Modified de Broglie approach applied to the Schrödinger and Klein-Gordon equations

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The goal of this paper is an extension of the de Broglie wave mechanics model for a single spinless particle in an electromagnetic field. The analysis indicates that the motion of a particle separates naturally into particle dynamics through the classical Hamilton-Jacobi equation and quantum wave behavior through a pilot or interaction wave equation. The interaction wave equation travels at the classical particle velocity. We study gauge invariance and interpret it in the light of the interaction wave. The Heisenberg uncertainty relations are shown to be implicit in the interaction wave. We also develop a complex quantum-mechanical, relativistic energy-momentum conservation expression using a complex quantum-mechanical four-vector.

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I. INTRODUCTION

The goal of this paper is to study the wave-particle interaction in quantum-mechanical systems using a modification of the de Broglie–Bohm Hamilton-Jacobi approach applied to the Schrödinger and Klein-Gordon equations in the presence of electromagnetic fields. Conservation of probability is replaced with conservation of energy momentum in the interaction process.

The features include the derivation of an interaction wave equation, reinterpreting the expression commonly used for probability conservation as an expression for conservation of interaction energy, and the derivation of a complex quantum-mechanical energy-momentum equation. However, of course, it is well known that a probability interpretation is entirely adequate. We use the word interaction to describe the coupling of the particle to the measurement interaction.

Over the years there have been a number of approaches used to derive and interpret the Schrödinger and Klein-Gordon equations. Schrödinger's approach was based on an intuitive generalization of Hamilton-Jacobi theory. Feynman's [1] approach focused on the many-path interpretation of Hamilton's principal function and path integrals. Nelson developed a stochastic theory of quantum mechanics [2]. In 1927, de Broglie developed a pilot-wave theory where he proposed that a particle is guided by the quantum-mechanical wave function. This theory was later rediscovered and extended by Bohm [3]. The de Broglie–Bohm approach is based on the assumption of a specific functional form for the wave function then substituting it into the Schrödinger and Klein-Gordon equations and separating out the real and imaginary component equations for energy and probability conservation [3–6]. In this paper we follow this procedure, but instead decompose the Schrödinger and Klein-Gordon equations into the classical Hamilton-Jacobi equation plus a nonlinear complex interaction wave equation. More recently there has been a revived interest in hidden-variable theories and the de Broglie–Bohm approach. Holland, Grossing, and others have performed extensive research in this area [6–8]. Barut and Hestenes developed theories of *Zitterbewegung* that are related to self-interacting or pilot waves traveling with particles [9,10]. Hall and Reginatto have studied quantum-mechanical uncertainty using Fisher information [11–13].

We start our analysis by reviewing Schrödinger's wave equation for a single particle of zero spin in an electromagnetic field. We then identify and separate the quantum-mechanical phase into an associated classical phase plus a nonclassical fluctuation in phase. In the analysis we argue that instead of viewing the evolution of the probability density equation as a primary conservation equation, we can derive a better understanding by considering that the equation represents momentum conservation in the interaction process. We therefore cast the probability conservation equation into an equation for conservation of interaction energy momentum, which is conserved over all space. In the past, researchers have canceled a common factor of $\hbar$ from the equation representing the imaginary part of the Schrödinger equation and therefore viewed it as a probability conservation equation. However, in the classical limit as $\hbar \rightarrow 0$ this is not allowed and as a result, in the classical limit, this probability equation does not apply and the Schrödinger equation then reduces to the classical Hamilton-Jacobi equation.

In Sec. III we subtract the classical Hamilton-Jacobi equation from a form of the Schrödinger equation and thereby develop a reduced form of Schrödinger's wave equation for fluctuations of nonclassical energy momentum related to *Zitterbewegung*. The effects of the electromagnetic and other potentials are contained in the classical Hamilton-Jacobi equation. Solving the Schrödinger equation becomes equivalent to solving the classical Hamilton-Jacobi equation and the interaction wave equation. In Appendix A we show how the Heisenberg uncertainty relations are contained within the interaction wave equation. The interaction wave function is shown to possess interaction energy, as well as energy in fluctuations in the particle energy. The analysis becomes more transparent in Sec. IV, where the interaction wave function for the Klein-Gordon equation is derived. Here we show in the last section that the Klein-Gordon equation can be interpreted as an equation for the conservation of complex four-momentum. We also study the relationship of unitary transformations of the Schrödinger wave function to electromagnetic gauge transformations and the interaction wave

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function. Finally, in Sec. V we develop an expression from the Klein-Gordon equation for a quantum-mechanical energy-momentum four-vector $\mathbf{p}_t=(\mathbf{p}_q+i\mathbf{p}_s)$ and derive a relativistically invariant energy-momentum conservation expression that we will reproduce here:

$$p_{t(\mu)}p_t^{\mu}-m_0^2c^2=i\hbar\,\partial_\mu p_t^{\mu}, \tag{1}$$

where Einstein summation is used and $\mathbf{p}_q$ is a point wise, rather than an expectation value of the quantum-mechanical particle four-momentum and $\mathbf{p}_s$ is the point-wise four-momentum of the interaction process. In the classical limit this reduces to the classical relativistic equation.

