Chapter 7

Theoretical Aspects of Neutrino Masses and Mixings

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Neutrino oscillation experiments have yielded valuable information on the nature of neutrino masses and mixings and have provided the first glimpse of new physics beyond the standard model. Even though we are far from a complete understanding of the new physics implied by them, some tell-tale hints are emerging which have narrowed the direction of the new physics and have provided some insight into the flavor problem. In these lectures, I provide a panoramic overview of the current thinkings in neutrino model building.

7.1. Introduction

For a long time, it was believed that neutrinos are massless, spin half particles, making them drastically different from their other standard model spin half cousins such as the charged leptons (e, μ, τ) and the quarks (u, d, s, c, t, b), which are known to have mass. This myth has however been shattered by the accumulating evidence for neutrino mass from the solar and atmospheric neutrino observations compiled in the nineties as well as several terrestrial experiments in the new century. One must therefore now be free to look beyond the massless neutrino idea to explore new physics as we proceed to understand the neutrino mass.

The possibility of a nonzero neutrino mass at phenomenological level goes back almost 50 years. In the context of gauge theories, they were discussed extensively in the 70's and 80's long before there was any firm evidence for it. For instance the left-right symmetric theories of weak interactions introduced in 1974 and discussed in those days in connection with the structure of neutral current weak interactions, predicted nonzero neutrino mass as a necessary consequence of parity invariance and quark lepton symmetry.

The existence of a nonzero neutrino mass makes neutrinos more like the quarks, and allows for mixing between the different neutrino species leading to the phenomenon of neutrino oscillation, an idea first discussed by Pontecorvo[1] and Maki et al.[1] in the 1960's, unleasing a whole new realm of particle physics phenomena to explore beyond the standard model. At the present time, we are of course far from a complete picture of the masses and mixings of the various neutrinos and cannot therefore have a full outline of the theory of neutrino masses. However there exist enough information that we can surmise some viable possibilities for the theories beyond the standard model. Combined with other ideas outside the neutrino arena such as supersymmetry and unification, the possibility narrows even further. Many clever experiments now under way will soon clarify or rule out many of the allowed models. In these lectures, I give a panoramic view of what we may have learned about physics beyond the standard model and new symmetries of flavor as we attempt to understand neutrino masses.[2]

For simplicity, I will focus on a widely discussed framework for understanding of the small neutrino masses, the seesaw mechanism, which employs a minimal extension of the standard model by adding two or three right handed neutrinos, which are super-heavy and Majorana type. I will touch briefly on some specific models that are based on the above general framework but attempt to provide an understanding of the detailed mass and mixing patterns using family symmetries which must supplement the seesaw mechanism. We will also present a class of SO(10) grand unified theories where there is no need for family symmetries to understand large mixings. These works are instructive for several reasons: first they provide an existence proof that this is a sensible way to proceed in tackling the hard problem of understanding large lepton mixings; second they often illustrate the kind of assumptions needed and through that provide a unique insight into which directions the next step should be; finally of course nature may be generous in picking one of those models as the final message bearer.

The fact that the neutrino has no electric charge endows it with certain properties not shared by other fermions of the standard model. One can write two kinds of Lorentz invariant mass terms for the neutrino, the Dirac and Majorana masses, whereas for the charged fermions, conservation of electric charge allows only Dirac type mass terms. In the four component notation for the fermions, the Dirac mass has the form $\bar{\psi}\psi$, whereas the Majorana mass is of the form $\psi^T C^{-1}\psi$, where ψ is the four component spinor and C is the charge conjugation matrix. One can also discuss the two different kinds of mass terms using the two component notation for the spinors, which provides a very useful way to discuss neutrino masses. We therefore present some of the salient concepts behind the two component description of the neutrino.

7.1.1. *Two component notation for neutrinos*

Before we start the discussion of the 2-component neutrino, let us write down the Dirac equation for an electron:[3]

$$i\gamma^{\lambda}\partial_{\lambda}\psi - m\psi = 0 \tag{7.1}$$

This equation follows from a free Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^{\lambda}\partial_{\lambda}\psi - m\bar{\psi}\psi \tag{7.2}$$

and leads to the relativistic energy momentum relation $p^{\lambda}p_{\lambda} = m^2$ for the spin-half particle only if the four γ_{λ}'s anticommute. If we take γ_{λ}'s to be $n \times n$ matrices , the smallest value of n for which four anticommuting matrices exist is four. Therefore ψ must be a four component spinor. The physical meaning of the four components is as follows: two components for particle spin up and down and same for the antiparticle.

A spin-half particle is said to be a Majorana particle if the spinor field ψ satisfies the condition of being self charge conjugate, i.e.

$$\psi = \psi^c \equiv C\bar{\psi}^T, \tag{7.3}$$

where C is the charge conjugation matrix and has the property $C\gamma_{\lambda}C^{-1} = -\gamma^{\lambda T}$. This constraint reduces the number of independent components of the spinor by a factor of two, since the particle and the antiparticle are now the same particle. Using this condition, the mass term in the Lagrangian in Eq. (2) can be written as $\psi^T C^{-1}\psi$, where we have used the fact that C is a unitary matrix. Writing the mass term in this way makes it clear that if a field carries a $U(1)$ charge and the theory is invariant under those $U(1)$ transformations, then the mass term breaks this symmetry. This means that one cannot impose the Majorana condition on a particle that has a gauge charge. Since the neutrinos do not have electric charge, they can be Majorana particles unlike the quarks, electron or the muon. It is of course well known that the gauge boson interactions in a gauge theory Lagrangian conserve a global $U(1)$ symmetry known as lepton number with the neutrino and electron carrying the same lepton number. If lepton number were to be established as an exact symmetry of nature, the Majorana mass for the neutrino would be forbidden and the neutrino, like the electron, would be a Dirac particle.

The properties of a Majorana fermion can be seen in its free field expansion in terms of creation and annihilation operators:

$$\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \Sigma_s \left(a_s(\mathbf{p}) u_s(\mathbf{p}) e^{-ip\cdot x} + a_s^\dagger v_s(\mathbf{p}) e^{ip\cdot x} \right). \quad (7.4)$$

In the gamma matrix convention where $\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$ and $\gamma_0 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}$, the u_s and v_s are given by

$$u_s(\mathbf{p}) = \frac{m}{\sqrt{E}} \begin{pmatrix} \alpha_s \\ \frac{E-\sigma\cdot\mathbf{p}}{m}\alpha_s \end{pmatrix} \quad (7.5)$$

and

$$v_s(\mathbf{p}) = \frac{m}{\sqrt{E}} \begin{pmatrix} -\frac{E+\sigma\cdot\mathbf{p}}{m}\alpha_s' \\ \alpha_s' \end{pmatrix}. \quad (7.6)$$

α_s and α_s' are two component spinors.

If we choose $\alpha_s' = \sigma_2 \alpha_s$, we get the relation among the spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ $C\gamma_0 u_s^*(\mathbf{p}) = v_s(\mathbf{p})$ and the Majorana condition follows. Note that if ψ were to describe a Dirac spinor, then we would have had a different creation operator $b^\dagger$ in the second term in the free field expansion above.

The origin of the two component neutrino is rooted in the isomorphism between the Lorentz group and the SL(2, C) group. The latter is defined as the set of 2×2 complex matrices with unit determinant, whose generators satisfy the same Lie algebra as that of the Lorentz group. Its basic representations are 2 and 2^* dimensional. These are the spinor representations and can be used to describe spin half particles.

We can therefore write the familiar 4-component Dirac spinor used in the text books to describe an electron can be written as $\psi = \begin{pmatrix} \phi \\ i\sigma_2\chi^* \end{pmatrix}$, where χ and ϕ two two component spinors. A Dirac mass is the given by $\chi^T\sigma_2\phi$ whereas a Majorana mass is given by $\chi^T\sigma_2\chi$, where σ_a are the Pauli matrices. To make correspondence with the four component notation, we point out that ϕ and $i\sigma_2\chi^*$ are nothing but the ψ_L and ψ_R respectively. It is then clear that χ and ϕ have opposite electric charges; therefore the Dirac mass $\chi^T\sigma_2\phi$ maintains electric charge conservation (as well as any other kind of charge like lepton number etc.).

2-component neutrino is described by the following Lagrangian:

$$\mathcal{L} \; = \; \nu^\dagger i\sigma^\lambda \partial_\lambda \nu - \frac{im}{2} e^{i\delta} \nu^T \sigma_2 \nu + \frac{im}{2} e^{-i\delta} \nu^\dagger \sigma_2 \nu^* . \quad (7.7)$$

This leads to the following equation of motion for the field χ

$$i\sigma^{\lambda}\partial_{\lambda}\chi - im\sigma_2\chi^* = 0 \tag{7.8}$$

. As is conventionally done in field theories, we can now give a free field expansion of the two component Majorana field in terms of the creation and annihilation operators:

$$\chi(x,t) = \sum_{p,s}[a_{\mathbf{p},s}\alpha_{p,s}e^{-ip.x} + a^{\dagger}_{\mathbf{p},s}\beta_{p,s}e^{ip.x}], \tag{7.9}$$

where the sum on s goes over the spin up and down states.
Exercise 1: Using the field equations for a free massive two component Majorana spinor, show that its expansion in terms of the creation and annihilation operators and two component spinors $\alpha = \begin{pmatrix}1\\0\end{pmatrix}$ and $\beta = \begin{pmatrix}0\\1\end{pmatrix}$ is given by the following expression:

$$\begin{aligned}\chi(x,t) &= \sum_{p}[a_{\mathbf{p},+}e^{-ip.x} - a^{\dagger}_{\mathbf{p},-}e^{ip.x}]\alpha\sqrt{E+p} \\ &+ \sum_{p}[a_{\mathbf{p},-}e^{-p.x} + a^{\dagger}_{\mathbf{p},+}e^{ip.x}]\beta\sqrt{E-p}.\end{aligned} \tag{7.10}$$

Note that in a beta decay process, where a neutron is annihilated and proton is created, the leptonic weak current that is involved is $\bar{e}\nu$ (dropping gamma matrices); therefore, along with the electron, what is created predominantly is a right handed particle (with a wave function α), the amplitude being of order $\sqrt{E+p} \approx \sqrt{2E}$. This is the right handed anti-neutrino. The left handed neutrino is produced with a much smaller amplitude $\sqrt{E-p} \approx m_{\nu}/E$. Similarly, in the fusion reaction in the core of the Sun, what is produced is a left handed state of the neutrino with a very tiny i.e. $O(m_{\nu}/E)$ admixture of the right handed helicity (or the "anti-neutrino" component).

7.1.2. *Neutrinoless double beta decay and neutrino Majorana mass*

As already noted a Majorana neutrino breaks lepton number by two units. This has the experimentally testable prediction that it leads to the process of neutrino-less double beta decay, that involves the decay of an even-even nucleus i.e. $(Z,N) \to (Z+2,N-2)+2e^-$. We will now show by using the above property of the Majorana neutrino that if light neutrino exchange is responsible for this process, then the amplitude is proportional to the neutrino mass.

Double beta decay involves the change of two neutrons to two protons and therefore has to be a second order weak interaction process. Since each weak interaction process emits an antineutrino, in second order weak interaction, the final state will involve two anti-neutrinos. But in neutrino-less double beta decay, there are no neutrinos in the final state; therefore the two neutrinos must go into the vacuum state. Vacuum state by definition has no spin whereas the antineutrino emitted in a beta decay has spin. Consider the antineutrino from one of the decays: it must be predominantly right handed. But to disappear into vacuum, it must combine with a lefthanded antineutrino so that the left and right handed spin projections add up to zero. In the previous paragraph, we showed that the fraction of left handed spin projection in a neutrino emitted in beta decay is m_ν/E. Therefore, $\bar{\nu}_e\bar{\nu}_e \to |0>$ must be proportional to the neutrino mass. Thus neutrino-less double beta decay is therefore a very sensitive measure of neutrino mass.

7.1.3. *Neutrino mass in two component notation*

Let us now discuss the general neutrino mass for Majorana neutrinos. We saw earlier that for Majorana neutrinos, there are two different ways to write a mass term consistent with relativistic invariance. This richness in the possibility for neutrino masses also has a down side in the sense that in general, there are more parameters describing the masses of the neutrinos than those for the quarks and leptons. For instance for the electron and quarks, dynamics (electric charge conservation) reduces the number of parameters in their mass matrix. As an example, using the two component notation for all fermions, for the case of two two component spinors, a charged fermion mass will be described only by one parameters whereas for a neutrino, there will be three parameters. This difference increases rapidly e.g. for 2N spinors, to describe charged fermion masses, we need N^2 parameters (ignoring CP violation) whereas for neutrinos, we need $\frac{2N(2N+1)}{2}$ parameters. What is more interesting is that for a neutrino like particle, one can have both even and odd number of two component objects and have a consistent theory.

In this article, we will use two component notation for neutrinos. Thus when we say that there are N neutrinos, we will mean N two-component neutrinos.

In the two component language, all massive neutrinos are Majorana particles and what is conventionally called a Dirac neutrino is really a very specific choice of mass parameters for the Majorana neutrino. Let us give some examples: If there is only one two component neutrino (we will drop the prefix two component henceforth), it can have a mass $m\nu^T\sigma_2\nu$(to be called $\equiv m\nu\nu$ in shorthanded notation). The neutrino is now a self conjugate object which can be seen if we write an equivalent 4-component spinor ψ:

$$\psi = \begin{pmatrix} \nu \\ i\sigma_2\nu^* \end{pmatrix} \tag{7.11}$$

Note that this 4-component spinor satisfies the condition

$$\psi \ = \ \psi^c \equiv C\bar{\psi}^T \tag{7.12}$$

This condition implies that the neutrino is its own anti-particle, a fact more transparent in the four- rather than the two-component notation. The above exercise illustrates an important point i.e. given any two component spinor, one can always write a self conjugate (or Majorana) 4-component spinor. Whether a particle is really its own antiparticle or not is therefore determined by its interactions. To see this for the electrons, one may solve the following exercise i.e. if we wrote two Majorana spinors using the two two-component spinors that describe the electron, then until we turn on the electromagnetic interactions and the mass term, we will not know whether the electron is its own antiparticle or not. Once we turn on the electromagnetism, this ambiguity is resolved since electric charge conservation will allow a mass term that connects the two 2-component spinors and no mass term connecting either of the two component spinors with themselves.

