddot{h}}{h}a^2$$

$$O(\theta^2): \quad 0 = a\ddot{a} + \frac{1-2\nu}{2}\dot{a}^2 + 2\frac{\dot{h}}{h}a\dot{a} - \frac{\nu}{\nu-1}\frac{\ddot{h}}{h}a^2.$$

For $\nu >> 1$, 1st eqn. reduces to Kidder's eqn., so may approximate

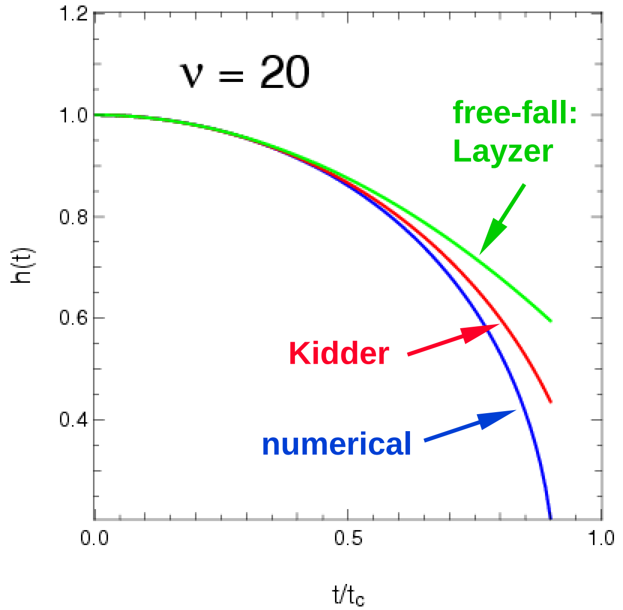
$h \approx h_{\text{Kidder}}$. The 2nd eqn. may then be solved for $a(t)$ by the WKB method

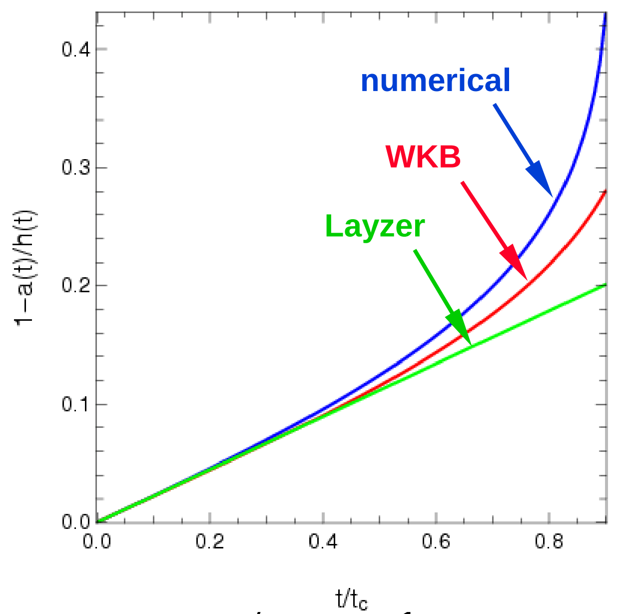
$$a(t) \sim \exp\left\{2\Lambda(\nu)\int dt\sqrt{-\ddot{h}/h}\right\} \sim \left(\frac{1+t/t_c}{1-t/t_c}\right)^{\Lambda(\nu)}$$

where $\Lambda(\nu) = \sqrt{\nu(3-2\nu)/2(1-\nu)-1}/(3-2\nu)$ and for $\gamma = 5/3$.

Numerical & WKB solutions predict greater bubble growth than in Layzer model

Numerically integrating ODE's confirms $h \approx h_{\mathrm{Kidder}}$ for $\nu \geq 20$ as well as "faster-than-Layzer" growth for $t/t_c \geq 1/2$ — presumably due to greater effective acceleration $\ddot{h}(t)$ at late times.