All results of the paper are consistent with predictions of the Schrödinger and Klein-Gordon equations; however the interpretations, we believe, are different. Although based on the de Broglie approach, this paper differs significantly from de Broglie's theory of the double solution, where he attempted to use a singularity to describe the particle. Our approach also deviates from Bohm's approach since ours is not based on the guidance principle and instead we subtract classical energy from the relevant Schrödinger and Klein-Gordon energies [3,4].

II. NONRELATIVISTIC QUANTUM THEORY

The goal of this section is to introduce our reinterpretations of nonrelativistic quantum theory that we will later generalize to the relativistic Klein-Gordon equation.

The Schrödinger wave equation as expressed in SI units with electromagnetic potentials ϕ and $\mathbf{A}$ and other potential V is given by

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{1}{2m_0}\left(\frac{\hbar}{i}\boldsymbol{\nabla}-e\mathbf{A}\right)\cdot\left(\frac{\hbar}{i}\boldsymbol{\nabla}-e\mathbf{A}\right)\psi+e\phi\psi+V\psi, \tag{2}$$

where e is the electromagnetic charge, $\hbar$ is Planck's constant, m_0 is the rest mass, and c is the speed of light in vacuum. de Broglie, Bohm, and many others have expressed ψ as [3,4,13]

$$\psi=\sqrt{\rho}e^{iS_q/\hbar}, \tag{3}$$

where ρ is a probability density and S_q is the quantum-mechanical phase or action. Note that we use the symbol S for Hamilton's principal function. In contrast to the commonly used representation of the wave function, Eq. (3), we have found that another equivalent representation of the wave function yields a consistent theory

$$\psi=\sqrt{C}e^{(iS_q-S_p)/\hbar}, \tag{4}$$

where S_p has units of action and characterizes the interaction in the measurement of the particle parameters. S_p is related to particle localization through the potential interaction, and C is a normalization constant. In the Schrödinger equation, the connection between the representations given in Eqs. (3) and (4) is $S_p=-(\hbar/2)\ln(\rho/C)$. Since $-\ln\rho$ is an uncertainty in a probability density, we conclude that S_p relates to uncertainty in the measurement interaction. It is well known that there are problems with probability interpretations in the Klein-Gordon equation. Although Eqs. (3) and (4) are equivalent, by using S_p, conservation of four-momentum follows naturally. The equation of motion for S_p, which is equivalent to the equation of motion for probability conservation, is an expression for conservation of energy momentum in the interaction process. In Klein-Gordon theory we will see that four-momentum generated from S_p is the part of the total quantum-mechanical four-momentum from the interaction process, which together with the four-momentum from S_q, the particle momentum, satisfies a balance equation. $\sqrt{C}e^{-S_p/\hbar}$ can be thought of as a distribution much like the Maxwell-Boltzmann distribution and is easily derived from a maximum-entropy variational problem with a constraint on S_p and associated Lagrange multiplier $1/\hbar$. We will argue that $-\partial S_p/\partial t$ is the interaction energy at a particular point and time. When the expectation value is taken, the net interaction energy is 0, since energy momentum is conserved in the interaction process.

Following Hall [13] and Feynman [1] we break up the quantum phase into the classical phase S_c, with associated momentum $\boldsymbol{\nabla}S_c$, and a nonclassical fluctuation in phase S_f with a corresponding nonclassical fluctuation in particle momentum $\boldsymbol{\nabla}S_f$, so that $S_q=S_c+S_f$. We interpret $-\partial S_f/\partial t$ as the fluctuations in the energy of the particle and $\boldsymbol{\nabla}S_f$ as the fluctuations in momentum of the particle. We assume fluctuations originate due to the interaction measurement energy and momentum.

We will also see that $\mathbf{\Phi}=\sqrt{C}e^{(iS_f-S_p)/\hbar}$ satisfies a linear wave equation and S_f+iS_p is a complex action with the real part related to fluctuations in the particle action and S_p related to the interaction process. Therefore $-\partial S_f/\partial t$ and $\boldsymbol{\nabla}S_f$ relates to fluctuations in particle energy and momentum, and $-\partial S_p/\partial t$ and $\boldsymbol{\nabla}S_p$ relates to interaction energy and momentum. The particle fluctuations and interaction momenta are coupled.

To reiterate, for purpose of analysis, we have now introduced four variables: the classical phase S_c, the particle phase $S_q=S_c+S_f$, where S_f is the fluctuation in particle phase and relates to fluctuations in the particle kinetic energy, and S_p, which relates to the interaction.

III. CLASSICAL HAMILTON-JACOBI EQUATION AND THE INTERACTION WAVE EQUATION

We now show how the Schrödinger wave equation can be separated into a classical Hamilton-Jacobi equation for classical particle energy and another wave equation for nonclassical energy momentum modeling the interaction. We begin by assuming a solution for the wave function ψ as a product of a classical particle phase component and a nonclassical wave component

$$\psi=e^{iS_c/\hbar}\mathbf{\Phi}=\psi_c\mathbf{\Phi}, \tag{5}$$

where the classical phase S_c satisfies the classical, nonrelativistic Hamilton-Jacobi equation

$$\frac{\partial S_c}{\partial t}+\frac{1}{2m_0}(\boldsymbol{\nabla}S_c-e\mathbf{A})\cdot(\boldsymbol{\nabla}S_c-e\mathbf{A})+e\phi+V=0. \tag{6}$$