Let us now go one step further and consider two 2-component neutrinos (ν_1, ν_2). The general mass matrix for this case is given by:

$$\mathcal{M}_{2\times 2} \ = \ \begin{pmatrix} m_1 & m_3 \\ m_3 & m_2 \end{pmatrix} \tag{7.13}$$

Note first that this is a symmetric matrix and can be diagonalised by orthogonal transformations. The eigen-states which will be certain admixtures of the original neutrinos now describe self conjugate particles. One can look at some special cases:

Case i:

If we have $m_{1,2} = 0$ and $m_3 \neq 0$, then one can assign a charge +1 to ν_1 and -1 to ν_2 under some $U(1)$ symmetry other than electromagnetism and the theory is invariant under this extra $U(1)$ symmetry which can be identified as the lepton number and the particle is then called a Dirac neutrino. The point to be noted is that the Dirac neutrino is a special case of for two Majorana neutrinos. In fact instead of calling this a Dirac neutrino, we could call this a case with two Majorana neutrinos with equal and opposite (in sign) mass. Since a complex mass term in general refers to its C transformation property (i.e. $\psi^c = e^{i\delta_m}\psi$, (where δ_m is the phase of the complex mass term), the two two-component fields of a Dirac neutrino having opposite sign mass would be equivalent to having opposite charge conjugation properties.

Case ii:

If we have $m_{1,2} \ll m_3$, this case is called pseudo-Dirac neutrino since this is a slight departure from case (i). In reality, in this case also the neutrinos are Majorana neutrinos with their masses $\pm m_0 + \delta$ with $\delta \ll m_0$. The two component neutrinos will be maximally mixed. Thus this case is of great current physical interest in view of the atmospheric (and perhaps solar) neutrino data.

Case iii:

There is third case where one may have $m_1 = 0$ and $m_3 \ll m_2$. In this case the eigenvalues of the neutrino mass matrix are given respectively by: $m_\nu \simeq -\frac{m_3^2}{m_2}$ and $M \simeq m_2$. One may wonder under what conditions such a situation may arise in a realistic gauge model. It turns out that if ν_1 transforms as an $SU(2)_L$ doublet and ν_2 is an $SU(2)_L$ singlet, then the value of m_3 is limited by the weak scale whereas m_2 has no such limit and $m_1 = 0$ if the theory has no $SU(2)_L$ triplet field (as for instance is the case in the standard model). Choosing $m_2 \gg m_3$ then provides a natural way to understand the smallness of the neutrino masses. This is known as the seesaw mechanism.[4] Since this case is very different from the case (i) and (ii), it is generally said that in grand unified theories, one expects the neutrinos to be Majorana particles. The reason is that in most grand unified theories there is a higher scale which under appropriate situations provides a natural home for the large mass m_2.

While we have so far used only two neutrinos to exemplify the various cases including the seesaw mechanism, these discussions generalize when $m_{1,2,3}$ are each $N \times N$ matrices (which we denote by $m_3 \equiv m_D$ and $m_2 \equiv M_R$). For example, the seesaw formula for this general situation can be written as

$$\mathcal{M}_\nu \simeq m_1 - m_D^T M_R^{-1} m_D \tag{7.14}$$

where the subscripts D and R are used in anticipation of their origin in gauge theories where M_D turns out to be the Dirac matrix and M_R is the mass matrix of the right handed neutrinos and all eigenvalues of M_R are much larger than the elements of M_D. It is also worth pointing out that Eq. 7.14 can be written in a more general form where the Dirac matrices are not necessarily square matrices but $N \times M$ matrices with $N \neq M$. We give such examples below.

Although there is no experimental proof that the neutrino is a Majorana particle, the general belief is that since the seesaw mechanism provides such a simple way to understand the glaring differences between the masses of the neutrinos and the charged fermions, neutrino is indeed most likely to be a Majorana particles as implied by it.

Even though in many situations, the difference between the Dirac and Majorana neutrinos is not manifest, there are some physical processes where differences becomes explicit: one such process is when the two neutrinos annihilate. For Dirac neutrinos, the particle and the antiparticle are distinct and therefore their annihilation is not restricted by Pauli principle in any manner. However, for the case of Majorana neutrinos, the identity of neutrinos and antineutrinos plays an important role and one finds that the annihilation to the Z-bosons occurs only via the P-waves. Similarly in the decay of the neutrino to any final state, the decay rate for the Majorana neutrino is a factor of two higher than for the Dirac neutrino.

7.1.4. *Experimental indications for neutrino masses*

There have been other lectures at this school on the experimental evidences for neutrino masses and their analyses to determine the current favorite values for the various mass differences as well as mixing angles. I will therefore only summarize the main results: (For detailed discussion and references, see[5] and lectures by J. Conrad[6]).

The evidence for neutrino masses and mixings have come from neutrino oscillation experiments involving neutrinos from the Sun, the cosmic rays as well as from accelerators. Neutrino oscillation is a phenomenon where neutrinos of one flavor transmute to neutrinos of another flavor. Since such transmutation can occur only if the neutrinos have masses and mixings, these experiments provide evidence for neutrino mass. To see this, note the expression for vacuum oscillation probability for neutrinos of a given energy E that have travelled a distance L is given by:

$$P_{\alpha\beta} \;=\; \sum_{i,j} |U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}| cos \left(\frac{\Delta_{ij} L}{2E} - \phi_{\alpha\beta, ij} \right) \tag{7.15}$$

This can be derived on the basis of simple quantum mechanical superposition principle and the equations of time evolution of free particles. The observed neutrino oscillation probabilities therefore yield information about the mass difference squares of the neutrinos ($\Delta_{ij} = m_i^2 - m_j^2$) and mixing angles $U_{\alpha i}$. If the neutrino propagates in dense matter, the oscillation probability is changed by the so-called Mikheyev-Smirnov-Wolfenstein effect however the new probability depends on the same parameters i.e. mass difference square and the $U_{\alpha i}$ in addition to depending on the density of matter through which neutrinos travel. The analysis of the data for the atmospheric neutrinos where the neutrino propagates in vacuum and that for solar neutrinos where the effect of dense matter in the Sun is included lead to the following picture values for mass differences and mixings:

7.1.4.1. *Atmospheric neutrinos:*

The Super-Kamiokande experiment observed the oscillations of the atmospheric muon neutrinos to tau neutrinos (although the tau neutrinos from the oscillations have not been confirmed yet). This oscillation of ν_μ to ν_τ has now been confirmed by the accelerator observations in the K2K and the MINOS experiments. From the existing data several important conclusions can be drawn: (i) the data cannot be fit assuming oscillation between ν_μ and ν_e nor $\nu_\mu - \nu_s$, where ν_s is a sterile neutrino which does not any direct weak interaction; (ii) the oscillation scenario that fit the data best is $\nu_\mu - \nu_\tau$ for the mass and mixing parameters

$$\begin{aligned} \Delta m^2_{\nu_\mu - \nu_\tau} &\simeq (2 \;-\; 8) \times 10^{-3} \; eV^2; \\ sin^2 2\theta_{\mu-\tau} &\geq 0.92 \end{aligned} \tag{7.16}$$

7.1.4.2. *Solar neutrinos*

The second evidence for neutrino oscillation comes from the several experiments that have observed a deficit in the flux of neutrinos from the Sun as compared to the predictions of the standard solar model championed by Bahcall and his collaborators[5] and more recently studied by many groups. The experiments responsible for this discovery are the Chlorine experiment of Ray Davis, Kamiokande, Gallex, SAGE, Super-Kamiokande, SNO, GNO experiments conducted at the Homestake mine, Kamioka in Japan, Gran Sasso in Italy and Baksan in Russia and Sudbery in Canada respectively. The different experiments see different parts of the solar neutrino spectrum. The details of these considerations are discussed in other lectures.

As far as the final state goes, it can either be one of the two remaining active neutrinos , ν_μ and ν_τ or it can be the sterile neutrino ν_s. SNO neutral current data announced recently has very strongly constrained the second possibility (i.e. the sterile neutrino in the final state). The global analyses of all solar neutrino data seem to favor the so called large mixing angle MSW solution with parameters: $\Delta m^2 \simeq 1.2 \times 10^{-5} - 3.1 \times 10^{-4} eV^2$; $sin^2 2\theta \simeq 0.58 \ - \ 0.95$.

This result has been confirmed by the terrestrial KamLand experiment which looked for oscillation for reactor neutrinos from several reactors with a detector at an average distance of about 100 Km from the various sources. It eliminated solutions to the solar neutrino puzzle based e.g. on spin flavor precession as well as the so called low solution and confirmed the large angle MSW resolution as the most plausible one.

7.1.4.3. *Search for the mixing angle θ_{13}*

The remaining mixing angle θ_{13} has been probed by the reactor experiments that used the French reactor CHOOZ and the US reactor in Palo-Verde by looking for the oscillation of reactor electron anti-neutrinos with a detector at a distance of about a kilo-meter. In the simple three neutrino picture, the dominant oscillation in this case would be to the tau neutrinos since $\Delta m^2_{31} \gg \Delta m^2_{21}$. The absence of a signal put an upper limit on the mixing angle of about $\theta_{13} \leq 0.17$. There are several reactor as well as long base-line experiments now being prepared which will conduct a higher precision search for θ_{13} during the next decade.[7]

7.1.4.4. *LSND and MiniBooNe*

Finally, we come to the last indication of neutrino oscillation from the Los Alamos Liquid Scintillation Detector (LSND)[6] experiment , where neutrino oscillations both from a stopped muon (DAR) as well as the one accompanying the muon in pion decay (known as the DIF) have been observed. The evidence from the DAR is statistically more significant and is an oscillation from $\bar{\nu}_\mu$ to $\bar{\nu}_e$. The mass and mixing parameter range that fits data is:[6]

$$LSND : \Delta m^2 \simeq 0.2 - 2eV^2; sin^2 2\theta \simeq 0.003 - 0.03 \tag{7.17}$$

There are also points at higher masses specifically at 6 eV^2 which are also allowed by the present LSND data for small mixings. KARMEN experiment at the Rutherford laboratory has very strongly constrained the allowed parameter range of the LSND data. Recently the Miniboone experiment at Fermilab has announced the results of its search for $\nu_\mu - \nu_e$ oscillation. They have not found any evidence for oscillation with characteristic mass diffrerence square in the eV range.[8,9]

7.1.4.5. *Neutrino-less double beta decay and Tritium decay experiment*

Oscillation experiments only depend on the difference of mass squares of the different neutrinos and the mixing angles. Therefore, in order to have a complete picture of neutrino masses, we need other experiments. Two such experiments are the neutrino-less double beta decay searches and the search for neutrino mass from the analysis of the end point of the electron energy spectrum in tritium beta decay. They have been discussed at this institute by P. Vogel,[11] which is referred to for more details and references.

Neutrino-less double beta decay measures the following combination of masses and mixing angles:

$$< m >_{\beta\beta} \quad = \quad \sum_i U_{ei}^2 m_i \tag{7.18}$$

Therefore naively speaking it is sensitive to the overall neutrino mass scale. But in practice, as we will see below, for the case of both normal and inverted hierarchies, it is unlikely to settle the question of the overall mass scale at the presently contemplated level of sensitivity in double beta decay searches. Only if the neutrino mass patterns are inverse hierarchical does one expect a visible signal in $\beta\beta_{0\nu}$ decay. We do not get into great details into this issue except to mention that in drawing any conclusions about neutrino mass from this process, one has to first have a good calculation of nuclear matrix elements of the various nuclei involves such as ^{76}Ge, ^{136}Xe, ^{100}Mo etc.; secondly, another confusing issue has to do with alternative physics contributions to $\beta\beta_{0\nu}$ which are unrelated to neutrino mass. Nevertheless, neutrino-less double beta decay is a fundamental experiment and a nonzero signal will establish the important result that neutrino is a Majorana particle and that lepton number symmetry is violated. Regardless of whether it tells us anything about the neutrino masses, it would provide an indication in favor of the seesaw mechanism. Presently two experiments

Heidelberg-Moscow and IGEX that use enriched ^{76}Ge have published limits of ≤ 0.3 eV. More recently, evidence for a double beta signal in the Heidelberg-Moscow data has been claimed.[12] Several experiments are now under planning e.g. EXO, Majorana and Cuore etc. which are expected to improve the sensitivity to the Majorana mass of the neutrino to the level of 20 milli-eV. This can for example test the hypothesis that neutrino mass ordering may be inverted if they are Majorana fermions.

Another important result in further understanding of neutrino mass physics could come from the tritium end point searches for neutrino masses. This experiment will measure the parameter $m_\nu = \sqrt{\sum_i |U_{ei}|^2 m_i^2}$. This involves a different combination of masses and mixing angles than $< m >_{\beta\beta}$. Presently, the KATRIN proposal for a high sensitive search for for m_ν has been made and it is expected that it can reach a sensitivity of 0.2 eV.

A third source of information on neutrino mass will come from cosmology, where more detailed study of structure in the universe is expected to provide an upper limit on $\sum_i m_i$ of less than an eV. Present WMAP data appears to provide a limit of $\sum_i m_{\nu_i} \leq 0.6 - 1$ eV.[13]

Our goal now is to study the theoretical implications of these discoveries. We will proceed towards this goal in the following manner: we will isolate the mass patterns that fit the above data and search for patterns and symmetries that lead to observed mixing angles. We then look for plausible models that can first lead to the general feature that neutrinos have tiny masses; then we would try to understand in simple manner some of the features indicated by data in the hope that these general ideas will be part of our final understanding of the neutrino masses. As mentioned earlier on, to understand the neutrino masses one has to go beyond the standard model. First we will sharpen what we mean by this statement. Then we will present some ideas which may form the basic framework for constructing the detailed models.

7.2. Physics of neutrino mass

There are two distinct aspects to neutrino mass physics: first is the absolute overall magnitude and second is the flavor structure. Understanding the first will reveal the gross features of the new physics such as the presence of a new symmetry and its scale responsible for the smallness of neutrino mass compared to masses of other fundamental fermions, whereas understanding the flavor pattern is likely to throw light on possible new family symmetries of matter which may in turn be relevant to unravelling the mystery of quark flavor. It could be (probably likely) that both are related to each other.

To begin this discussion, we first list the puzzles of neutrino mass physics; we then discuss the neutrino mass matrix which is the starting point of many attempts to understand the neutrino mixings and then discuss ideas that have been proposed to understand these patterns before going to a discussion of the overall scale.

7.2.1. *Puzzles of neutrino mass physics*

The present neutrino discoveries have posed a list of puzzles for physics beyond the standard model, whose resolution will provide an unmistakable path beyond it. Below we give a list of these puzzles.

- *Ultra-light-ness of neutrinos:* Why are the neutrino masses so much lighter than the quark and charged lepton masses ?
- *Bi-large mixing:* How to understand simultaneously two large mixing angles one for the $\mu - \tau$ and another for $e - \mu$?
- *Smallness of* $\Delta m^2_\odot/\Delta m^2_A$ *:* Experimentally, $\Delta m^2_\odot \simeq 10^{-2}\Delta m^2_A$. How does one understand this in a natural manner ?
- *Smallness of* U_{e3}*:* The reactor results also seem to indicate that the angle $\theta_{13} \equiv U_{e3}$ is a very small number. One must also understand this in a framework that simultaneously explains all other puzzles.