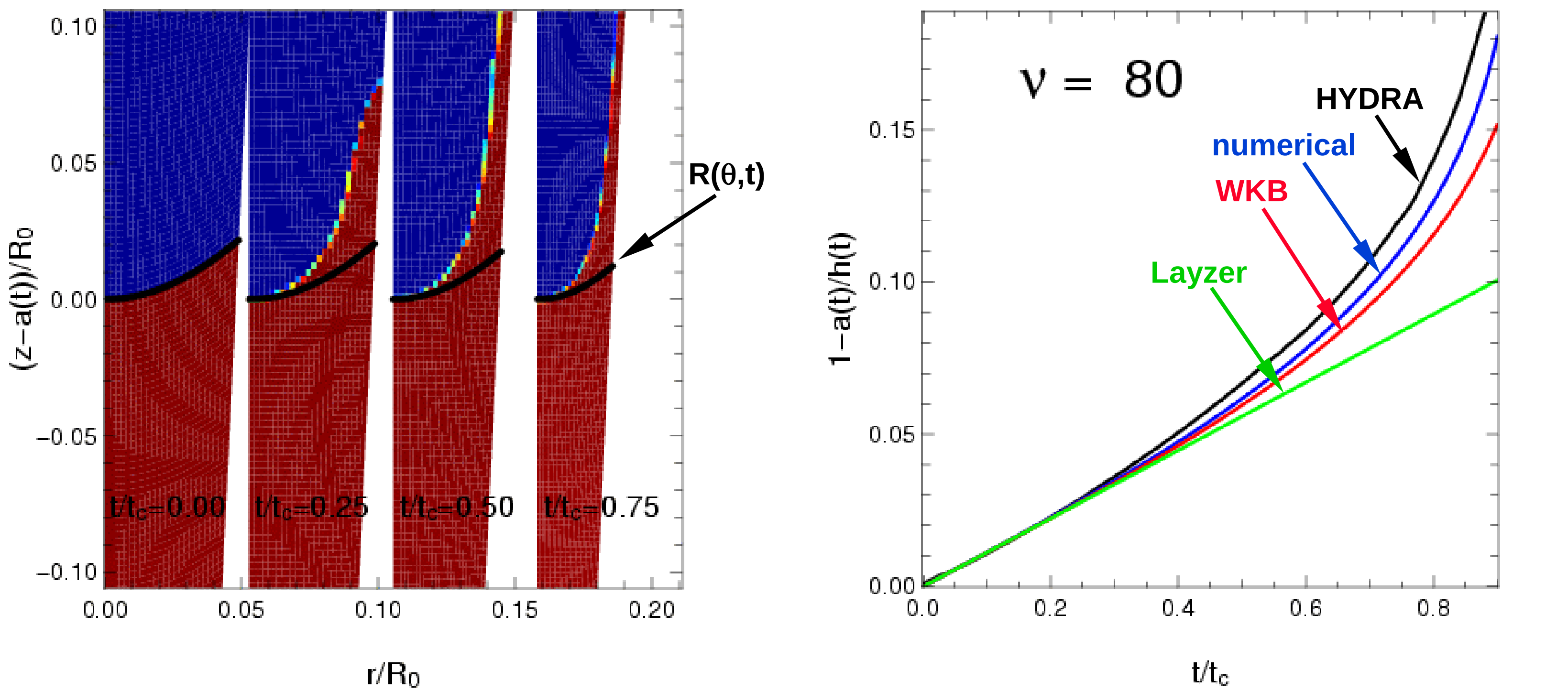


In the cylindrical limit ($\nu \to \infty$) and for early times ($t/t_c << 1$), Layzer-like bubble growth and bubble curvature are recovered

$$1-\frac{a(t)}{R_0} \sim 1-\left(\frac{1+t/t_c}{1-t/t_c}\right)^{-1/2\sqrt{\nu}} \sim \frac{1}{\sqrt{\nu}}\frac{t}{t_c} \quad \text{and} \quad \frac{1}{2}\frac{R_{\theta\theta}}{R_0} \sim \frac{\nu}{8}.$$