This equation is assumed to completely model the classical evolution in applied electromagnetic fields. The wave function $\boldsymbol{\Phi}=\sqrt{C}e^{(iS_f-S_p)/\hbar}$ contains effects of fluctuations of the particle and the interaction in the measurement process. Using Eq. (5) in Eq. (2) we obtain

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\partial S_c}{\partial t}e^{iS_c/\hbar}\boldsymbol{\Phi}+i\hbar e^{iS_c/\hbar}\frac{\partial\boldsymbol{\Phi}}{\partial t}. \tag{7}$$

Hence, the quantum-mechanical energy can be decomposed into classical and nonclassical components. In this paper the point-wise energy and momentum each have two coupled components, one energy term relating to the particle aspects, and one energy term for the measurement interaction. The energy at a specific point and time is complex denoting particle energy and interaction energy

$$E_q=\frac{i\hbar}{\psi}\frac{\partial\psi}{\partial t}=-\frac{\partial S_c}{\partial t}+\frac{i\hbar}{\psi^*\psi}\frac{\partial\boldsymbol{\Phi}}{\partial t}\boldsymbol{\Phi}^*$$
$$\equiv-\frac{\partial S_c}{\partial t}+\left(-\frac{\partial S_f}{\partial t}-i\frac{\partial S_p}{\partial t}\right)=-\frac{\partial S_q}{\partial t}-i\frac{\partial S_p}{\partial t}. \tag{8}$$

The energy operator in the Schrödinger equation is Hermitian so the expected energy must be real, which we see to be true by taking the expectation of Eq. (8)

$$\langle E_q\rangle=\langle-\partial S_q/\partial t\rangle, \tag{9}$$

where the brackets denote $\langle f\rangle=\int\psi f\psi^* dV=C\int f\exp(-2S_p/\hbar)$. This shows that $\langle-\partial S_p/\partial t\rangle=0$ since the total energy in the interaction and observer is conserved.

The term $-\partial S_c/\partial t$ on the right-hand side (RHS) of Eq. (8) is the classical energy of the particle and $-\partial S_f/\partial t$ is the nonclassical contribution and is related to particle fluctuations. The imaginary part $-\partial S_p/\partial t$ is the interaction energy, which has an expectation value 0. This is reminiscent of the treatment of the dissipative or interaction energies in electromagnetics by use of complex functions. Each of these two energies are themselves real, but they are coupled in the interaction process. The three-vector momentum in the Schrödinger equation is similarly decomposed into classical and nonclassical components. It is important to note that in the Klein-Gordon analysis the momentum will be a four-vector.

When Eq. (5) is substituted into Eq. (2), and Eq. (6) is used, we obtain a linear interaction wave equation for Φ, which is closely related to the concept of *Zitterbewegung*:

$$i\hbar\left[\frac{\partial\boldsymbol{\Phi}}{\partial t}+\boldsymbol{\nabla}\boldsymbol{\Phi}\cdot\frac{(\boldsymbol{\nabla}S_c-e\mathbf{A})}{m_0}\right]$$
$$=-\frac{\hbar^2}{2m_0}\boldsymbol{\nabla}^2\boldsymbol{\Phi}-\frac{i\hbar}{2m_0}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}S_c-e\mathbf{A})\boldsymbol{\Phi}. \tag{10}$$

This equation and Eq. (8) are the primary results of this section of the paper; the rest of the section deals with interpretation of the equation. The key idea in this part of the paper is that once the classical energy is subtracted from the energy contained in Schrödinger's equation, what is left forms a wave equation that evolves through correlations between fluctuations of the particle's kinetic energy and the interaction energy. The $\boldsymbol{\Phi}$ wave, which is just a phase transformation of the wave function, contains energy momentum from the interaction process on the RHS in the form of a complex potential. What is notable is the combination of the interaction and the resulting fluctuations in particle kinetic energy form a wave. Note that the evanescence of the wave is caused by the last term, which is the divergence of the classical momentum. In the absence of this term the equation would describe a free particle. Equation (10) is a reduced form of Schrödinger's equation and is related to Hestenes' self-interaction and de Broglie's concept of a pilot wave [4,9,10]. Equation (10) contains all the nonclassical contributions, and the envelope travels with a velocity equal to the classical particle velocity $(\boldsymbol{\nabla}S_c-e\mathbf{A})/m_0$. Note that the left-hand side of Eq. (10) contains a convective derivative or the derivative with respect to a coordinate frame traveling with the particle. In this approach, the classical phase is derived from Eq. (6). The solution to the Schrödinger or Klein-Gordon equations reduces to solving the classical Hamilton-Jacobi equation and then the interaction wave equation. Note that $\boldsymbol{\Phi}$ combines S_f and S_p, each of which alone, as we will see, satisfy nonlinear differential equations, into a linear Eq. (10). Later, when we apply our approach to the Klein-Gordon relativistic theory, we will develop a more general interaction wave equation with space and time symmetry. S_p and S_f are conjugate variables in the sense of Heisenberg's uncertainty relations. In Appendix A we relate various derivatives of $S_p\leftrightarrow\Delta t,\Delta\mathbf{q}$ and derivatives of $S_f\leftrightarrow\Delta E,\Delta\mathbf{p}$ to the derive the Heisenberg uncertainty principle. This expresses the fact that the phase and magnitude of the interaction wave are related.