Possible other puzzles include a proper understanding of neutrino mass degeneracy if there is a large positive signal for the neutrinoless double beta decay and of course, when we have evidence for CP violating phases in the mass matrix, we must understand their magnitude.

In order to take the first step towards understanding these puzzles, we discuss the flavor pattern of leptons that may be at the root of large mixings and defer the discussion of the origin of mass scale to the next section and finally focus on specific unification models which address both the issues as examples how one may proceed to unravel the grand picture of physics beyond standard model inspired by neutrino physics. It could of course be that large neutrino mixings result from a joint effect of both the charged lepton and the neutrino matrix since $U_{PMNS} = U^\dagger_l U_\nu$ and we do not know apriori whether the large neutrino mixings come from the charged lepton sector or the neutrino sector or both. However, we first follow the line of thinking that in the fundamental theory, charged lepton mass matrix is diagonal or near diagonal and all mixings result from the neutrino mass matrix. This point of view is not so unreasonable since charged lepton mass matrix is likely to be similar to the quark sector and the small observed CKM mixings pretty much guarantees that quark mass matrices are near diagonal. We will also present grand unified models where this conjecture is borne out.

7.2.2. *Notation*

We will assume two component neutrinos and therefore their masses will in general be Majorana type. Let us also give our notation to facilitate further discussion: the neutrinos emitted in weak processes such as the beta decay or muon decay are weak eigenstates and are not mass eigenstates. The mass eigenstates determine how a neutrino state evolves in time. Similarly, in the detection process, it is the weak eigenstate that is picked out. This is of course the key idea behind neutrino oscillation and the formula presented in the last section. To set the notation, let us express the weak eigenstates in terms of the mass eigenstates. We will denote the weak eigenstate by the symbol α, β or simply e, μ, τ etc whereas the mass eigenstate will be denoted by the symbols i, j, k etc. To relate the weak eigenstates to the mass eigenstates, let us start with the mass terms in the Lagrangian for the neutrino and the charged leptons:

$$\mathcal{L}_m \;=\; \nu_L^T \mathcal{M}_\nu \nu_L + \bar{E}_L M_\ell E_R + h.c. \tag{7.19}$$

Here the ν and E which denote the column vectors for neutrinos and charged leptons are in the weak basis. To go to the mass basis, we diagonalize these matrices as follows:

$$\begin{aligned} U_L^T \mathcal{M}_\nu U_L &= d_\nu \\ V_L M_\ell V_R^\dagger &= d_\ell \end{aligned} \tag{7.20}$$

The physical neutrino mixing matrix is then given by:

$$\mathbf{U} \;=\; V_L U_L \tag{7.21}$$

$\mathbf{U}_{\alpha i}$ and relate the two sets of eigenstates (weak and mass) as follows:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \;=\; U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{7.22}$$

Using this equation, one can derive the well known oscillation formulae for the survival probability of a particular weak eigenstate α discussed in the previous section.

To see the general structure of the mixing matrix $\mathbf{U}$, let us recall that the matrix $\mathcal{M}_\nu$ is complex and symmetric and therefore has six complex parameters describing it for the case of three generations. But since the neutrino is described by a complex field, we can redefine the phases of three fields to remove three parameters. That leaves nine parameters. In terms of observables, there are three mass eigenvalues (m_1, m_2, m_3) and three mixing angles and phases in the mixing matrix $\mathbf{U}$. The three phases can be split into one Dirac phase, which is analogous to the phase in the quark mixing matrix and two Majorana phases. We can then write the matrix $\mathbf{U}$ as

$$\mathbf{U} \;=\; U^{(0)} \begin{pmatrix} 1 & & \\ & e^{i\phi_1} & \\ & & e^{i\phi_2} \end{pmatrix} \tag{7.23}$$

The matrix $\mathbf{U^{(0)}}$ has three real angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a phase. The goal of experiments is to determine all nine of these parameters. The knowledge of the nine observables allows one to construct the mass matrix for the neutrinos and from there one can go in search of the new physics beyond the standard model that leads to such a mass matrix.

The neutrino mass observables given above can be separated into two classes: (i) oscillation observables and (ii) non-oscillation observables. The first class of observables are those accessible to neutrino oscillation experiments and are the two mass differences $\Delta m^2_\odot$ and Δm^2_A; three mixing angles θ_{12} (or $\theta_\odot$); θ_{23} (or θ_A) and θ_{13} (the reactor angle, also called U_{e3}) and the CP phase δ in $U^{(0)}$. The remaining three observables i.e. the lightest mass of the three neutrinos and the two Majorana phases $\phi_{1,2}$ can only be probed by nonoscillation experiments such as $\beta\beta_{0\nu}$ decay, beta decay spectrum at the endpoint and cosmological observations etc.

7.3. Neutrino mixing matrix and mass patterns

A good starting point for the exploration of new physics such as new symmetries, new scaleshidden in neutrino observations is to construct the neutrino mass matrix in the basis where the charged leptons are mass eigenstates (or near mass eigen states.) One can then look for symmetries which could be responsible for this form of the neutrino mass matrix in extensions of the standard model.

In order to construct the neutrino mass matrix, we will use the following experimental numbers: $\Delta m^2_A \simeq 0.0021$ eV^2; for solar neutrinos, it gives

$\Delta m^2_{\odot} \simeq (7.21 - 8.63) \times 10^{-5}\,\mathrm{eV}^2$. It also provides information on the angles in $\mathbf{U}$ which can be summarized by the following mixing matrix (neglecting all CP phases):

$$\mathbf{U} \;=\; \begin{pmatrix} c & s & \epsilon \\ -\frac{s+c\epsilon}{\sqrt{2}} & \frac{c-s\epsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{s-c\epsilon}{\sqrt{2}} & \frac{-c-s\epsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{7.24}$$

where from the discussion above $\epsilon \leq 0.17$; $s = \sin\theta_{12}$ is in the range $0.267 \leq sin^2\theta_{12} \leq 0.371$ with the central value being near 0.314. We have chosen the atmospheric mixing angle θ_{23} to be maximal.

A particularly interesting form of the mixing matrix which seems to be in accord with data is the so-called tri-bi-maximal form[15] where:

$$U_{PMNS} \;=\; \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{7.25}$$

As far as the mass pattern goes however, there are three possibilities:

- (i) normal hierarchy: $m_1 \ll m_2 \ll m_3$;
- (ii) inverted hierarchy : $m_1 \simeq -m_2 \gg m_3$ and
- (iii) approximately degenerate pattern[14] $m_1 \simeq m_2 \simeq m_3$,

where m_i are the eigenvalues of the neutrino mass matrix. In the first case, the atmospheric and the solar neutrino data give direct information on m_3 and m_2 respectively. On the other hand, in the last case, the mass differences between the first and the second eigenvalues will be chosen to fit the solar neutrino data and the second and the third to fit the atmospheric neutrino data.

7.3.1. *Neutrino mass textures*

From the mixing matrix in Eq. 7.24, we can write down the allowed neutrino mass matrix for any arbitrary mass pattern assuming the neutrino is a Majorana fermion. Denoting the matrix elements of $\mathcal{M}_\nu$ as $\mu_{\alpha\beta}$ for $\alpha, \beta =$

1, 2, 3, we have(Recall that $\mu_{\alpha\beta} = \mu_{\beta\alpha}$):

$$\begin{aligned}
\mu_{11} &= [c^2 m_1 + s^2 m_2 + \epsilon^2 m_3] \\
\mu_{12} &= \frac{1}{\sqrt{2}}[-c(s + c\epsilon)m_1 + s(c - s\epsilon)m_2 + \epsilon m_3] \\
\mu_{13} &= \frac{1}{\sqrt{2}}[-c(s - c\epsilon)m_1 - s(c + s\epsilon)m_2 + \epsilon m_3] \\
\mu_{22} &= \frac{1}{2}[(s + c\epsilon)^2 m_1 + (c - s\epsilon)^2 m_2 + m_3] \\
\mu_{23} &= \frac{1}{2}[-(s^2 - c^2\epsilon^2)m_1 - (c^2 - s^2\epsilon^2)m_2 + m_3] \\
\mu_{33} &= \frac{1}{2}[(s - c\epsilon)^2 m_1 + (c + s\epsilon)^2 m_2 + m_3].
\end{aligned} \tag{7.26}$$

In certain limits for the mixing angles in the above mass matrix, symmetries leptons can manifest. Below I give some examples:

7.3.2. *$\mu - \tau$ exchange symmetry*

The current observations that θ_{23} is very close to maximal (with $\theta_{23} = \frac{\pi}{4}$ being the best fit solution in many analyses) and the fact that θ_{13} could be vanishing can be understood if the neutrino Majorana mass matix has a Z_2 symmetry that interchanges $\mu - \tau$.[16] The corresponding mass matrix is the special case of the matrix below with $c = 1$, and $a = b$.

$$\mathcal{M}_\nu = \sqrt{\Delta m^2_A}\begin{pmatrix} d\epsilon^n & b\epsilon & a\epsilon \\ b\epsilon & 1+\epsilon & 1 \\ a\epsilon & 1 & 1+c\epsilon \end{pmatrix} \tag{7.27}$$

where a, b, c, d are parameters of order one and $\epsilon \sim \sqrt{\frac{\Delta m^2_\odot}{\Delta m^2_A}} \sim 0.2$. Since at the moment it is not certain whether $\mu - \tau$ symmetry is exact or approximate, the mass matrix in Eq.7.27 includes small breaking terms characterized by $a \neq b$ and $c \neq 1$. These small departures lead to non-zero values for θ_{13} and $\theta_{23} - \frac{\pi}{4}$ which are correlated with each other.[17] We have ignored leptonic CP violation in this discussion. One can have other ways of introducing $\mu - \tau$ symmetry breaking using CP phases.[18]

Overall, this symmetry has a good chance be part of the final theory of neutrino mixing since there seems to be good experimental support for it. This has therefore led to a great deal of model building activity[19] most of whom predict departures from the exact symmetry limit and can provide insight into which way to proceed in assimilating this symmetry as part of the quark-lepton world.

7.3.3. *Mass matrix for tri-bi-maximal mixing and associated approximate flavor symmetries*

In the $\mu - \tau$ symmetric models, the value of the solar mixing angle θ_{12} remains large but arbitrary. The indication that the value of $\sin\theta_{12} \simeq \frac{1}{\sqrt{3}}$ may be an indication of higher symmetries of the lepton world. This is called tri-bi-maximal mixing pattern.[15] Clearly these must have $\mu - \tau$ symmetry as a subgroup. A typical neutrino mass matrix that leads to the tri-bi-maximal mixing pattern is:

$$\mathcal{M}_\nu = \begin{pmatrix} a & b & b \\ b & a+c & b-c \\ b & b-c & a+c \end{pmatrix} \tag{7.28}$$

where a, b, c are arbitrary parameters. The charged lepton mass matrix is chosen to be diagonal. Diagonalizing this matrix leads to the U_{PMNS} in the tri-bi-maximal form and the neutrino masses: $m_1 = a - b; m_2 = a + 2b$ and $m_3 = a - b + 2c$. Clearly if $a \simeq b \ll c$, we get a normal hierarchy for masses. A symmetry that has been found to lead to this mass matrix is S_3.[20]

Another way to get the tri-bi-maximal form is to use specific forms for the charged lepton and neutrino mass matrices in such a way that the combined diagonalization leads to the desired lepton mixing matrix. An example is to have a charged lepton mass matrix of the form:

$$\mathcal{M}_\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \tag{7.29}$$

$$M_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$. This for may appear much too contrived to arise from some symmetries, but remarkably enough it has been shown that this can emerge under certain assumptions if the assumed symmetry is A_4,[21] which is group of even permutations of four elements.

Detailed theory for all these cases involves many Higgs multiplets and probably should not be taken literally but the important message is the presence of the hidden symmetry and its implications. Generally such symmetries are hard to grand unify and further research along this direction is going to be important.

7.3.4. *Inverted hierarchy and $L_e - L_\mu - L_\tau$ symmetry*

In this sub-section, we consider another interesting clue to model building present in neutrino data if the mass arrangement is inverted. It starts with the observation that if the neutrino mass matrix has the form

$$\mathcal{M}_\nu \;=\; \begin{pmatrix} 0 & A & A \\ A & 0 & 0 \\ A & 0b & 0 \end{pmatrix} \tag{7.30}$$

this leads to two degenerate neutrinos with mass $\pm\sqrt{2}A$ and one massless neutrino. The atmospheric mass difference is given by $\Delta m_A^2 = 2A^2$ and mixing angle $\theta_A = \pi/4$. As far as the solar ν_e oscillation is concerned, the $sin^2 2\theta_\odot = 1$ but $\Delta m_\odot^2 = 0$. While this is unphysical, this raises the hope that as corrections to this mass matrix are taken into account, it may be possible understand the smallness of $\Delta m_\odot^2/\Delta m_A^2$ naturally.

In fact this hope is fortified by the observation that this mass matrix has the leptonic symmetry $L_e - L_\mu - L_\tau$; therefore one might hope that as this symmetry is broken by small terms, one will end up with a situation that fits data well.

This question was studied in two papers.[25,26] To proceed with the discussion, let us consider the following mass matrix for neutrinos where small $L_e - L_\mu - L_\tau$ violating terms have been added.

$$\mathcal{M}_\nu = m \begin{pmatrix} z & c & s \\ c & y & d \\ s & d & x \end{pmatrix}. \tag{7.31}$$

The charged lepton mass matrix is chosen to have a diagonal form in this basis and $L_e - L_\mu - L_\tau$ symmetric. In the perturbative approximation, there are sum rules involving the neutrino observables and the elements of the neutrino mass matrix,[26] which first of all imply (i) a close connection between the measured value of the solar mixing angle and the neutrino mass measured in neutrino-less double beta decay; (ii) the present values for the solar mixing angle can be used to predict the $m_{\beta\beta}$ for a value of the $\Delta m_\odot^2$. For instance, for $sin^2 2\theta_\odot = 0.9$, we would predict $(\frac{\Delta m_\odot^2}{4\Delta m_A^2} - z) = 0.3$. For small $\Delta m_\odot^2$, this implies $m_{\beta\beta} \simeq 0.01$ eV. This is expected to be within the reach of new double beta decay experiments contemplated.[27] In fact now there exist more thorough numerical analyses[23] of general $L_e - L_\mu - L_\tau$ broken models which imply that the inverted hierarchy for neutrinos can be tested in neutrinoless double beta experiments in the next decade or so. This way of breaking $L_e - L_\mu - L_\tau$ symmetry also implies a value for $sin^2 2\theta_{12} \geq 0.9$, which is now almost ruled out. Of course there could be other ways of breaking this symmetry using charged lepton sector etc. which can still lead to lower solar angle.

