Expected returns, risk premia, and volatility surfaces implicit in option market prices

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Abstract

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This article presents a pure exchange economy that extends Rubinstein (1976) to show how the jump-diffusion option pricing model of Merton (1976) is altered when jumps are correlated with diffusive risks. All correlations are statistically different from zero. In equilibrium, the equity risk premium depends not only on the risk premium factors of the traditional jump-diffusion models with systematic jump and diffusion risks, but also on both the covariance of the diffusive pricing kernel with price jumps and the covariance of the jumps of the pricing kernel with the diffusive price. These two covariances are positive, and they help to explain the *sneers* that we observe in the marketplace. The expected stock return is not given by the sum of the diffusive expected return and the expected return due to jumps, but it takes also into account the covariance between the diffusive return and price jumps. Our evidence is consistent with a negative covariance, which leads to a nonmonotonic term structure of implied volatilities. This leads to an asset pricing model and an option pricing model where the level of the market prices is correlated with the size of the jumps.

1. Introduction

The pathbreaking article of Merton (1976) on jump-diffusion option pricing motivated a series of empirical studies and theoretical extensions. The earlier papers by Jarrow and Rosenfeld (1984), Ball and Torus (1985), and Jorion (1988) provide empirical evidence that there are jumps in asset market prices. Our evidence is consistent with slightly less than two jumps per calendar year. The Merton (1976) model assumes that jumps are idiosyncratic risk which can be diversified away and, therefore, is not rewarded. This work was extended among others by Naik and Lee (1990), Ahn (1992), and Amin and Ng (1993) who derived option pricing formulae of the Merton's type assuming that jump risk is systematic risk. Recently, theoretical advances to the model have been made by Duffie, Pan, and Singleton (2000) who extend the theory by allowing jumps in volatility, Kou (2002) who assumes that jumps have a double exponential distribution, and Santa-Clara and Yan (2006) who allow the jump intensity to follow its own stochastic process.[1]

The strand of the literature represented by Naik and Lee (1990), Ahn (1992), and Amin and Ng (1993) assumes that systematic jump risk results from correlated jumps in the asset price and the pricing kernel that results from simultaneous jumps in the asset price and the pricing kernel. In these models, systematic diffusive risk results from the covariance of the stock price Brownian motion with the pricing kernel Brownian motion. One drawback of this previous literature is that it ignores the possibility that the covariance between the diffusive pricing kernel and price jumps and the covariance between jumps in the pricing kernel and the diffusive price might be important for explaining risky equity returns and option prices. Kou (2002, p. 1087) stresses that the usual assumption that the Brownian motion and jumps are independent "can be relaxed". This suggests that new factors of systematic equity risk can be added to option pricing models. For example,

[1]Other recent contributions on jump-diffusions include Anderson, Benzoni, and Lund (2002), Eraker, Johannes, and Polson (2003), Kou and Wang (2004), Maheu and McCurdy (2004), Liu, Pan, and Wang (2004), and Ramezani and Zeng (2007).

the systematic jump risk that results from simultaneous jumps in the stock price and the pricing kernel only accounts for 55 percent of our systematic jump risk.

We investigate the effects of the covariance structure of the underlying uncertainty to aggregate consumption and stock price on expected returns, equity risk premia and option prices. In our model, the equity risk premium can be decomposed in diffusive risk premium and jump risk premium. The diffusive risk premium arises from both the covariance of the diffusive price with the diffusive pricing kernel, and the covariance of the diffusive price with the jumps of the pricing kernel. The new term, that takes into account the covariance of the diffusive price with the jumps of the pricing kernel, adds 4.9 percent to the annual diffusive risk premium. The jump risk premium arises from both the covariance of price jumps with the jumps of the pricing kernel, and the covariance of price jumps with the diffusive pricing kernel. Our evidence is consistent with an annual equity jump risk premium of 12.12 percent. In equilibrium, all these four sources of the equity risk premium affect asset prices, and the two new factors play a determinant role in the shape of the *smiles* and *sneers* that we observe in the options market.

Empirical evidence that investors require a jump risk premium in addition to the diffusion risk premium is provided by Pan (2002), Eraker (2004), Santa-Clara and Yan (2006), and Broadie, Chernov, and Johannes (2007), but these authors ignore the risk premium born with the correlation between Brownian motions and jumps. According to our empirical results, the effect of the covariance between the diffusive pricing kernel and price jumps represents around 45 percent of the jump risk premium.