It is well known that an equation for the total quantum-mechanical energy balance can be obtained by substituting Eq. (4) into Eq. (2) and taking the real part to obtain [3]

$$\frac{\partial S_q}{\partial t}+\frac{1}{2m_0}(\boldsymbol{\nabla}S_q-e\mathbf{A})\cdot(\boldsymbol{\nabla}S_q-e\mathbf{A})+e\phi+V+\frac{\hbar}{2m_0}\boldsymbol{\nabla}^2S_p$$
$$-\frac{\boldsymbol{\nabla}S_p\cdot\boldsymbol{\nabla}S_p}{2m_0}=0. \tag{11}$$

Similarly, the real part of Eq. (10) is the change in energy due to the nonclassical particle momentum fluctuations

$$\frac{\partial S_f}{\partial t}+\frac{[(\boldsymbol{\nabla}S_c-e\mathbf{A})+\boldsymbol{\nabla}S_f]\cdot[(\boldsymbol{\nabla}S_c-e\mathbf{A})+\boldsymbol{\nabla}S_f]}{2m_0}$$
$$-\frac{(\boldsymbol{\nabla}S_c-e\mathbf{A})\cdot(\boldsymbol{\nabla}S_c-e\mathbf{A})}{2m_0}+\frac{\hbar}{2m_0}\boldsymbol{\nabla}^2S_p-\frac{\boldsymbol{\nabla}S_p\cdot\boldsymbol{\nabla}S_p}{2m_0}$$
$$=0. \tag{12}$$

Our interpretation of Eq. (12) is that it is a nonlinear differential equation relating to total energy conservation of a particle. The relativistic version of this equation, as derived from the Klein-Gordon equation, does not depend explicitly on the rest mass. We will see this in Sec. IV, where the relativistic generalization of this equation is derived.

The imaginary part of Eq. (2) or Eq. (10) can be written as

$$\frac{\partial S_p}{\partial t}+\nabla S_p\cdot\frac{(\nabla S_q-e\mathbf{A})}{m_0}=\frac{\hbar}{2m_0}\nabla\cdot(\nabla S_q-e\mathbf{A}). \quad (13)$$

Equation (13) is a balance equation for the interaction energy momentum. Since Eqs. (12) and (13) are coupled, energy-momentum cycles between the interaction energy and fluctuations in particle kinetic energy. This is analogous to the energy and the nonreactive energy in electromagnetic systems. Equation (13) is equivalent to the conservation of probability equation used by Bohm and de Broglie if one cancels a common factor of $\hbar$ from Eq. (13) when one converts to probability density

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\frac{(\nabla S_q-e\mathbf{A})}{m_0}\right)=0. \quad (14)$$

The imaginary part of Eq. (10) satisfies either Eq. (13) or Eq. (14). In this paper we work with Eq. (13) rather than Eq. (14). This interaction energy acts as a source to the quantum-mechanical energy around the particle site. The total interaction energy is 0 when the expectation over all space is taken. In the classical limit, S_p and S_f are constants and $\mathbf{\Phi}\to 1$. Therefore in the classical limit the Schrödinger equation reduces to only a single equation, Eq. (6), and there is no probability-conservation equation or S_p equation.

Even though the Schrödinger equation is nonrelativistic, for purposes of analysis we can better understand Eq. (13) by recasting it as an expression for conservation of the effective energy-momentum four-vector in the interaction

$$\frac{1}{c}\frac{\partial}{\partial t}(m_0 c\Theta^2)+\nabla\cdot(\Theta^2(\nabla S_q-e\mathbf{A}))=0, \quad (15)$$

where $\Theta=\sqrt{\rho}=\sqrt{C}e^{-S_p/\hbar}$ and in this classical limit, the time component of the four-momentum is $-(\partial S_q/\partial t+V+e\phi)/c\to m_0 c$. We interpret Eq. (15) as an equation of energy-momentum conservation. For the case of constant rest mass, we define the four-vector $\mathbf{p}_q=(m_0 c,\nabla S_q-e\mathbf{A})$. Also, $\partial^\mu\equiv\partial/\partial_\mu=(\partial/\partial_\mu,-\nabla)$, $\partial_\mu\equiv\partial/\partial^\mu=(\partial/\partial_\mu,\nabla)$. We define the interaction-wave momentum four-vector as

$$\mathbf{p}_s=\left(-\frac{1}{c}\frac{\partial S_p}{\partial t},\nabla S_p\right). \quad (16)$$

With these definitions, Eq. (15) can be written as an equation for interaction-momentum conservation:

$$\partial_\mu(p_q^\mu\Theta^2)=0 \quad (17)$$

or

$$2p_{q(\mu)}p_s^\mu\equiv 2\mathbf{p}_q\cdot\mathbf{p}_s=\hbar\,\partial_\mu p_q^\mu. \quad (18)$$

Equation (17) shows that the transfer of momentum between the interaction and the particle is given by the projection of $\mathbf{p}_s$ onto $\mathbf{p}_q$. If S_p has four-momentum perpendicular to the particle's momentum or vanishes, then they are independent and there is no coupling or interaction. In the classical limit, as $\hbar\to 0$ we see that $\mathbf{p}_s$ is orthogonal to $\mathbf{p}_q$ or $\partial_\mu p_q^\mu=\partial_\mu p_c^\mu=0$. Equation (18) highlights the interaction of the particle-momentum fluctuations and interaction energy momentum. This shows that in the classical limit the interaction momentum can be decoupled from the particle momentum by taking them into account separately, whereas in quantum theory they are coupled in a complicated way.