If the value of $sin^2 2\theta_\odot$ is ultimately determined to be less than 0.9, the question one may ask is whether the idea of $L_e - L_\mu - L_\tau$ symmetry is dead. The answer is in the negative since so far we have explored the breaking of $L_e - L_\mu - L_\tau$ symmetry only in the neutrino mass matrix. It was shown in[25] that if the symmetry is broken in the charged lepton mass, one can lower the $sin^2 2\theta_\odot$ as long as the value of U_{e3} is sizable. However given the present upper limit on U_{e3}, the smallest value is somewhere around $sin^2 2\theta_\odot \simeq 0.8$.

7.3.5. *Scale invariant mass matrix*

Most of the above forms for the mass matrices are scale dependent in the sense that once radiative corrections are taken into account, their forms can change. This is specially relevant because in many theories neutrino mass matrix is predicted at a high scale due to physics at this scale. They have to be extrapolated to the weak scale to compare with experiments. This process relies on the nature of physics between the neutrino mass generation scale and the weak scale. Thus connection between fundamental physics responsible for neutrino masses and observations gets interrupted. Luckily the radiative correction effects are not significant if neutrino mass hierarchy is normal (see discussion later). However both for inverted and degenerate spectra, they are. It is therefore interesting to search for neutrino flavor structure that is not affected by such effects. One such form arises when the neutrino mass matrices satisfy certain scaling properties.[28] An example of such a mass matrix is:

$$\mathcal{M}_\nu = m_0 \begin{pmatrix} A & B & B/c \\ B & D & D/c \\ B/c & D/c & D/c^2 \end{pmatrix} . \tag{7.32}$$

It is called $\mu - \tau$ scaling whose most important phenomenological is that it leads to an inverted hierarchy with $m_3 = 0$ and $U_{e3} = 0$. Atmospheric neutrino mixing is governed by the "scaling factor" c via $\tan^2\theta_{23} = 1/c^2$, i.e., is in general non-maximal because c is naturally of order, but not equal to, one. These results are scale independent predictions and do not depend on extraneous physics between the neutrino mass generation scale and the weak scale. It is interesting to note that current data analyzes (though at the present stage statistically not very significant) yield non-maximal $\tan^2\theta_{23} = 0.89$ as the best-fit point.[5]

7.3.6. *CP violation*

A not very well explored aspect of neutrino physics at the moment is CP violation in lepton physics. Unlike the quark sector, CP violation for Majorana neutrinos allows for more phases than in the quark sector. Since the Majorana neutrino mass matrix is symmetric, for N generations of neutrinos, there are in general $\frac{N(N+1)}{2}$ phases in it. When the mass matrix is diagonalized, these phases will appear in the unitary matrix U_L that does the diagonalization (i.e. $U^T\mathcal{M}_\nu U_L = d_\nu$). If we are working in a basis where the charged lepton mass matrix is diagonal, then U_L is the leptonic weak mixing matrix. As we saw this has $N(N+1)/2$ phases. Out of them, redefinition of the charged lepton fields in the weak current allows the removal of N phases; so there are $N(N-1)/2$ phases in the neutrino masses. In the quark sector, both up and down fields could be redefined allowing for the number of physical phases that appear in the end to be smaller. However for Majorana neutrinos, redefinition of the fields does not remove the phases entirely from the theory but rather shifts them to other places where they can manifest themselves physically.[29] The detailed discussion of CP violation is in other lectures at this school.

7.4. Neutrino mass scale and physics beyond the standard model

In the standard model (SM), the neutrino mass vanishes to all orders in perturbation theory as well as nonperturbatively implying that observation of neutrino masses is the first laboratory evidence for physics beyond the standard model. To clarify this point, note that the SM is based on the gauge group $SU(3)_c\times SU(2)_L\times U(1)_Y$ under which the quarks and leptons, Higgs bosons and gauge bosons transform as described in the Table I.

Table I

Field	gauge transformation
Quarks Q_L	$(3, 2, \frac{1}{3})$
Righthanded up quarks u_R	$(3, 1, \frac{4}{3})$
Righthanded down quarks d_R	$(3, 1, -\frac{2}{3})$
Lefthanded Leptons L	$(1, 2-1)$
Righthanded leptons e_R	$(1, 1, -2)$
Higgs Boson $\mathbf{H}$	$(1, 2, +1)$
Color Gauge Fields G_a	$(8, 1, 0)$
Weak Gauge Fields $W^\pm$, Z, γ	$(1, 3+1, 0)$

Table caption: The assignment of particles to the standard model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$.

The electro-weak symmetry $SU(2)_L \times U(1)_Y$ is broken by the vacuum expectation of the Higgs doublet $< H^0 >= \frac{v_{wk}}{\sqrt{2}} \simeq 186$ GeV, which gives mass to the gauge bosons. The fermion masses arise from the Yukawa couplings:

$$\mathcal{L}_Y \;=\; h_u \bar{Q}_L H u_R + h_d \bar{Q}_L \tilde{H} d_R + h_e \bar{L} \tilde{H} e_R \;+\; h.c. \tag{7.33}$$

when H^0 acquires a vev. Note that since there are no right handed neutrinos in the theory, there is no term in Eq.7.33 that can give mass to the neutrinos. Thus they remain massless at the tree level.

There are several questions that arise at this stage. What happens when one goes beyond the above simple tree level approximation ? Secondly, do non-perturbative effects change this tree level result ? Finally, how to judge whether this result will be modified when the quantum gravity effects are included ?

The first and second questions are easily answered by using the B-L symmetry of the standard model. The point is that since the standard model has no $SU(2)_L$ singlet neutrino-like field, the only possible mass terms that are allowed by Lorentz invariance are of the form $\nu^T_{iL} C^{-1} \nu_{jL}$, where i, j stand for the generation index and C is the Lorentz charge conjugation matrix. Since the ν_{iL} is part of the $SU(2)_L$ doublet field and has lepton number $+1$, the above neutrino mass term transforms as an $SU(2)_L$ triplet and furthermore, it violates total lepton number (defined as $L \equiv L_e + L_\mu + L_\tau$) by two units. However, a quick look at the standard model Lagrangian convinces one that the model has exact lepton number symmetry after symmetry breaking; therefore such terms can never arise in perturbation theory. Thus to all orders in perturbation theory, the neutrinos are massless. As far as the nonperturbative effects go, the only known source is the weak instanton effects. Such effects could change the result if they broke the lepton number symmetry. One way to see if such breaking occurs is to look for anomalies in lepton number current conservation from triangle diagrams. Indeed it is easy to convince oneself that $\partial_\mu j^\mu_\ell = cW\tilde{W} + c'B\tilde{B}$ due to the contribution of the leptons to the triangle involving the lepton number current and W's or B's. Luckily, it turns out that the anomaly contribution to the baryon number current nonconservation has also an identical form, so that the $B-L$ current j^μ_{B-L} is conserved to all orders in the gauge couplings. As a consequence, nonperturbative effects from the gauge sector cannot induce $B-L$ violation. Since the neutrino mass operator described above violates also $B-L$, this proves that neutrino masses remain zero even in the presence of nonperturbative effects.

Let us now turn to the effect of gravity. Clearly as long as we treat gravity in perturbation theory, the above symmetry arguments hold since all gravity coupling respect $B-L$ symmetry. However, once nonperturbative gravitational effects e.g black holes and worm holes are included, there is no guarantee that global symmetries will be respected in the low energy theory. The intuitive way to appreciate the argument is to note that throwing baryons into a black hole does not lead to any detectable consequence except through a net change in the baryon number of the universe. Since one can throw in an arbitrary number of baryons into the black hole, an arbitrary information loss about the net number of missing baryons would prevent us from defining a baryon number of the visible universe- thus baryon number in the presence of a black hole can not be an exact symmetry. Similar arguments can be made for any global charge such as lepton number in the standard model. A field theoretic parameterization of this statement is that the effective low energy Lagrangian for the standard model in the presence of black holes and worm holes etc must contain baryon and lepton number violating terms. In the context of the standard model, the only such terms that one can construct are nonrenormalizable terms of the form $LHLH/M_{P\ell}$. After gauge symmetry breaking, they lead to neutrino masses; however these masses are at most of order $v^2_{wk}/M_{P\ell} \simeq 10^{-5}$ eV.[30] But as we discussed in the previous section, in order to solve the atmospheric neutrino problem, one needs masses at least three orders of magnitude higher.

Thus one must seek physics beyond the standard model to explain observed evidences for neutrino masses. While there are many possibilities that lead to small neutrino masses of both Majorana as well as Dirac kind, here we focus on the possibility that there is a heavy right handed neutrino (or neutrinos) that lead to a small neutrino mass. The resulting mechanism is known as the seesaw mechanism[4] and leads to neutrino being a Majorana particle.

The nature and origin of the seesaw mechanism can also be tested in other experiments and we will discuss them below. This will be dependent on the kind of operators that play a role in generating neutrino masses. If the leading order operator is of dimension 5, then the scale necessarily is very high (of order 10^{12} GeV or greater). On the other hand, in theories with extra space dimensions, this operator may be forbidden and one may be forced to go to higher dimensional operators, in which case the scale could be lower or it could be that neutrino Dirac Yukawa couplings are of order 10^{-6} (similar to the electron Yukawa coupling in the standard model) in which case even the seesaw scale could be in the TeV range.

The seesaw mechanism raises a very important question: since we require the mass of the right handed neutrino to be much less than the Planck scale, a key question is "what symmetry keeps the right handed neutrino mass lighter?" We will give two examples of symmetries that can do this.

7.4.1. *Seesaw and the right handed neutrino*

The simplest possibility extension of the standard model that leads to nonzero mass for the neutrino is one where only a right handed neutrino is added to the standard model. In this case ν_L and ν_R can form a mass term; but apriori, this mass term is like the mass terms for charged leptons or quark masses and will therefore involve the weak scale. If we call the corresponding Yukawa coupling to be Y_ν, then the neutrino mass is $m_D = Y_\nu v/\sqrt{2}$. For a neutrino mass in the eV range requires that $Y_\nu \simeq 10^{-11}$ or less which is far below even the small electron Yukawa coupling of SM. Introduction of such small coupling constants into a theory is generally considered unnatural and a sound theory must find a symmetry reason for such smallness. As already already alluded to before, seesaw mechanism,[4] where we introduce a singlet Majorana mass term for the right handed neutrino is one way to achieve this goal. The effective mass terms for the (ν_L, ν_R) system is given by (suppressing the generation index):

$$\mathcal{L}_m \;=\; m_D^* \bar{\nu}_L \nu_R \;+\; M_R^* \nu_R^T C^{-1} \nu_R \;+ h.c. \tag{7.34}$$

where $m_D = Y_\nu v_{wk}$, Y_ν is the lepton doublet coupling to the right handed neutrinos (defined as $\bar{\nu}_R Y_\nu L$) . Suppose we write ν spinor in terms of its two component spinors as $\nu = \begin{pmatrix}\nu \\ i\sigma_2 N^*\end{pmatrix}$, then $\nu_L = \begin{pmatrix}\nu \\ 0\end{pmatrix}$ and $\nu_R = \begin{pmatrix}0 \\ i\sigma_2 N^*\end{pmatrix}$. This gives remembering that $\gamma_0 = \begin{pmatrix}0 & I \\ I & 0\end{pmatrix}$ and $C^{-1} = \begin{pmatrix}i\sigma_2 & 0 \\ 0 & -i\sigma_2\end{pmatrix}$

$$\mathcal{L}_m = im_D^* \nu^\dagger \sigma_2 N^* + iM_R^* N^\dagger \sigma_2 N^* + h.c. \tag{7.35}$$

. We can write the neutrino mass matrix as:

$$\mathcal{L}_m = i \left(\nu^T \; N^T\right) \sigma_2 \mathbf{M} \begin{pmatrix}\nu \\ N\end{pmatrix} + h.c. \tag{7.36}$$

where $\mathbf{M}$ has the form:

$$\mathbf{M} = \begin{pmatrix}0 & m_D \\ m_D^T & M_R\end{pmatrix} \tag{7.37}$$

Since M_R is not constrained by the standard model symmetries, it is natural to choose it to be at a scale much higher than the weak scale. Now diagonalizing this mass matrix , we get a set of heavy eigenstate N_R and a set of light eigenstates with mass matrix given by:

$$\mathcal{M}_\nu \simeq -m_D^T M_R^{-1} m_D \tag{7.38}$$

This provides a natural way to understand a small neutrino mass without any unnatural adjustment of parameters of a theory. In a subsequent section, we will discuss a theory which connects the scale M_R to a new symmetry of nature beyond the standard model. This formula for neutrino masses is called type I seesaw formula.

7.4.1.1. *Why is $M_R \ll M_{P\ell}$?*

The question "why $M_R \ll M_{P\ell}$?" is in many ways similar to the question in the standard model i.e. "why is $M_{Higgs} \ll M_{P\ell}$?" It is well known that searches for answer to this latter question has led to many interesting possibilities for physics beyond the standard model including supersymmetry, extra dimensions and technicolor. It is hoped that answering this question for ν_R can also lead us to new insight into new symmetries beyond the standard model. There are two interesting answers to our question that I will elaborate later on.

$B - L$:

If one adds three right handed neutrinos to implement the seesaw mechanism, the model admits an anomaly free new symmetry i.e. $B-L$. One can therefore extend the standard model symmetry to either $SU(2)_L\times U(1)_{I_{3R}}\times U(1)_{B-L}$ or its left-right symmetric extension $SU(2)_L\times SU(2)_R\times U(1)_{B-L}$. In either case the right handed neutrino carries the B-L quamtum number and its Majorana mass breaks this symmetry. Therefore, the mass of the ν_R can at most be the scale of $B-L$ symmetry breaking, hence answering the question "why $M_{\nu_R} \ll M_{P\ell}$?".

$SU(2)_H$**:**

While local $B-L$ is perhaps the most straight forward and natural symmetry that keeps ν_R lighter than the Planck scale, another possibility has recently been suggested in ref.[24] The main observation here is that is the standard model is extended by including a local $SU(2)_H$ symmetry acting on the first two lepton generations including the right handed charged leptons, then global Witten anomaly freedom dictates that there must be at least two right handed neutrinos which transform as a doublet under the $SU(2)_H$ local symmetry. In this class of models, in the limit of exact $SU(2)_H$ symmetry, the ν_R's are massless and as soon as the $SU(2)_H$ symmetry is broken, they pick up mass. Therefore "lightness" of the ν_R's compared to the Planck scale in these models is related to an $SU(2)_H$ symmetry. These comments are elaborated with explicit examples later on in this review.