A standard assumption of all previous jump-diffusion literature is that the Brownian motions and the jumps are independent. The assumption that the correlation between the diffusive pricing kernel and jumps in the pricing kernel as well as the correlation between the stock price level and the size of price jumps are unimportant for option pricing is at odds with the recent empirical and theoretical research. For example, Santa-Clara and Yan (2006) present empirical evidence showing that jump risk is correlated with the stock index. This suggests that the stock price level is not independent of the size of the price jump. Duffie, Pan, and Singleton (2000) also remark that one potential explanation for some stylized facts in option markets might be the fact that option pricing models unnecessarily restrict the correlations of the state variables. These works help to motivate our assumption that Brownian motions and jumps are correlated in our model. In our general model, equilibrium option prices also depend on the correlation between the diffusive pricing kernel and the jumps of the pricing kernel, and on the correlation between the diffusive stock price and the jumps of the stock price. The correlation between the diffusive pricing kernel and the jumps of the pricing kernel is positive, while the correlation between the diffusive stock price and price jumps is negative.

This article assumes that there is a representative agent with a power utility function of consumption who sets prices in equilibrium, and that aggregate consumption and the stock price follow jump-diffusion processes with simultaneous random jumping times. Our evidence is consistent with a coefficient of proportional risk aversion of 6.55, which is within the range of estimated parameter values by Bliss and Panigirtzoglou (2003). In equilibrium the pricing kernel also follows a jump-diffusion process with the same random jumping times. The equilibrium interest rate is determined not only by the usual parameters of the traditional jump-diffusion models with systematic jump and diffusion risks, but also by the covariance between the diffusive pricing kernel and the jumps of the pricing kernel. This, in general, leads to a non-flat term structure of interest rates.

We derive a consumption capital asset pricing model for our jump diffusion model. We obtain this relationship in closed form which, due to its nonlinear nature, tells us implicitly that the expected rate of return of the stock in equilibrium is given by the interest rate plus the stock risk premium. This equity risk premium has four distinct factors as previously remarked. Our general jump-diffusion model, which considers a full covariance structure of the underlying uncertainty to the pricing kernel and stock price, applies the technique of pricing by substitution in equilibrium introduced by Rubinstein (1976) to extend the Merton's (1976) option pricing model and derive formulae of the Merton's type. This means that first we derive the price of the call option in equilibrium, and then use our consumption capital asset pricing equation in the equilibrium valuation equation of the call to eliminate one of the four sources of risk premium. Our general option pricing equation depends not only on the parameters of the traditional option pricing models with systematic diffusion risk and systematic jump risk, but also on the covariance between the diffusive price and price jumps, the covariance between the diffusive pricing kernel and price jumps, the covariance between the diffusive price and jumps in the pricing kernel, and the covariance between the diffusive pricing kernel and jumps in the pricing kernel. All these covariances are statistically significant at the 1 percent level.

The covariance between the diffusive price and price jumps plays an important role in the model that is not shared by any existing jump-diffusion option pricing model. It makes the model able to generate increasing, decreasing, and non-monotone term structures of implied volatilities of at-the-money options. This is relevant since the term structure of implied volatilities of at-the-money forward options in the traditional jump-diffusion models is always an increasing function of the time-to-maturity. This fact, as mentioned by Das and Sundaram (1999) puts traditional jump-diffusion option pricing models at odds with the data, since decreasing and non-monotone term structures of implied volatilities of at-the-money options frequently arise in practice. In our sample period, January 1996 through April 2006, the covariance between the diffusive price and price jumps is negative which generates a nonmonotonic term structure of implied volatilities. This negative correlation means that higher prices are associated with price jumps down. This result is timely and of significant interest since recently on February 27, 2007, the Chinese stock market jumped down 8.84 percent, less than 24 hours after the Shanghai stock index had reached a new record high.

The remainder of the paper is organized as follows. Section 2 presents the standard Euler equations. Section 3 derives our general jump-diffusion model. Section 4 presents the empirical tests and links the empirical results to the theoretical model.