In Schrödinger's equation, the time derivative of the first term in p_q^0 is 0 because in a nonrelativistic approximation the time component of this momentum is approximately $m_0 c$, which is constant. The derivative in the second term in Eq. (18) is due to mass-energy changes from the interaction and is carried in the wave moving with the particle. The term Θ^2 originates as a weighting factor, since some of the momentum is in the particle and some is carried in the interaction-field momentum ∇S_p. Equations (13) and (17) are more fundamental than Eq. (14) because they are equations for conservation of the mass-energy and extend without modification into relativistic quantum mechanics. As we will see, the relativistic Klein-Gordon generalization amounts to replacing the field momentum $m_0 c$ with the relativistic time component of an energy-momentum four-vector, $m_0 c\to -(1/c)(\partial S_q/\partial t+e\phi+V)$. Therefore, we conclude a dual interpretation of Eq. (13), first as an equation for conservation of probability and second as an equation for mass-energy conservation in the interaction. However, in the Schrödinger equation, the mass is constant and Eq. (15) is also fully equivalent to a conservation of probability density equation with $\rho=C\exp(-2S_p/\hbar)$. In the Klein-Gordon equation this analogy does not hold and the relativistic mass depends on velocity, but obeys the same equation as Eq. (17).

We can also reduce Eq. (12) to a nonlinear wave equation. The real part of Eq. (10) can be written as

$$\left[2m_0\frac{\partial S_f}{\partial t}+[(\nabla S_c-e\mathbf{A})+\nabla S_f]\cdot[(\nabla S_c-e\mathbf{A})+\nabla S_f]-(\nabla S_c-e\mathbf{A})\cdot(\nabla S_c-e\mathbf{A})\right]\Theta=\hbar^2\nabla^2\Theta. \quad (19)$$

The wave is driven by the difference between quantum-mechanical and classical energy. When integrated over all space the RHS goes to zero, indicating energy conservation over space. In the Klein-Gordon equation the analogous equation does not depend explicitly on rest mass and is symmetric in time and space.

We now study Eqs. (2) and (10) under unitary transformations such as $e^{i\Delta S_c/\hbar}$ and the associated electromagnetic gauge transformations. It is well known that Maxwell's equations are invariant under gauge transformations where $\mathbf{A}\to\mathbf{A}+\nabla\chi$ and $\phi\to\phi-\partial\chi/\partial t$, and Schrödinger's equation in the presence of an electromagnetic field is invariant to phase translations. If we take a transformation in Eq. (5) such that $S_c\to S_c+\Delta S_c$, then the freedom of the gauge makes Eq. (6)

invariant to the transformation. The same gauge transformation that makes Eq. (6) invariant cancels any phase change in S_c in Eq. (10). Therefore, when Eq. (5) is multiplied by an arbitrary unitary transformation, all the effects are manifested in a phase shift in $e^{iS_c/\hbar}\rightarrow e^{i(S_c+\Delta S_c)/\hbar}$. The arbitrary phase in the Schrödinger wave function is seen as being due to the arbitrariness in the classical momentum and energy, whereas the quantum behavior is contained in the interaction wave. Φ is a unitary transformation of ψ.

IV. INTERACTION WAVE EQUATION IN RELATIVISTIC QUANTUM THEORY

In this section we present the theory of the interaction wave equation in a relativistically invariant format using the Klein-Gordon equation for a spinless particle. We will see that in this case the analysis is symmetric in time and space and the interpretations are clearer.

The Klein-Gordon equation in flat space-time is

$$\frac{1}{m_0c^2}\left(i\hbar\frac{\partial}{\partial t}-e\phi-V\right)\left(i\hbar\frac{\partial}{\partial t}-e\phi-V\right)\psi = \frac{1}{m_0}\left(\frac{\hbar}{i}\boldsymbol{\nabla}-e\mathbf{A}\right)\cdot\left(\frac{\hbar}{i}\boldsymbol{\nabla}-e\mathbf{A}\right)\psi+m_0c^2\psi=0. \tag{20}$$

The generalized relativistic Hamilton-Jacobi equation can be written as

$$g^{kl}p_{k(c)}p_{l(c)}-m_0^2c^2=0, \tag{21}$$

where g^{kl} are the metric coefficients and we use the Einstein summation convention. The classical Hamilton-Jacobi four-momentum in an electromagnetic field is

$$\mathbf{p}_c=(E_c/c,\mathbf{p}_{c(3)})=\left(-\frac{1}{c}\left(\frac{\partial S_c}{\partial t}+e\phi+V\right),(\boldsymbol{\nabla}S_c-e\mathbf{A})\right). \tag{22}$$

For the special case of flat spacetime, the relativistic Hamilton-Jacobi equation is [16]

$$\frac{1}{m_0c^2}\left(\frac{\partial S_c}{\partial t}+e\phi+V\right)^2-\frac{(\boldsymbol{\nabla}S_c-e\mathbf{A})\cdot(\boldsymbol{\nabla}S_c-e\mathbf{A})}{m_0}-m_0c^2 =0. \tag{23}$$

The case of a general metric is presented in Appendix B.