7.4.2. *Double seesaw mechanism with ν_R and B-L singlet neutral fermions*

As we saw from the previous discussion, the conventional seesaw mechanism requires rather high mass for the right handed neutrino and therefore a correspondingly high scale for B-L symmetry breaking. The right handed neutrinos are B-L non-singlet fields. There is however no way at present to know what the scale of B-L symmetry breaking is. For lower B-L scale models, one must either find a mechanism to suppress the Dirac mass in the conventional seesaw formula or extend the theory some other way. A particularly simple way is to introduce, B-L singlet heavy neutrinos S and use a double seesaw mechanism suggested in Ref.[31] where one writes a three by three neutrino mass matrix in the basis (ν, N, S) of the form:

$$M \;=\; \begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & M \\ 0 & M & \mu \end{pmatrix} \tag{7.39}$$

This comes from an effective mass Lagrangian of the form:

$$\mathcal{L}'_m \;=\; m^\dagger_D \bar{\nu}_L N_R + M^\dagger \bar{N}_R S \;+ \mu S^T_L C^{-1} S_L \;+\; h.c. \tag{7.40}$$

It is possible to have extra symmetries that guarantees the above form for the Lagrangian. (It will be a good exercise to discover these symmetries.) For the case $\mu \ll M \approx M_{B-L}$, (where M_{B-L} is the $B-L$ breaking scale) this matrix has one light and two heavy neutrinos per generation and the latter two form a pseudo-Dirac pair with mass of order M_{B-L}. The important thing for us is that the light mass eigenvalue is given by $m^2_D \mu / M^2$; for $m_D \approx \mu \simeq$ GeV, a 10 TeV $B-L$ scale is enough to give neutrino masses in the eV range. For the case of three generations, the formula for the light neutrino mass matrix is given by:

$$\mathcal{M}_\nu \;=\; m^T_D M^{-1} \mu M^{-1} m_D. \tag{7.41}$$

7.4.3. *High mass Higgs triplet induced neutrino masses*

As already noted, one way to generate nonzero neutrino masses without using righthanded neutrinos is to extend the standard model by the addition of an $SU(2)_L$ triplet Higgs field with $Y = 2$ so that the electric charge profile of the members of the multiplet is given as follows: $(\Delta^{++}, \Delta^+, \Delta^0)$. This allows an additional Yukawa coupling of the form $f_L L^T \tau_2 \tau L.\mathbf{\Delta}$, where the Δ^0 couples to the neutrinos. Clearly Δ field has $L = 2$. When Δ^0 field has a nonzero vev, it breaks lepton number by two units and leads to Majorana mass for the neutrinos. There are two questions that arise now: one, how does the vev arise in a model and how does one understand the smallness of the neutrino masses in this scheme. There are two answers to the first question: One can maintain exact lepton number symmetry in the model and generate the vev of the triplet field via the usual "mexican hat" potential. There are two problems with this case. This leads to the triplet Majoron which has been ruled out by LEP data on Z-width. In any case, in this model smallness of the neutrino mass is not naturally understood.

Another way to generate the induced vev is to keep a large but positive mass (M_Δ) for the triplet Higgs boson and allowing for a lepton number violating coupling $M\Delta^* HH$.[2] In this case, minimization of the potential induces a vev for the Δ^0 field when the doublet field acquires a vev:[36]

$$v_T \equiv < \Delta^0 > = \frac{M v_{wk}^2}{M_\Delta^2} \tag{7.42}$$

Since the mass of the Δ field is invariant under $SU(2)_L \times U(1)_Y$, it can be very large connected perhaps with some new scale of physics. If we assume that $M_\Delta \sim M \sim 10^{13}$ GeV or so, we get $v_T \sim$ eV. Now in the Yukawa coupling $f_L L^T \tau_2 \tau L.\mathbf{\Delta}$, since the Δ^0 couples to the neutrinos, its vev leads to a neutrino mass We will see later when we discuss the seesaw models that unlike those models, the neutrino mass in this case is not hierarchically dependent on the charged fermion masses. Note further the high mass suppression in Eq.7.42 leading to a new kind of seesaw suppression. This is called type II seesaw.

7.5. Left right symmetric unification: a natural realization of the seesaw

Let us now explore the implications of including the righthanded neutrinos into the extensions of the standard model to understand the small neutrino mass by the seesaw mechanism. As already emphasized, if we assume that there are no new symmetries beyond the standard model, the right handed neutrino will have a natural mass of order of the Planck scale making the light neutrino masses too small to be of interest in understanding the observed oscillations. We must therefore search for new symmetries that can keep the RH neutrinos at a lower scale than the Planck scale. A new symmetry always helps in making this natural.

To study this question, let us note that the inclusion of the right handed neutrinos transforms the dynamics of the gauge models in a profound way. To clarify what we mean, note that in the standard model (that does not contain a ν_R) the $B-L$ symmetry is only linearly anomaly free i.e. $Tr[(B-L)Q_a^2] = 0$ where Q_a are the gauge generators of the standard model but $Tr(B-L)^3 \neq 0$. This means that $B-L$ is only a global symmetry and cannot be gauged. However as soon as the ν_R is added to the standard model, one gets $Tr[(B-L)^3] = 0$ implying that the B-L symmetry is now gaugeable and one could choose the gauge group of nature to be either $SU(2)_L \times U(1)_{I_{3R}} \times U(1)_{B-L}$ or $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, the latter being the gauge group of the left-right symmetric models.[37] Furthermore the presence of the ν_R makes the model quark lepton symmetric and leads to a Gell-Mann-Nishijima like formula for the electric charges[39] i.e.

$$Q = I_{3L} + I_{3R} + \frac{B-L}{2} \tag{7.43}$$

The advantage of this formula over the charge formula in the standard model charge formula is that in this case all entries have a physical meaning. Furthermore, it leads naturally to Majorana nature of neutrinos as can be seen by looking at the distance scale where the $SU(2)_L \times U(1)_Y$ symmetry is valid but the left-right gauge group is broken. In that case, one gets

$$\begin{aligned} \Delta Q = 0 = \Delta I_{3L} : \\ \Delta I_{3R} \;=\; -\Delta\frac{B-L}{2} \end{aligned} \tag{7.44}$$

We see that if the Higgs fields that break the left-right gauge group carry righthanded isospin of one, one must have $|\Delta L| = 2$ which means that the neutrino mass must be Majorana type and the theory will break lepton number by two units.

Let us now proceed to give a few details of the left-right symmetric model and demonstrate how the seesaw mechanism emerges in this model.

The gauge group of the theory is $\mathrm{SU}(2)_L \times$ $\mathrm{SU}(2)_R \times$ $\mathrm{U}(1)_{B-L}$ with quarks and leptons transforming as doublets under $\mathrm{SU}(2)_{L,R}$. In Table 3, we denote the quark, lepton and Higgs fields in the theory along with their transformation properties under the gauge group.

Table II

Fields	$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$ representation
Q_L	$(2,1,+\frac{1}{3})$
Q_R	$(1,2,\frac{1}{3})$
L_L	$(2,1,-1)$
L_R	$(1,2,-1)$
ϕ	$(2,2,0)$
Δ_L	$(3,1,+2)$
Δ_R	$(1,3,+2)$

Table caption Assignment of the fermion and Higgs fields to the representation of the left-right symmetry group.

The first task is to specify how the left-right symmetry group breaks to the standard model i.e. how one breaks the $SU(2)_R \times U(1)_{B-L}$ symmetry so that the successes of the standard model including the observed predominant V-A structure of weak interactions at low energies is reproduced. Another question of naturalness that also arises simultaneously is that since the charged fermions and the neutrinos are treated completely symmetrically (quark-lepton symmetry) in this model, how does one understand the smallness of the neutrino masses compared to the other fermion masses.

It turns out that both the above problems of the LR model have a common solution. The process of spontaneous breaking of the $SU(2)_R$ symmetry that suppresses the V+A currents at low energies also solves the problem of ultralight neutrino masses. To see this let us write the Higgs fields explicitly:

$$\Delta \;=\; \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{pmatrix}; \;\; \phi \;=\; \begin{pmatrix} \phi_1^0 & \phi_2^+ \\ \phi_1^- & \phi_2^0 \end{pmatrix} \tag{7.45}$$

All these Higgs fields have Yukawa couplings to the fermions given symbolically as below.

$$\begin{aligned} \mathcal{L}_{\mathcal{Y}} &= h_1 \bar{L}_L \phi L_R + h_2 \bar{L}_L \tilde{\phi} L_R \\ &\quad + h_1' \bar{Q}_L \phi Q_R + h_2' \bar{Q}_L \tilde{\phi} Q_R \\ &+ f(L_L L_L \Delta_L + L_R L_R \Delta_R) + \; h.c. \end{aligned} \tag{7.46}$$

The $SU(2)_R \times U(1)_{B-L}$ is broken down to the standard model hypercharge $U(1)_Y$ by choosing $< \Delta_R^0 >= v_R \neq 0$ since this carries both $SU(2)_R$ and $U(1)_{B-L}$ quantum numbers. It gives mass to the charged and neutral righthanded gauge bosons i.e. $M_{W_R} = gv_R$ and $M_{Z'} = \sqrt{2} g v_R cos\theta_W / \sqrt{cos 2\theta_W}$. Thus by adjusting the value of v_R one can suppress the right handed current effects in both neutral and charged current interactions arbitrarily leading to an effective near maximal left-handed form for the charged current weak interactions.

The fact that at the same time the neutrino masses also become small can be seen by looking at the form of the Yukawa couplings. Note that the f-term leads to a mass for the right handed neutrinos only at the scale v_R. Next as we break the standard model symmetry by turning on the vev's for the ϕ fields as $Diag < \phi >= (\kappa, \kappa')$, we not only give masses to the W_L and the Z bosons but also to the quarks and the leptons. In the neutrino sector the above Yukawa couplings after $SU(2)_L$ breaking by $< \phi > \neq 0$ lead to the so called Dirac masses for the neutrino connecting the left and right handed neutrinos. In the two component neutrino language, this leads to the following mass matrix for the ν, N using the notation earlier with the four component $\nu = \begin{pmatrix} \nu \\ i\sigma_2 N^* \end{pmatrix}$.

$$M = \begin{pmatrix} 0 & h\kappa \\ h\kappa & fv_R \end{pmatrix} \tag{7.47}$$

Note that m_D in previous discussions of the seesaw formula (see Eq. ()) is given by $m_D = h\kappa$, which links it to the weak scale and the mass of the RH neutrinos is given by $M_R = fv_R$, which is linked to the local B-L symmetry. This justifies keeping RH neutrino mass at a scale lower than the Planck mass. It is therefore fair to assume that seesaw mechanism coupled with observations of neutrino oscillations are a strong indication of the existence of a local B-L symmetry far below the Planck scale.

By diagonalizing this 2×2 matrix, we get the light neutrino eigenvalue to be $m_\nu \simeq \frac{(h\kappa)^2}{fv_R}$ and the heavy one to be fv_R. Note that typical charged fermion masses are given by $h'\kappa$ etc. So since $v_R \gg \kappa, \kappa'$, the light neutrino mass is automatically suppressed. This way of suppressing the neutrino masses is called the seesaw mechanism.[4] Thus in one stroke, one explains the smallness of the neutrino mass as well as the suppression of the V+A currents.

In deriving the above seesaw formula for neutrino masses, it has been assumed that the vev of the lefthanded triplet is zero so that the $\nu_L\nu_L$ entry of the neutrino mass matrix is zero. However, in most explicit models such as the left-right model which provide an explicit derivation of this formula, there is an induced vev for the Δ^0_L of order $< \Delta^0_L >= v_T \simeq \frac{v^2_{wk}}{v_R}$. In the left-right models, this this arises from the presence of a coupling in the Higgs potential of the form $\Delta_L\phi\Delta^\dagger_R\phi^\dagger$. In the presence of the Δ_L vev, the seesaw formula undergoes a fundamental change and takes the form

$$M_\nu \ = \ fv_L - h^T_\nu f^{-1}_R h_\nu \left(\frac{v^2_{wk}}{v_R}\right) \tag{7.48}$$

which includes both the type I and the type II seesaw contributions. In Fig. 1, the two mdiagrams responsible for type I and type II seesaw are given:

This left-right symmetric seesaw formula has recently been shown to exhibit some interesting duality properties[38]which can perhaps be used to restrict some of the arbitrariness in its applications.

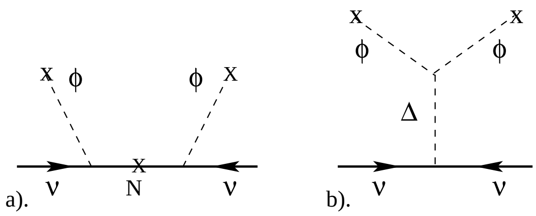

Fig. 7.1. The type I and type II contribution to seesaw formula for neutrino masses.

Note that in the type I seesaw formula, what appears is the square of the Dirac neutrino mass matrix which in general expected to have the same hierarchical structure as the corresponding charged fermion mass matrix. In fact in some specific GUT models such as SO(10), $M_D = M_u$. This is the origin of the common statement that neutrino masses given by the seesaw formula are hierarchical i.e. $m_{\nu_e} \ll m_{\nu_\mu} \ll m_{\nu_\tau}$ and even a more model dependent statement that $m_{\nu_e} : m_{\nu_\mu} : m_{\nu_\tau} = m_u^2 : m_c^2 : m_t^2$.

On the other hand if one uses the type II seesaw formula, there is no reason to expect a hierarchy and in fact if the neutrino masses turn out to be degenerate as discussed before as one possibility, one possible way to understand this may be to use the type II seesaw formula.

Secondly, the type II seesaw formula is a reflection of the parity invariance of the theory at high energies. Evidence for it would point more strongly towards left-right symmetry at high energies.

7.5.1. *Understanding detailed mixing pattern for neutrinos using the seesaw formula*

Let us now address the question: to what extent one can understand the details of the neutrino masses and mixings using the seesaw formulae. The answer to this question is quite model dependent. While there exist many models which fit the observations, none (except a few) are completely predictive and almost always they need to invoke new symmetries or new assumptions. The problem in general is that the seesaw formula of type I, has 12 parameters in the absence of CP violation (six parameters for a symmetric Dirac mass matrix and six for the M_R) which is why its predictive power is so limited. In the presence of CP violation, the number of parameters double making the situation worse. Specific predictions can be made only under additional assumptions.