Full 2-D hydrodynamics simulations confirm acceleration of bubble growth

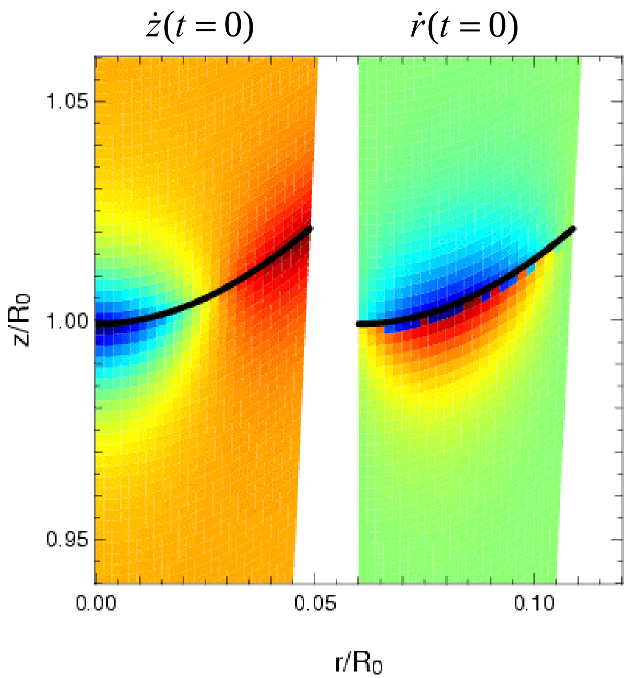
Apply external Kidder pressure source in HYDRA simulations with imposed nonlinear perturbation and measure bubble growth.

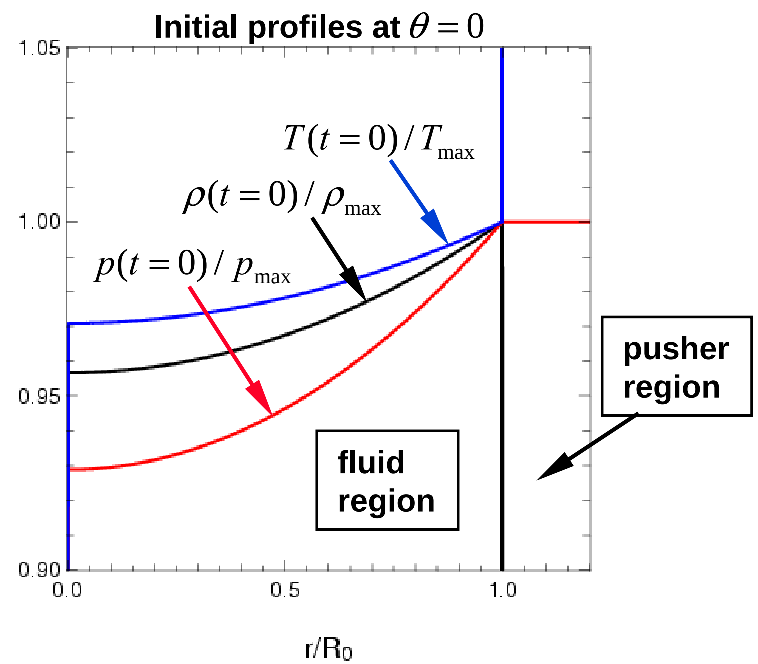


Agreement with the theoretical bubble curvature on axis and bubble rise rate are evident. Similar results were found for larger and smaller values of ν.

Careful specification of the nonlinear initial conditions was required

Careful specification of the initial conditions according to the theoretical $\rho(t=0)$, $p(t=0)$, and $\phi(t=0)$ (in the dense fluid as well as the low density pusher) was required to avoid shock formation and obtain "correct" nonlinear bubble growth.





Substantial ALE relaxation of the mesh was also required throughout the simulation.

Summary & Conclusions

- Layzer's model quite accurately describes nonlinear RT bubble growth but is strictly valid only in planar geometry.
- By combining Kidder's self-similar implosion with a Layzer-like bubble rise model in spherical geometry, a bubble model for spherical interfaces more relevant to IFE can be developed.
- Model reveals initial bubble growth linear in time like Layzer, followed by late time acceleration.
- Comparison with full, 2-D numerical simulations confirms bubble curvature and bubble acceleration predictions but requires care in specifying initial conditions.
- Extending the theory from the imploding solid sphere considered so far to imploding shells and non-self-similar flows is underway.