The interaction wave equation is determined by substituting Eq. (5) into Eq. (20) and using Eq. (23)

$$\frac{\hbar^2}{2}\left[\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}-\nabla^2\boldsymbol{\Phi}\right]-i\hbar\left[-\frac{1}{c^2}\left(\frac{\partial S_c}{\partial t}+e\phi+V\right)\frac{\partial\Phi}{\partial t} +(\boldsymbol{\nabla}S_c-e\mathbf{A})\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}\right]+\frac{i\hbar}{2}\left[\frac{1}{c^2}\left(\frac{\partial^2 S_c}{\partial t^2}+\frac{\partial(e\phi+V)}{\partial t}\right) -\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}S_c-e\mathbf{A})\right]\boldsymbol{\Phi}=0. \tag{24}$$

The last term causes localization and would vanish only for particles in a region with no potential, that satisfy the condition $(1/c^2)\partial^2 S_c/\partial t^2-\nabla^2 S_c=0$, in the Lorenz gauge. Equation (24) can be written in more compact notation as

$$i\hbar\partial_\mu(i\hbar\partial^\mu\boldsymbol{\Phi})-2p_c^\mu(i\hbar\partial_\mu\boldsymbol{\Phi})-i\hbar(\partial^\mu p_{\mu(c)})\boldsymbol{\Phi}=0. \tag{25}$$

Note that this equation does not explicitly contain the rest mass. The real part of the interaction wave equation (24) yields an energy-momentum relation

$$-\frac{1}{c^2}\frac{\partial S_f}{\partial t}\left(\frac{\partial S_c}{\partial t}+e\phi+V\right)+\boldsymbol{\nabla}S_f\cdot(\boldsymbol{\nabla}S_c-e\mathbf{A}) +\frac{1}{2}\left[-\frac{1}{c^2}\left(\frac{\partial S_f}{\partial t}\right)^2+\boldsymbol{\nabla}S_f\cdot\boldsymbol{\nabla}S_f\right] +\frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial S_p}{\partial t}\right)^2-\boldsymbol{\nabla}S_p\cdot\boldsymbol{\nabla}S_p\right] +\frac{\hbar}{2}\left[-\frac{1}{c^2}\frac{\partial^2 S_p}{\partial t^2}+\nabla^2 S_p\right]=0. \tag{26}$$

In more compact notation, the energy momentum in the interaction wave function satisfies a conservation wave equation for Θ:

$$[(\partial_\mu S_q+eA_\mu)(\partial^\mu S_q+eA^\mu)-(\partial_\mu S_c+eA_\mu)(\partial^\mu S_c+eA^\mu)]\Theta =\hbar^2\partial_\mu\partial^\mu\Theta, \tag{27}$$

where we combined V with $e\phi$. This can be written as

$$\hbar^2\partial_\mu\partial^\mu\Theta=[p_{\mu(q)}p_q^\mu-m_0^2c^2]\Theta, \tag{28}$$

where

$$\mathbf{p}_q=\left(-\frac{1}{c}\left(\frac{\partial S_q}{\partial t}+e\phi+V\right),(\boldsymbol{\nabla}S_q-e\mathbf{A})\right). \tag{29}$$

This equation is nonlinear since $\mathbf{p}_q$ depends on S_f, which depends on S_p. This equation is close to that derived by de Broglie [4,6], but differs in interpretation. Equation (28) indicates that the difference in quantum and classical energy-momentum drives the pulse represented by Θ. In the limit where the classical and quantum energy-momentum become equal, the RHS of Eq. (28) vanishes and then Θ satisfies a homogenous wave equation. Equation (28) is analogous to that of a string embedded in an elastic medium and has solutions similar to that for waves in dispersive media. Since we know that Θ is real, the waves are damped. If the RHS of Eq. (28) is less than zero then Θ becomes a propagating wave function, which means the particle is not localized.

Equation (28) is equivalent to the energy-momentum conservation equation:

$$p_{q(\mu)}p_q^{\mu}-m_0^2c^2-p_{s(\mu)}p_s^{\mu}=p_{f(\mu)}p_f^{\mu}+2p_{f(\mu)}p_c^{\mu}-p_{s(\mu)}p_s^{\mu}$$
$$=-\hbar\,\partial_{\mu}\partial^{\mu}S_p\,, \qquad (30)$$

where

$$\mathbf{p}_f=\left(-\frac{1}{c}\frac{\partial S_f}{\partial t},\boldsymbol{\nabla}S_f\right). \qquad (31)$$

Equation (30) is a balance equation for energy momentum added or subtracted from the interaction wave equation. We can rewrite this equation as

$$p_{f(\mu)}p_f^{\mu}+2p_{f(\mu)}p_c^{\mu}=\hbar^2\frac{\partial_{\mu}\partial^{\mu}\Theta}{\Theta}\equiv\Delta m^2c^2. \qquad (32)$$

Δm^2 is what de Broglie called the square of equivalent mass [4]. The problem is that Δm^2 can be positive or negative and therefore it is not useful to treat it as a mass. This is a result of the nonlinear coupling between $\mathbf{p}_f$ and $\mathbf{p}_s$. In the last section of the paper we will derive a complete relationship for the mass-energy using a complex momentum that does not have this nonphysical nature.