For instance, in a class of seesaw models based on the SO(10) group that embodies the left-right symmetric unification model or the SU(4)-color, the mass the tau neutrino mass can be estimated provided one assumes the normal mass hierarchy for neutrinos and a certain parameter accompanying a higher dimensional operator to be of order one. To see this, let us assume that in the SO(10) theory, the B-L symmetry is broken by a **16**-dim. Higgs boson. The RH neutrino mass in such a model arises from the nonrenormalizable operator $\lambda(\mathbf{16_F \bar{16}_H})^2/\mathbf{M_{P\ell}}$. In a supersymmetric theory, if **16**-Higgs is also responsible for GUT symmetry breaking, then after symmetry breaking, one obtains the RH neutrino mass $M_R \simeq \lambda(2 \times 10^{16})^2/M_{P\ell} \simeq 4\lambda 10^{14}$ GeV. In models with $SU(4)_c$ symmetry, $m_{\nu_\tau,D} \simeq m_t(M_U) \sim 100$ GeV. Using the seesaw formula then, one obtains for $\lambda = 1$, tau neutrino mass $m_{\nu_\tau} \simeq .025$ eV, which is close to the presently preferred value of 0.05 eV. The situation with respect to other neutrino masses is however less certain and here one has to make assumptions.

The situation with respect to mixing angles is much more complicated. In generic seesaw models, one needs additional family symmetries to understand the largeness of both solar and atmospheric mixing angles, as has been commented before. It could of course very well be that the Dirac coupling in the seesaw formula is similar to the quark Yukawas but large neutrino mixings owe their origin to the flavor structure of right handed neutrino mass matrix. Or it could be that it is the type II seesaw term (the triplet Higgs contribution) dominates the neutrino mass decoupling neutrino masses completely from the charged lepton and quark mixings.

Essentially, one has to arrive at matrices similar to the above examples. There are however some exceptional situations such as in a class of minimal SO(10) models described below where the overall unification constraints on Yukawa textures is enough to explain desired large mixings, without the need for any family symmetry.

7.5.2. *General consequences of the seesaw formula for neutrino masses*

In this section, we will consider some implications of the seesaw mechanism for understanding neutrino masses. We will discuss two main points. One is the nature of the right handed neutrino spectrum as dictated by the seesaw mechanism and secondly, ways to get an approximate $L_e - L_\mu - L_\tau$ symmetric neutrino mass matrix using the seesaw mechanism and its possible implications for physics beyond the standard model.[33]

For this purpose, we use the type I seesaw formula along with the assumption of a diagonal Dirac neutrino mass matrix to obtain the right handed neutrino mass matrix M_R:

$$\mathcal{M}_{R,ij} \ = \ m_{D,i}\mu^{-1}_{ij} m_{D,j} \tag{7.49}$$

with

$$\begin{aligned}
\mu^{-1}_{11} &= \frac{c^2}{m_1} + \frac{s^2}{m_2} + \frac{\epsilon^2}{m_3} \qquad (7.50)\\
\mu^{-1}_{12} &= -\frac{c(s+c\epsilon)}{\sqrt{2}m_1} + \frac{s(c-s\epsilon)}{\sqrt{2}m_2} + \frac{\epsilon}{\sqrt{2}m_3}\\
\mu^{-1}_{13} &= \frac{c(s-c\epsilon)}{\sqrt{2}m_1} - \frac{s(c+s\epsilon)}{\sqrt{2}m_2} + \frac{\epsilon}{\sqrt{2}m_3}\\
\mu^{-1}_{22} &= \frac{(s+c\epsilon)^2}{2m_1} + \frac{(c_s\epsilon)^2}{2m_2} + \frac{1}{2m_3}\\
\mu^{-1}_{23} &= -\frac{(s^2-c^2\epsilon^2)}{2m_1} - \frac{(c^2-s^2\epsilon^2)}{2m_2} + \frac{1}{2m_3}\\
\mu^{-1}_{33} &= \frac{(s-c\epsilon)^2}{2m_1} + \frac{(c+s\epsilon)^2}{2m_2} + \frac{1}{2m_3}.
\end{aligned}$$

Since for the cases of normal and inverted hierarchy, we have no information on the mass of the lightest neutrino m_1, we could assume it in principle to be quite small. In that case, the above equation enables us to conclude that quite likely one of the three right handed neutrinos is much heavier than the other two, leading to the so-called two right handed neutrino dominance model.[32] The situation is of course completely different for the degenerate case. This kind of separation of the RH neutrino spectrum is very suggestive of a symmetry. In fact we have recently argued that,[24] this indicates the possible existence of an $SU(2)_H$ horizontal symmetry, that leads in the simplest case to an inverted mass pattern for light neutrinos. A scenario which realizes this is given below.

7.5.2.1. *Approximate $L_e - L_\mu - L_\tau$ symmetric mass matrix from seesaw*

In this section, we discuss how an approximate $L_e - L_\mu - L_\tau$ symmetric neutrino mass matrix may arise within a seesaw framework. Consider a simple extension of the standard model by adding two additional singlet right handed neutrinos,[34] N_1, N_2 assigning them $L_e - L_\mu - L_\tau$ quantum numbers of $+1$ and -1 respectively. Denoting the standard model lepton doublets by $\psi_{e,\mu,\tau}$, the $L_e - L_\mu - L_\tau$ symmetry allows the following new couplings to the Lagrangian of the standard model:

$$\mathcal{L}' = (h_3\bar{\psi_\tau} + h_2\bar{\psi_\mu})HN_2 + h_1\bar{\psi}_e HN_1 + MN_1^T C^{-1} N_2 + h.c. \quad (7.51)$$

where H is the Higgs doublet of the standard model; C^{-1} is the Dirac charge conjugation matrix. We add to it the symmetry breaking mass terms for the right handed neutrinos, which are soft terms, i.e.

$$\mathcal{L}_B = \epsilon(M_1 N_1^T C^{-1} N_1 + M_2 N_2^T C^{-1} N_2) + h.c. \quad (7.52)$$

with $\epsilon \ll 1$. These terms break $L_e - L_\mu - L_\tau$ by two units but since they are dimension 3 terms, they are soft and do not induce any

with $\epsilon \ll 1$. These terms break $L_e - L_\mu - L_\tau$ by two units but since they are dimension 3 terms, they are soft and do not induce any new terms into the theory.

It is clear from the resulting mass matrix for the ν_L, N system that the linear combination $h_2\nu_\tau - h_3\nu_\mu)$ is massless and the atmospheric oscillation angle is given by $tan\theta_A = h_2/h_3$; for $h_3 \sim h_2$, the θ_A is maximal. The seesaw mass matrix then takes the following form (in the basis $(\nu_e, \tilde{\nu}_\mu, N_1, N_2)$ with $\tilde{\nu}_\mu \equiv h_2\nu_\mu + h_3\nu_\tau$):

$$M = \begin{pmatrix} 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \\ m_1 & 0 & \epsilon M_1 & M \\ 0 & m_2 & M & \epsilon M_2 \end{pmatrix} \quad (7.53)$$

The diagonalization of this mass matrix leads to the mass matrix of the form discussed before.

7.5.2.2. *Tri-bi-maximal mixing from seesaw*

In this section, we present an example of a seesaw model for the mass matrix for neutrinos (Eq.7.28) that leads to the tri-bi-maximal mixing.[35] It was shown that the Majorana neutrino mass matrix in 7.28 can be realized in a combined type I type II seesaw model with soft-broken S_3 family symmetry for leptons. The type II contribution comes from an S_3 invariant coupling of lepton doublets to the triplet field Δ i.e. $f_{\alpha\beta}L_\alpha L_\beta\Delta$. The most general S_3 invariant form for f is:

$$f = \begin{pmatrix} f_a & f_b & f_b \\ f_b & f_a & f_b \\ f_b & f_b & f_a \end{pmatrix} \quad (7.54)$$

After the triplet Higgs field Δ gets vev and decouples, its contribution to the light neutrino mass can written as

$$M_{II} = \begin{pmatrix} a' & b' & b' \\ b' & a' & b' \\ b' & b' & a' \end{pmatrix} \tag{7.55}$$

where $a' = \frac{v^2 \sin^2\beta\lambda}{M_T} f_a$ and $b' = \frac{v^2 \sin^2\beta\lambda}{M_T} f_b$. We denote M_T as the mass of the triplet Higgs and λ as the coupling constant between the triplet and doublets in the superpotential.

Coming to the type I contribution, the Dirac mass matrix for neutrinos comes from an S_3 invariant Yukawa coupling of the form:

$$\mathcal{L}_D = h_\nu[\overline{\nu_{R1}}H(L_e - L_\mu) + \overline{\nu_{R2}}H(L_\mu - L_\tau) + \overline{\nu_{R3}}H(L_\tau - L_e)] + h.c. \tag{7.56}$$

leading to

$$Y_\nu = \begin{pmatrix} h & -h & 0 \\ 0 & h & -h \\ -h & 0 & h \end{pmatrix}. \tag{7.57}$$

In the limit of $|M_{R1,R3}| \gg |M_{R2}|$, where a single right-handed neutrino dominates the type I contribution, the mixed type I+II seesaw formula

$$\mathcal{M}_\nu = M_{II} - M_D^T M_{\nu R}^{-1} M_D, \tag{7.58}$$

then leads to Eq.7.28 which gives the tri-bi-maximal mixing matrix. It turns out that,[35] the charged lepton mass matrix in this case can be made diagonal if one of the two lepton Yukawa couplings is set to zero.

7.6. Neutrino mass and grand unification

One of the interesting features of the seesaw mechanism is that if one assumes the Dirac masses to be roughly of order of the up-quark masses, then the atmospheric neutrino mass difference would directly measure the mass of m_3 and can be used to get a rough idea of how high the seesaw scale is. In order to do this one can use the rough relation $m_{\nu_3} \sim \frac{m_t^2}{M_R}$ which then yields $M_R \sim 10^{14}$ GeV. This is of course not a rigorous argument at all and can therefore only be used as a suggestive one. If however one takes this seriously, then it suggests that the seesaw scale could be related to the scale of grand unification which from arguments of coupling constant unification is also of order 10^{16} GeV. In view of other theoretical arguments in favor of GUTs, one may try to understand the neutrino masses within a grand unified theory framework.

The minimal GUT group that appears to have many desirable properties is the SO(10) group,[41] whose spinor representation is **16** dimensional and is just right for all then SM fermions of one generation plus the right handed neutrino needed for implementing the seesaw mechanism. This has therefore been extensively studied as a way to understand neutrino properties.

7.6.1. *SO(10) Grand Unification of seesaw mechanism and predictions for neutrino masses*

In addition to the fermion unification by the **16** dimensional spinor representation, SO(10) contains the B-L as a subgroup and seesaw mechanism requires that the process of symmetry breaking down to the standard model must break the B-L at a high scale. One implication of this is a natural understanding of the seesaw scale as being connected to the GUT scale. Secondly in the context of supersymmetric SO(10) models, the way B-L breaks has profound consequences for low energy physics. For instance, if B-L is broken by a Higgs field belonging to the **16** dimensional Higgs field, then the field that acquires a nonzero vev has the quantum numbers of the ν_R field i.e. B-L breaks by one unit. In this case higher dimensional operators of the form $\Psi\Psi\Psi\Psi_H$ will lead to R-parity violating operators in the effective low energy MSSM theory such as $QLd^c, u^cd^cd^c$ etc which can lead to large breaking of lepton and baryon number symmetry and hence unacceptable rates for proton decay. This theory also has no dark matter candidate without making additional assumptions. Furthermore, since non-renormalizable operators are an essential part of this approach, there are many more parameters, making it non-predictive in the absence of additional assumptions.[42]

On the other hand, one may break B-L by a **126** dimensional Higgs field.[43,44] The member of this multiplet that acquires vev has $B-L = 2$ and leaves R-parity as an automatic symmetry of the low energy Lagrangian. This then gives a naturally stable dark matter. Furthermore, in this approach, since one considers only renormalizable couplings, the number of Yukawa parameters are quite limited so that the model is quite predictive.[43] The predictivity clearly arises from one irreducible **16** dimensional spinor multiplet containing all fermions of each family or complete fermion unification.

In order to study the predictions of the model, we first note that since the SO(10) model contains the left-right subgroup, the seesaw formula takes the modified form as in Eq.7.48 that we repeat below.[36]

$$M_\nu \;=\; f v_L - h_\nu^T f_R^{-1} h_\nu \left(\frac{v_{wk}^2}{v_R}\right) \tag{7.59}$$

It turns out that if the B-L symmetry is broken by **16** Higgs fields, the first term in the type II seesaw (effective triplet vev induced term) becomes very small compared to the type I term. On the other hand, if B-L is broken by a **126** field, then the first term in the type II seesaw formula is not necessarily small and can in principle dominate in the seesaw formula. As we discuss below, this leads to predictions for neutrino masses and mixings that are in excellent agreement with experiments.

The basic ingredients of this model[43] are that one considers only two Higgs multiplets that contribute to fermion masses i.e. one **10** and one **126**. A unique property of the **126** multiplet is that it not only breaks the B-L symmetry and therefore contributes to right handed neutrino masses, but it also contributes to charged fermion masses by virtue of the fact that it contains MSSM doublets which mix with those from the **10** dimensional multiplets and survive down to the MSSM scale. This leads to a tremendous reduction of the number of arbitrary parameters.

There are only two Yukawa coupling matrices in this model: (i) h for the **10** Higgs and (ii) f for the **126** Higgs. SO(10) has the property that the Yukawa couplings involving the **10** and **126** Higgs representations are symmetric. Therefore if we assume that CP violation arises from other sectors of the theory (e.g. squark masses) and work in a basis where one of these two sets of Yukawa coupling matrices is diagonal, then it will have only nine parameters. Noting the fact that the (2,2,15) submultiplet of **126** has a pair of standard model doublets that contributes to charged fermion masses, one can write the quark and lepton mass matrices as follows:[43]

$$\begin{aligned} M_u &= h\kappa_u + f v_u \\ M_d &= h\kappa_d + f v_d \\ M_\ell &= h\kappa_d - 3 f v_d \\ M_{\nu_D} &= h\kappa_u - 3 f v_u \end{aligned} \tag{7.60}$$

where $\kappa_{u,d}$ are the vev's of the up and down standard model type Higgs fields in the **10** multiplet and $v_{u,d}$ are the corresponding vevs for the same doublets in **126**. Note that there are 13 parameters in the above equations and there are 13 inputs (six quark masses, three lepton masses and three quark mixing angles and weak scale). Thus all parameters of the model that go into fermion masses are determined.