In the classical limit the RHS of Eq. (32) goes to zero, and there is no difference between the quantum-mechanical and classical energies. If Eq. (32) is multiplied by Θ^2 and integrated over space, we obtain a balance equation between quantum and classical energy momentum. As a simple, idealized example that models S_p, we assume that Δm is constant, $S_p\propto(\hbar kz\pm\hbar\omega t)$ and then Eq. (28) satisfies the dispersion relation $(\hbar\omega)^2/c^4-(\hbar k)^2/c^2=\Delta m^2$. In this example, the energy is $\mp\partial S_p/\partial t=\pm\hbar\omega$ and momentum $\boldsymbol{\nabla}S_p=\hbar k$. As expected, Eq. (27), unlike the nonrelativistic case in Eq. (19) has time and space symmetry.

The relativistic version of Eq. (13), is

$$-\frac{1}{c^2}\left(\frac{\partial S_q}{\partial t}+e\phi+V\right)\frac{\partial S_p}{\partial t}+(\boldsymbol{\nabla}S_q-e\mathbf{A})\cdot\boldsymbol{\nabla}S_p$$
$$=-\frac{\hbar}{2}\left[\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial S_q}{\partial t}+e\phi+V\right)-\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}S_q-e\mathbf{A})\right]. \qquad (33)$$

The RHS of Eq. (33) is the interaction-energy source of the wave. We can express Eq. (33) compactly as

$$\partial_{\mu}(p_q^{\mu}\Theta^2)=0 \qquad (34)$$

or

$$2\mathbf{p}_q\cdot\mathbf{p}_s=\hbar\,\partial_{\mu}p_q^{\mu}=\hbar\,\partial_{\mu}\partial^{\mu}S_q\equiv\hbar\tilde{\boldsymbol{\nabla}}\cdot\mathbf{p}_q\,. \qquad (35)$$

This equation is analogous to Eq. (18), which is a balance equation for energy momentum added to or subtracted from the particle. The Klein-Gordon equation is then equivalent to Eqs. (23), (28), and (34) for the unknowns S_c, S_f, and S_p.

V. CONSERVATION OF FOUR-MOMENTUM AT SPACETIME POINTS

If we combine Eqs. (30) and (35) into a single complex expression, we can write an energy-momentum conservation equation, analogous to the relativistic relation, in terms of a total complex quantum-mechanical momentum at specific spacetimes positions $\mathbf{p}_t=(\mathbf{p}_q+i\mathbf{p}_s)$:

$$p_{t(\mu)}p_t^{\mu}-m_0^2c^2=i\hbar\,\partial_{\mu}p_t^{\mu}\,. \qquad (36)$$

This is the exact extension of the relativistic momentum conservation to quantum systems. This equation is equivalent to the Klein-Gordon equation and reduces to the classical relativistic expression $\mathbf{p}_c\cdot\mathbf{p}_c=m_0^2c^2$ as $\hbar\to 0$ and $\mathbf{p}_s,\mathbf{p}_f\to 0$. Note that Eq. (36) requires only $\mathbf{p}_q$ or $\mathbf{p}_f$, $\mathbf{p}_c$, and $\mathbf{p}_s$. The complex form of the momentum expresses the fact that the particle momentum and interaction momentum are coupled in the interaction process and must be expressed as two coupled equations. Note that the only place where $\hbar$ appears explicit is on the RHS of Eq. (36). Also note that unlike in the de Broglie theory, here the rest mass is always constant.

Similarly, the nonrelativistic version, equivalent to the Schrödinger equation, can be written in terms of the 3-vector form of $\mathbf{p}_t$, where $\mathbf{p}_q=\boldsymbol{\nabla}S-e\mathbf{A}$ and $\mathbf{p}_s=\boldsymbol{\nabla}S_p$:

$$\mathbf{p}_t\cdot\mathbf{p}_t+2m_0\left(\frac{\partial(S_q+iS_p)}{\partial t}+V\right)=i\hbar\boldsymbol{\nabla}\cdot\mathbf{p}_t\,. \qquad (37)$$

VI. DISCUSSION

We have modified the de Broglie approach to quantum theory by separating the Schrödinger equation into two coupled partial differential equations, one for classical phase (particlelike) and the other for the interaction (wavelike). The interaction wave equation envelope composed of nonclassical energy travels with the velocity of the classical particle. Solving the Schrödinger or Klein-Gordon equations reduces to solving the associated classical Hamilton-Jacobi equation and the interaction wave equation. The Klein-Gordon equation was also separated into classical and nonclassical equations. The interaction wave function for the Klein-Gordon equation does not depend on the rest mass explicitly. The conservation of momentum in the Klein-Gordon equation is given by Eq. (34), and the conservation of energy is given by Eq. (32). The Heisenberg uncertainty relations are contained implicitly in the interaction wave function. Since the energy and momentum are partitioned into particle and wave components at specific space-time points, the energy and momentum are represented as complex quantities. The real part relates to particle energy-momentum and the imaginary part to the interaction momentum. This is analogous to the partitioning of energy in electromagnetic fields into interaction-field energy and dissipative energy. We defined a complex quantum-mechanical momentum and wrote the Klein-Gordon equation as a momentum-conservation equation that reduces to the classical relativistic form as $\hbar \to 0$. This equation shows how the total momentum is related to S_p, the divergence of momentum, and the classical energy. Whereas classical energy-momentum conservation can be expressed as a single equation, in quantum mechanics an additional relation for the interaction itself is needed to balance the energy momentum.