To determine the light neutrino masses, we use the seesaw formula in Eq.7.48, where the **f** is nothing but the **126** Yukawa coupling. Thus all parameters that give neutrino mixings except an overall scale are determined. A simple way to see how large mixings arise in this model is to note that when the triplet term dominates the seesaw formula, we have the neutrino mass matrix $M_\nu \propto f$, where f matrix is the **126** coupling to fermions discussed earlier.

$$M_\nu \;=\; c(M_d - M_\ell) \tag{7.61}$$

All the quark mixing effects are then in the up quark mass matrix i.e. $M_u \;=\; U^T_{CKM} M^d_u U_{CKM}$. Note further that the minimality of the Higgs content leads to the following sum-rule among the mass matrices:

$$k\tilde{M}_\ell \;=\; r\tilde{M}_d + \tilde{M}_u \tag{7.62}$$

where the tilde denotes the fact that we have made the mass matrices dimensionless by dividing them by the heaviest mass of the species. We then find that we have

$$M_{d,\ell} \;\approx\; m_{b,\tau} \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \tag{7.63}$$

where $\lambda \sim 0.22$ and the matrix elements are supposed to give only the approximate order of magnitude. An important consequence of the relation between the charged lepton and the quark mass matrices in Eq. 7.62 is that the charged lepton contribution to the neutrino mixing matrix i.e. $U_\ell \simeq \mathbf{1} + O(\lambda)$ or close to identity matrix. As a result the neutrino mixing matrix is given by $U_{PMNS} \;=\; U^\dagger_\ell U_\nu \simeq U_\nu$, since in U_ℓ, all mixing angles are small. Thus the dominant contribution to large mixings will come from U_ν, which in turn will be dictated by the sum rule in Eq. 7.61.

As we extrapolate the quark masses to the GUT scale, due to the fact that $m_b - m_\tau \approx m_\tau\lambda^2$ for a wide range of values of tanβ, the neutrino mass matrix $M_\nu \;= c(M_d - M_\ell)$ takes roughly the form

$$M_\nu \;= c(M_d - M_\ell) \approx \; m_0 \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda^2 \end{pmatrix} \tag{7.64}$$

It is easy to see that both the θ_{12} (solar angle) and θ_{23} (the atmospheric angle) are now large. The detailed magnitudes of these angles of course depend on the details of the quark masses at the GUT scale. Using the extrapolated values of the quark masses and mixing angles to the GUT scale, the predictions of this model for various oscillation parameters are given in.[26] Some of the salient features are: (i) the atmospheric mixing angle θ_{23} is not maximal and the maximum value for it is around 38^0; (ii) the prediction for $sin\theta_{13} \equiv U_{e3}$ is near 0.18, a value within the reach of MINOS as well as other planned Long Base Line neutrino experiments such as Numi-Off-Axis, JPARC etc.

7.6.2. *CP violation in the minimal SO(10) model*

In the discussion given above, it was assumed that CP violation is non-CKM type and resides in the soft SUSY breaking terms of the Lagrangian. The overwhelming evidence from experiments seem to be that CP violation is perhaps of CKM type with CKM phase of about 60^0. Success of the above approach in understanding neutrino mixings suggests that we should consider extending the above simple model to accommodate CKM CP violation. Several such attempts have been made in recent literature.[19,47]

One approach discussed by us[47] employs a slight extension of the **10**+**126** model by adding a **120** Higgs field. A further Z_2 symmetry is imposed in such a way that the **10** and **126** couplings are real whereas the **120** couplings turn out to be imaginary. This will add a new piece to all fermion masses but in such a way that the $b-\tau$ mass convergence still leads to large atmospheric mixing as in the 10+126 case.

The new model is still predictive in the neutrino sector. Of the three new parameters, one is determined by the CP violating quark phase. the two others are determined by the solar mixing angle and the solar mass difference squared. Therefore we lose the prediction for these parameters. However, we can predict in addition to θ_A which is now close to maximal, $\theta_{13} \geq 0.1$ (see figure below) and the Dirac phase for the neutrinos, also close to 90^o.

In the above discussion, we assumed type II seesaw terms to dominate.

This model has been reanalyzed using type I seesaw term to dominant in a recent paper[19] and a fit to the fermion masses as well as neutrino mixings exists for this case.

Finally, a few comments on what really will constitute a true test of the grand unification theories: a key prediction of simple grand unified theories such as SU(5) and SO(10) is the existence of proton decay. In supersymmetric theories, proton decay turns from an exciting prediction to somewhat of a challenge since the presence of super-partners at the TeV scale generates "dangerous" operators such as $\tilde{Q}\tilde{Q}QL/M_U$ which could lead to very rapid proton decay. In fact it is the appearance of these kind of operators that has ruled out minimal SUSY SU(5) model. For this reason, in the last two papers by us,[47] a Yukawa texture was chosen that is in accord with current experimental bounds on proton lifetime resulting from dimension five operators as the one given above and a fit to neutrino masses as well as charged fermions etc was found with type II seesaw. A proton decay check is therefore needed for the type I fit to fermions carried out in ref.[48] and proton decay predictions for the model of[47] need to be worked out.

7.6.3. *Type II seesaw and Quasi-degenerate neutrinos*

In this subsection we like to discuss some issues related to the degenerate neutrino hypothesis, which will be necessary if there is evidence for neutrinoless double beta decay at a significant level(see for example the recent results from the Heidelberg-Moscow group[12]) and assuming that no other physics such as R-parity breaking or doubly charged Higgs etc are not the source of this effect). Thus it is appropriate to discuss how such models can arise in theoretical schemes and how stable they are under radiative corrections.

There are two aspects to this question: one is whether the degeneracy arises within a gauge theory framework without arbitrary adjustment of parameters and the second aspect being that given such a degeneracy arises at some scale naturally in a field theory, is this mass degeneracy stable under renormali9zation group extrapolation to the weak scale where we need the degeneracy to be present. In this section we comment on the first aspect.

It was pointed out long ago[14] that degenerate neutrinos arise naturally in models that employ the type II seesaw since the first term in the mass formula is not connected to the charged fermion masses. One way that has been discussed is to consider schemes where one uses symmetries such as SO(3) or SU(2) or permutation symmetry S_4 so that the Majorana Yukawa couplings f_i are all equal. This then leads to the dominant contribution to all neutrinos being equal. This symmetry however must be broken in the charged fermion sector in order to explain the observed quark and lepton masses. Such models consistent with known data have been constructed based on SO(10) as well as other groups. The interesting point about the SO(10) realization is that the dominant contributions to the Δm^2's in this model comes from the second term in the type II seesaw formula which in simple models is hierarchical. It is of course known that if the MSW solution to the solar neutrino puzzle is the right solution (or an energy independent solution), then we have $\Delta m^2_{solar} \ll \Delta m^2_{ATMOS}$. In fact if we use the fact true in SO(10) models that $M_u = M_D$, then we have $\Delta m^2_{ATMOS} \simeq m_0 \frac{m_t^2}{f v_R}$ and $\Delta m^2_{SOLAR} \simeq m_0 \frac{m_c^2}{f v_R}$ where m_0 is the common mass for the three neutrinos. It is interesting that for $m_0 \sim$ few eV and $f v_R \approx 10^{15}$ GeV, both the Δm^2's are of right order to the required values.

Outside the seesaw framework, there could also be electroweak symmetries that guarantee the mass degeneracy.

The second question of stability under RGE of such a pattern is discussed in a subsequent section.

7.7. Some other consequences of seesaw paradigm

Generic seesaw models have several other important implications that we go into now. For simplicity, we first consider the type I seesaw formula7.38. The first question one can ask is that given low energy information, to what extent we can discover the high scale physics associated with the seesaw mechanism such as the spectrum of right handed neutrinos, the structure of the Dirac mass matrix m_D (or equivalently Y_ν). A simple parameter counting shows that the neutrino masses and mixings (including CP phases) are characterized by nine observables whereas seesaw formula involves eighteen parameters (in the basis where RH neutrinos are mass eigenstates, there are three masses and 15 parameters characterizing m_D. Thus we need nine more pieces of low energy inputs to completely determine the seesaw physics (granted that all observables in the neutrino mass matrix are determined). Radiative leptonic decays such as $\mu \to e + \gamma, \tau \to \mu, e + \gamma$ including both CP violating and conserving channels could provide six pieces of information; three electric dipole moments of the charged leptons could provide the remaining three. Thus in principle, all the seesaw parameters could be determined from low energy observations.

In discussing the connection between high scale and low scale physics for neutrinos, it is often convenient to use a parameterization suggested by Casas and Ibarra.[49]

$$Y_\nu v_{wk} \;=\; i {M_R}^{1/2} O (\mathcal{M}^d_\nu)^{1/2} U^\dagger \tag{7.65}$$

where O is a complex matrix with the property that $OO^T = 1^{49}$ and U is the neutrino mixing matrix; M^d_ν is the diagonal neutrino mass matrix. The set of matrices O in fact form a group analogous to the complex extension of the Lorentz group. Note that six parameters (or three complex angles) characterize O, three needed each for M_R and $\mathcal{M}_\nu$ and six for U giving a total of 18 as we counted above. In special cases where there are symmetries e.g. $\mu - \tau$ symmetry, the number of complex angles in reduces to only one making the direct connection between high and low energy phases somewhat closer.

For the case of type II seesaw, the corresponding relation is:

$$Y_\nu v_{wk} \;=\; i{M_R}^{1/2} O [U^* \mathcal{M}^d_\nu U^\dagger - M_R \zeta]^{1/2} \tag{7.66}$$

where $\zeta = \frac{v_L}{v_R}$.

It is clear from Eq.7.65 that in general the neutrino mixing matrix is only indirectly related to the details of Y_ν due to the unknown matrix O. In a given model however, when Y_ν is given, O and U get related.

7.7.1. *SUSY seesaw and lepton flavor violation*

In the standard model, the masslessness of the neutrino implies that that there is no lepton flavor changing effects unlike in the quark sector. Thus the leptons are completely "flavor sterile" and do not throw any light on the flavor puzzle. Once one includes the right handed neutrinos N_R one for each family, there is lepton mixing and this activates the lepton flavor. A simple phenomenological consequence of this "flavor activation" is that there appear lepton flavor changing effects such as $\mu \to e + \gamma$, $\tau \to e, \mu + \gamma$ etc. However, a simple estimate of the one loop contribution to such effects shows that the amplitude is of order

$$A(\ell_j \to \ell_i + \gamma) \simeq \frac{e G_F m_{\ell_j} m_e m^2_\nu}{\pi^2 m^2_W} \mu_B \tag{7.67}$$

This leads to an unobservable branching ratio (of order $\sim 10^{-40}$) for the rare radiative decay modes for the leptons given above.

The situation however changes drastically as soon as the seesaw mechanism for neutrino masses is combined with supersymmetry. It has been noted in many papers already that in supersymmetric theories, the lepton flavor changing effects get significantly enhanced. They arise from the the mixings among sleptons (superpartners of leptons) of different flavor caused by the renormalization group extrapolations which via loop diagrams lead to lepton flavor violating (LFV) effects at low energies.[50]

The way this happens is as follows. In the simplest N=1 supergravity models,[51] the supersymmetry breaking terms at the Planck scale are taken to have only few parameters: a universal scalar mass m_0, universal A terms, one gaugino mass $m_{1/2}$ for all three types of gauginos. Clearly, a universal scalar mass implies that at Planck scale, there is no flavor violation anywhere except in the Yukawa couplings (or when the Yukawa terms are diagonalized, in the CKM angles). However as we extrapolate this theory to the weak scale, the flavor mixings in the Yukawa interactions induce non universal flavor violating scalar mass terms (i.e. flavor violating slepton and squark mass terms). In the absence of neutrino masses, the Yukawa matrices for leptons can be diagonalized so that there is no flavor violation in the lepton sector even after extrapolation down to the weak scale. On the other hand, when neutrino mixings are present or when the quarks and leptons are unified in such a way that this diagonalization becomes impossible, there is no basis where all leptonic flavor mixings can be made to disappear. In fact, in the most general case, of the three matrices Y_ℓ, the charged lepton coupling matrix, Y_ν, RH neutrino Yukawa coupling and M_{N_R}, the matrix characterizing the heavy RH neutrino mixing, only one can be diagonalizd by an appropriate choice of basis and the flavor mixing in the other two remain. In a somewhat restricted case where the right handed neutrinos do not have any interaction other than the Yukawa interaction and an interaction that generates the Majorana mass for the right handed neutrino, one can only diagonalize two out of the three matrices (i.e. Y_ν, Y_ℓ and M_R). Thus there will always be lepton flavor violating terms in the basic Lagrangian, no matter what basis one chooses. These LFV terms can then induce mixings between the sleptons of different flavor and lead to LFV processes. If we keep the M_ℓ diagonal by choice of basis, searches for LFV processes such as $\tau \to \mu + \gamma$ and/or $\mu \to e + \gamma$ can throw light on the RH neutrino mixings/or family mixings in M_D, as has already been observed.

Since in the absence of CP violation, there are at least six mixing angles (nine if M_D is not symmetric) in the seesaw formula and only three are observable in neutrino oscillation, to get useful information on the fundamental high scale theory from LFV processes, it is assumed that M_{N_R} is diagonal so that one has a direct correlation between the observed neutrino mixings and the fundamental high scale paramters of the theory. The important point is that the flavor mixings in Y_ν then reflect themselves in the slepton mixings that lead to the LFV processes via the RGEs.

From the point of view of the LFV analysis, there are essentially two classes of neutrino mass models that need to be considered: (i) the first class is where it is assumed that the RH neutrino mass M_{N_R} is either a mass term in the basic Lagrangian or arises from nonrenormalizable terms such as $\nu^c\chi^{c2}/M_{P\ell}$, as in a class of SO(10) models; (we will such models Dirac type) and (ii) a second class where the Majorana mass of the right handed neutrino itself arises from a renormalizable Yukawa coupling e.g. $f\nu^c\nu^c\Delta$ (we will call them Majorana type models). In Dirac type models, in principle, one could decide to have all the flavor mixing effects in the right handed neutrino mass matrix and keep the Y_ν diagonal. In that case, RGEs would not induce any LFV effects. However we will bar this possibility and consider the case where all flavor mixings are in the Y_ν so that RGEs can induce LFV effects . In Majorana type models on the other hand, there will always be an LFV effect, although its magnitude will depend on the choice of the seesaw scale (v_{BL}).

Examples of class two models are models for neutrino mixings such as SO(10) with a **126** Higgs field[43] or models with a triplet Higgs, whose vev is the seesaw scale.

In both these examples, the equations that determine the extent of lepton flavor violation in leading order, for the case $A = 0$ are:

$$\frac{dm_L^2}{dt} \simeq \frac{1}{16\pi^2}[3m_0^2(Y_\nu^\dagger Y_\nu)] \tag{7.68}$$

In the Majorana case, this equation will have contributions from the renormalizable f couplings that give Majorana masses to the right handed neutrinos[52] in the sense that generation mixing elements in Y_ν will be generated by f's even if they were absent in the beginning. Using these equations, one can obtain the branching ratios for the radiative lepton flavor violating processes using the formula below:

$$\mathcal{L} = iem_j\left(\bar\ell_{jL}\sigma_{\mu\nu}\ell_{iR}C_L + \bar\ell_{jR}\sigma_{\mu\nu}\ell_{iL}C_R\right)F^{\mu\nu} + h.c. \tag{7.69}$$

then the Branching ratio for the decay $\ell_j \to \ell_i + \gamma$ is given by the formula

$$B(\ell_j \to \ell_i + \gamma) = \frac{48\pi^3\alpha_{em}}{G_F^2}(|C_L|^2 + |C_R|^2)B(\ell_j \to \ell_i + 2\nu). \tag{7.70}$$

7.7.2. *Renormalization group evolution of the neutrino mass matrix*

In the seesaw models for neutrino masses, the neutrino mass arises from the effective operator

$$\mathcal{O}_\nu = -\frac{1}{4}\kappa_{\alpha\beta}\frac{L_\alpha H L_\beta H}{M} \tag{7.71}$$

after symmetry breaking $< H^0 >\neq 0$; here L and H are the leptonic and weak doublets respectively. α and β denote the weak flavor index. The matrix κ becomes the neutrino mass matrix after symmetry breaking i.e. $< H^0 >\neq 0$. This operator is defined at the scale M since it arises after the heavy field N_R is integrated out. On the other hand, in conventional oscillation experiments, the neutrino masses and mixings being probed are at the weak scale. One must therefore extrapolate the operator down from the seesaw scale M to the weak scale M_Z.[53] The form of the renormalization group extrapolation of course depends on the details of the theory. For simplicity we will consider only the supersymmetric theories, where the only contributions come from the wave function renormalization and is therefore easy to calculate. The equation governing the extrapolation of the $\kappa_{\alpha\beta}$ matrix is given in the case of MSSM by:

$$\frac{d\kappa}{dt} = [-3g_2^2 + 6Tr(Y_u^\dagger Y_u)]\kappa + \frac{1}{2}[\kappa(Y_e^\dagger Y_e) + (Y_e^\dagger Y_e)\kappa] \tag{7.72}$$

We note two kinds of effects on the neutrino mass matrix from the above formula: (i) one that is flavor independent and (ii) a part that is flavor specific. If we work in a basis where the charged leptons are diagonal, then the resulting correction to the neutrino mass matrox is given by:

$$\mathcal{M}_\nu(M_Z) \;=\; (1+\delta)\mathcal{M}(M_{B-L})(1+\delta) \tag{7.73}$$

where δ is a diagonal matrix with matrix elements $\delta_{\alpha\alpha} \simeq -\frac{m_\alpha^2 tan^2\beta}{16\pi^2 v^2}$ In more complicated theories, the corrections will be different. Let us now study some implications of this corrections. For this first note that in the MSSM, this effect can be sizable if $tan\beta$ is large (of order 10 or bigger).

7.7.3. *Radiative magnification of neutrino mixing angles*

A major puzzle of quark lepton physics is the diverse nature of the mixing angles. Whereas in the quark sector the mixing angles are small, for the neutrinos they are large. One possible suggestion in this connection is that perhaps the mixing angles in both quark and lepton sectors at similar at some high scale; but due to renormalization effects, they may become magnified at low scales. It was shown in ref.[54] that this indeed happens if the neutrino spectrum os degenerate. This can be seen in a simple way for the $\nu_\mu - \nu_\tau$ sector.[54]

Let us start with the mass matrix in the flavor basis:

$$\mathcal{M}_{\mathcal{F}} = \mathcal{U}^* \mathcal{M}_{\mathcal{D}} \mathcal{U}^\dagger = \begin{pmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 e^{-i\phi} \end{pmatrix} \begin{pmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{pmatrix}. \tag{7.74}$$

Let us examine the situation when $\phi = 0$ (i.e. CP is conserved), which corresponds to the case when the neutrinos ν_1 and ν_2 are in the same CP eigenstate. Due to the presence of radiative corrections to m_1 and m_2, the matrix $\mathcal{M}_{\mathcal{F}}$ gets modified to

$$\mathcal{M}_{\mathcal{F}} \rightarrow \begin{pmatrix} 1+\delta_\alpha & 0 \\ 0 & 1+\delta_\beta \end{pmatrix} \mathcal{M}_{\mathcal{F}} \begin{pmatrix} 1+\delta_\alpha & 0 \\ 0 & 1+\delta_\beta \end{pmatrix}. \tag{7.75}$$

The mixing angle $\bar{\theta}$ that now diagonalizes the matrix $\mathcal{M}_{\mathcal{F}}$ at the low scale μ (after radiative corrections) can be related to the old mixing angle θ

through the following expression:

$$\tan 2\bar{\theta} \;=\; \tan 2\theta \;(1 \;+\; \delta_\alpha + \delta_\beta)\,\frac{1}{\lambda}\;, \tag{7.76}$$

where

$$\lambda \equiv \frac{(m_2 \;-\; m_1)C_{2\theta} \;+\; 2\delta_\beta(m_1 S_\theta^2 \;+\; m_2 C_\theta^2) \;-\; 2\delta_\alpha(m_1 C_\theta^2 \;+\; m_2 S_\theta^2)}{(m_2 \;-\; m_1)C_{2\theta}} \tag{7.77}$$

If

$$(m_1 - m_2)\; C_{2\theta} \;=\; 2\delta_\beta(m_1 S_\theta^2 \;+\; m_2 C_\theta^2) \;-\; 2\delta_\alpha(m_1 C_\theta^2 \;+\; m_2 S_\theta^2) \tag{7.78}$$

then $\lambda = 0$ or equivalently $\bar{\theta} = \pi/4$; *i.e.* maximal mixing. Given the mass heirarchy of the charged leptons: $m_{l_\alpha} \ll m_{l_\beta}$, we expect $|\delta_\alpha| \ll |\delta_\beta|$, which reduces (7.78) to a simpler form:

$$\epsilon \;=\; \frac{\delta m C_{2\theta}}{(m_1 S_\theta^2 \;+\; m_2 C_\theta^2} \tag{7.79}$$

In the case of MSSM, the radiative magnification condition can be satisfied provided provided

$$h_\tau(MSSM) \;\approx\; \sqrt{\frac{8\pi^2|\Delta m^2(\Lambda)|C_{2\theta}}{ln(\frac{\Lambda}{\mu})m^2}}\;. \tag{7.80}$$

For $\Delta m^2|simeq\Delta m^2_A$, this condition can be satisfied for a very wide range of $tan\beta$.

It is important to emphasize that this magnification occurs only if at the seesaw scale the neutrino masses are nearly degenerate. A similar mechanism using the righthanded neutrino Yukawa couplings instead of the charged lepton ones has been carried out recently.[55] Here two conditions must be satisfied: (i) the neutrino spectrum must be nearly degenerate (i.e. $m_1 \simeq m_2$ as in ref.[54]) and (ii) there must be a hierarchy between the right handed neutrinos.

7.7.3.1. *An explicit example of a neutrino mass matrix unstable under RGE*

In this section, we give an explicit example of a neutrino mass matrix unstable under RGE effects. Consider the following mass matrix with degenerate neutrino masses and a bimaximal mixing.[56]

$$\mathcal{M}_\nu \;=\; \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{7.81}$$

The eigenvalues of this mass matrix are $(1, -1, 1)$ and the eigenvectors:

$$V_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}; V_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}; V_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{7.82}$$

After RGE to the weak scale, the mass matrix becomes

$$\mathcal{M}_\nu \;=\; \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}(1+\delta) \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2}(1+\delta) \\ \frac{1}{\sqrt{2}}(1+\delta) & \frac{1}{2}(1+\delta) & \frac{1}{2}(1+2\delta) \end{pmatrix} \tag{7.83}$$

It turns out that the eigenvectors of this matrix become totally different and are given by:

$$V_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \\ 0 \end{pmatrix}; V_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}; V_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{2\sqrt{3}} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{7.84}$$

We thus see that the neutrino mixing pattern has become totally altered. although the eigenvalues are only slightly perturbed from their unperturbed value.

7.7.4. *Seesaw paradigm and leptogenesis*

Finally let us comment that in models where the light neutrino mass is understood via the seesaw mechanism using heavy righthanded neutrinos, there is a very simple mechanism for the generation of baryon asymmetry of the universe. Since the righthanded neutrino has a high mass, it decays at a high temperature which in combination with CP violation in Y_ν generates a lepton asymmetry.[57] This lepton asymmetry is converted to baryon asymmetry via the sphaleron effects[58] above the electroweak phase transition temperature since sphalerons break B+L conservation. It also turns out that one of the necessary conditions for sufficient leptogenesis is that the right handed neutrinos must be heavy as is required by the seesaw mechanism. To see this note that one of Sakharov conditions for leptogenesis is that the right handed neutrino decay must be slower than the expansion rate of the universe at the temperature $T \sim M_{N_R}$. The corresponding condition is:

$$\frac{h_\ell^2 M_{N_R}}{16\pi} \leq \sqrt{g^*}\frac{M_{N_R}^2}{M_{P\ell}} \tag{7.85}$$

This implies that $M_{N_R} \geq \frac{h_\ell^2 M_{P\ell}}{16\pi\sqrt{g^*}}$. Translating this into a reliable bound on the masses of the right handed neutrinos is quite model dependent since the Yukawa texture i.e. h_ℓ values in the above equation depends on the particular way to understand large mixings as well as the neutrino mass hierarchy.

To proceed further, we start with the expression for lepton asymmetry in these scenarios:[59]

$$\varepsilon_i^I = -\frac{1}{8\pi}\frac{1}{[Y_\nu Y_\nu^\dagger]_{ii}}\sum_j \mathrm{Im}[\mathrm{Y}_\nu \mathrm{Y}_\nu^\dagger]_{\mathrm{ij}}^2 F(\frac{M_j^2}{M_i^2}), \tag{7.86}$$

One can draw several conclusions from this expression: first using Eq.7.65 and 7.86, we see that, ε_i^I is independent of the low energy CP phases. This implies that in principle observation of CP violation in neutrino oscillation may not throw any light on the origin of matter.

Second point to notice is that for hierarchical masses for RH neutrinos i.e. $M_1 \ll M_{2,3}$, one can write 7.86 as

$$\varepsilon_i^I = -\frac{1}{8\pi}\frac{1}{[Y_\nu Y_\nu^\dagger]_{ii}}\sum_j \mathrm{Im}[\mathrm{Y}_\nu \mathcal{M}_\nu^* \mathrm{Y}_\nu^{\mathrm{T}}]_{\mathrm{ii}}. \tag{7.87}$$

From this it follows that if the neutrino masses are strictly degenerate, then using 7.65, it is easy to see that $\varepsilon_i^I = 0$. This is an interesting result although this cannot strictly be used to rule out the possibility of degenerate neutrino masses since, it is more natural (as emphasized earlier) for a degenerate neutrino spectrum to arise from a type II seesaw rather than type I seesaw which has been used in drawing this conclusion.

Another consequence of 7.86 is that combining 7.85 and 7.86, it is possible to obtain a reliable lower bound on the lightest RH neutrino mass[60] and it turns out to be: $M_{N_1} \geq 10^9$ GeV. This is bound is somewhat strengthened if one further demands that there is a $\mu - \tau$ exchange symmetry in the left as well as the right handed neutrino sector for which there is some observational indication (since θ_{13} appears to be very small). The bound then becomes $M_{N_1} \geq 6 \times 10^9$ GeV.[61]

In the presence of triplet contributions to seesaw (type II), there are new contributions to the lepton asymmetry given by: The type II contribution has been calculated and is given in Ref.[45,62] to be

$$\varepsilon_i^{II} = \frac{3}{8\pi} \frac{\mathrm{Im}[Y_\nu f^* Y_\nu^T \mu]_{ii}}{[Y_\nu Y_\nu^\dagger]_{ii} M_i} \ln(1 + \frac{M_i^2}{M_T^2}), \tag{7.88}$$

where $\mu \equiv \lambda M_T$ and λ is the coupling between triplet and two doublets in the superpotential and f is the triplet coupling to leptons. Thus baryogenesis via leptogenesis is a very sensitive way to probe the neutrino mass mechanisms. For more details on this see the lectures by M.C.Chen.[63]

7.8. Conclusions and outlook

In summary, the neutrino oscillation experiments have provided the first evidence for new physics beyond the standard model. The field of neutrino physics, along with the search for the origin of mass, dark matter has therefore become central to the study of new physics at the TeV scale and beyond. Another area which is foremost in the minds of many theorists is supersymmetry which stabilizes the Higgs mass, provides a way to understand the electro-weak symmetry breaking and possibly a dark matter candidate. In discussing consequences of seesaw mechanism as well as in seeking theories of neutrino mass, we have assumed supersymmetry. An exception is the last section, where we consider low scale extra dimensional models for understanding Higgs mass and its possibility as an alternative to seesaw mechanism.

What have we learned so far ? One thing that seems very clear is that there is probably a set of three right handed neutrinos which restore quark lepton symmetry to physics; secondly there must be a local $B - L$ symmetry at some high scale beyond the standard model that keeps the RH neutrinos so far below the Planck scale. While there are very appealing arguments that the scale of $B-L$ symmetry is close to 10^{14}-10^{16} GeV's, in models with extra dimensions, one cannot rule out the possibility that it is around a few TeVs, although the present TeV scale models generally require many near TeV particles with sometimes undesirable consequences for flavor violation. Third thing that one may suspect is that the right handed neutrino spectrum may be split into a heavier one and two others which are nearby. If this suspicion is confirmed, that would point towards an $SU(2)_H$ horizontal symmetry or perhaps even an $SU(3)_H$ symmetry which breaks into an $SU(2)_H$ symmetry (although simple anomaly considerations prefer the first alternative).

The correct theory should explain:

(i) Why both the solar mixing angle is large and atmospheric mixing angle maximal ?

(ii) Why the $\Delta m^2_\odot \ll \Delta m^2_A$ and what is responsible for the smallness of U_{e3} ? While in the inverted hierarchy models and models with $\mu-\tau$ symmetry, the smallness of U_{e3} is natural, in general it is not. In fact high precision search for U_{e3} may hold the clue to possible family symmetries vrs simple grad unification.

(iii) What is the nature of CP phases in the lepton sector and what is their relation to the CP phases possibly responsible for baryogenesis via leptogenesis ?

(iv) What is the complete mass spectrum for neutrinos ?

These and other questions are likely to prove to be very exciting challenges to both theory and experiment in neutrino physics for the next two decades. These lectures are meant to be a very cursory overview of what seems to be the simplest way to understand neutrino masses and mixings i.e. seesaw mechanism and some related physics. Even at that, only a few selected topics are covered and for more details and references, we refer the reader to other excellent reviews in the literature.

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