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APPENDIX A: UNCERTAINTY EVOLUTION

In this section we relate the interaction wave to the Heisenberg uncertainty relations. In this process we show how the uncertainty relations are related to Eq. (10).

We begin by investigating the relationship of the total quantum-mechanical kinetic energy to the nonclassical kinetic energy. The expectation of the quantum-mechanical kinetic energy is

$$\begin{aligned}\langle \mathcal{T}_{qs} \rangle &= \frac{1}{2m_0} \left\langle \left(\frac{\hbar}{i} \nabla \psi - e\mathbf{A}\psi \right) \cdot \left(-\frac{\hbar}{i} \nabla \psi^* - e\mathbf{A}\psi^* \right) \right\rangle \\ &= \left\langle \frac{[(\nabla S_c - e\mathbf{A}) + \nabla S_f] \cdot [(\nabla S_c - e\mathbf{A}) + \nabla S_f]}{2m_0} \right\rangle \\ &\quad + \frac{\langle \nabla S_p \cdot \nabla S_p \rangle}{2m_0}. \end{aligned} \tag{A1}$$

We note that the quantum-mechanical kinetic energy contains both the particle's classical and fluctuating kinetic energy and the interaction-field energy. Therefore this kinetic energy contains classical and quantum contributions to phase and energy.

The kinetic energy in the interaction wave is the nonclassical energy

$$\langle \Delta \mathcal{T} \rangle = \langle \mathcal{T}_{qs} \rangle - \frac{\langle (\nabla S_c - e\mathbf{A}) \cdot (\nabla S_c - e\mathbf{A}) \rangle}{2m_0}. \tag{A2}$$

We begin by multiplying Eq. (10) by $\mathbf{\Phi}^*$ and then integrate over space. After a number of simple manipulations we obtain

$$\left(\delta E\, \delta t - \frac{\hbar}{2} \right) \delta q^2 = \left(\langle \Delta \mathcal{T} \rangle \delta q^2 - \frac{\hbar^2}{8m_0} \right) \delta t. \tag{A3}$$

We define the variance δq^2 in terms of the square of the pilot-wave momentum

$$\frac{1}{\delta q^2} = F = \frac{4}{\hbar^2} \langle \nabla S_p \cdot \nabla S_p \rangle, \tag{A4}$$

where F is the Fisher information calculated from the probability density ρ. Minimizing F is known to yield the largest variance subject to constraints [13–15]. Following Hall [13], we define $\sqrt{\delta q^2}$ by using the associated Fisher length

$$\sqrt{\delta q^2} = \frac{1}{\sqrt{F}} \leqslant \sqrt{\Delta q^2}, \tag{A5}$$

where Δq^2 is the position variance. We define an uncertainty in time in terms of the expectation value of the second derivative of S_p as follows

$$\delta t = \left| \frac{2m_0}{\langle \nabla^2 S_p \rangle} \right| \leqslant \Delta t \tag{A6}$$

and the particle energy uncertainty as

$$\delta E = \left\langle -\frac{\partial S_f}{\partial t} \right\rangle \leqslant \Delta E. \tag{A7}$$

For minimum uncertainty we require

$$\Delta E \Delta t \geqslant \delta E\, \delta t = \frac{\hbar}{2}. \tag{A8}$$

For an energy eigenstate $\Delta E \to 0$, and we see from Eq. (A6) and from the fact $\nabla^2 S_p \to 0$ that $\delta t \to \infty$.

If we define the uncertainty in particle kinetic energy as

$$\Delta \mathcal{T} = \frac{\delta \mathbf{p}^2}{2m_0} \geqslant \frac{\Delta \mathbf{p}^2}{2m_0}, \tag{A9}$$

where $\delta \mathbf{p} = \sqrt{\delta \mathbf{p}^2}$, we can obtain the momentum uncertainty relationship

$$\Delta p \Delta q \geqslant \delta p\, \delta q = \frac{\hbar}{2}. \tag{A10}$$

We see that Eq. (10) is an equation for the evolution of coupled uncertainties in energy and momentum. We note $\Delta \mathbf{p}$ and ΔE respectively refer to fluctuations in the particle momentum and energy and $\Delta \mathbf{q}$ and Δt refer to the fluctuations in the pilot-wave. The coupling in the uncertainty relationships is an expression of the interaction.

APPENDIX B: GENERALIZED-METRIC EQUATIONS

For a general metric g^{kl}, we have the generalized interaction wave equation (25),

$$\begin{aligned}&\frac{i\hbar}{\sqrt{-g}} \partial_k (i\hbar \sqrt{-g} g^{kl} \partial_l \mathbf{\Phi}) - 2 g^{kl} p_{k(c)} (i\hbar \partial_l \mathbf{\Phi}) \\ &\quad - \frac{i\hbar}{\sqrt{-g}} \partial_k (\sqrt{-g} g^{kl} p_{l(c)}) \mathbf{\Phi} = 0, \end{aligned} \tag{B1}$$

where g is the determinant of g_{kl}.

Equation (36) is

$$g^{kl} p_{k(t)} p_{l(t)} = m_0^2 c^2 + i\hbar \frac{1}{\sqrt{-g}} \partial_k (\sqrt{-g} g^{kl} p_{l(t)}). \tag{B2}$$

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