Our general set-up contemplates several special cases of particular interest. For example, it is possible to construct a model of asset prices with systematic jump risk even if aggregate consumption and the pricing kernel do not jump. Section 5 specializes the model by restricting the covariance structure of the underlying uncertainty to aggregate consumption and the stock price into two different directions, and assumes that aggregate consumption follows a geometric Brownian motion while the stock price follows a jump-diffusion process. In this section, we study the roles played by the covariance between the diffusive pricing kernel and price jumps. Section 6 specializes the model by restricting the covariance structure of the underlying uncertainty to aggregate consumption and the stock price into two other different directions, and assumes that aggregate consumption follows a jump-diffusion process while the stock price follows a geometric Brownian motion. In this section, we investigate the roles played by the covariance between the jumps of the pricing kernel and the diffusive price. Section 7 extends the Merton (1976) option pricing model assuming that the stock price level and stock price jumps are correlated. Here we illustrate the roles played by the covariance between the price level and price jumps. In section 8, we present the conclusions of the paper.

2. The Euler equation

The results of this paper are obtained in economies that extend the pure exchange economy of Rubinstein (1976). There is a representative agent who maximizes his expected utility of consumption when he makes his consumption and investment decisions. We assume that this representative agent is nonsatiated and risk averse. Then, from the first order conditions of the representative agent, we can derive the valuation equations of the assets of the economy. These well known results are presented in this short section, and later used in the proofs of our results.

The representative investor maximizes:

$$E\left[\sum_{t=0}^{T} U_t(C_t) \mid \mathcal{F}_0\right], \tag{1}$$

where E is the expectation operator, $U_t(C_t)$ is the utility function of consumption at date t, and $\mathcal{F}_0$ is the information set available to the investor at date $t = 0$.

In this article, we price a riskless bond, a stock, and a call option, with terminal payoffs at time T of \$1, S_T, and $(S_T - K)^+$, where K is the exercise price of the option.[2] The current price of an arbitrary asset, P_0, is given by the following valuation equation:

$$P_0 = E\left[\frac{U'(C_T)}{U'(C_0)}\phi(S_T) \mid \mathcal{F}_0\right], \tag{2}$$

where the marginal rate of substitution $\frac{U'(C_T)}{U'(C_0)}$ is the pricing kernel or stochastic discount factor, and $\phi(S_T)$ is the terminal payoff of the arbitrary asset. Equation (2) is the Euler equation.

Throughout this article we assume that the representative agent has a power utility function given by:

$$U_t(C_t) = \rho^t C_t^{1-b}/(1-b), \tag{3}$$

where ρ is the time discount factor, and b is the coefficient of proportional risk aversion. In this case, the pricing kernel is given by:

$$\psi(C) = \frac{U'(C_T)}{U'(C_0)} = \rho^T \left(\frac{C_T}{C_0}\right)^{-b}. \tag{4}$$

3. General jump-diffusion option pricing model

This section presents our general jump-diffusion option pricing model. It is assumed that both aggregate consumption and stock price follow jump-diffusion processes with simultaneous random

[2] The price of the put can be obtained either using the call-put parity or following similar steps to the ones that lead to the price of the call.

jumping times. Let C_T and S_T be the aggregate consumption and stock price with jump–diffusion processes from n–dimensional Brownian motions:

$$C_T = \exp(a_c(T) + \beta_c \cdot \mathbf{B}_c(T) + Y_c), \tag{5}$$

$$S_T = \exp(a(T) + \beta \cdot \mathbf{B}(T) + Y), \tag{6}$$

where $a_c(T)$ and $a(T)$ are drifts processes, $\beta_c = (\beta_{c,1}, \cdots, \beta_{c,n})$ and $\beta = (\beta_1, \cdots, \beta_n)$ are diffusion processes of the logarithms of aggregate consumption and stock price,

$$\mathbf{B}_c(T) = \{B_{c,1}(T), \cdots, B_{c,n}(T)\} \text{ and } \mathbf{B}(T) = \{B_1(T), \cdots, B_n(T)\}$$

are n–dimensional Brownian motions vectors, and $Y_c = \sum_{i=1}^{N(T)} Y_{c,i}$ and $Y = \sum_{i=1}^{N(T)} Y_i$ are logarithms of aggregate consumption jumps and stock price jumps respectively, where $N(T)$ is the Poisson process of both aggregate consumption and stock price with intensity λ. For every i with $1 \leq i \leq n$, the pairs $B_i(T)$ and $N(T)$, $B_{c,i}(T)$ and $N(T)$, Y_i and $N(T)$, and $Y_{c,i}$ and $N(T)$ are independent.

We assume that:

1. The covariance between aggregate consumption Brownian motion $B_{c,i}$ and aggregate consumption jump $Y_{c,j}$ is given by $cov(B_{c,i}(T), Y_{c,j}) = \gamma_c \rho_{cy_c} \sqrt{T}$ for each $i, j = 1, 2, \cdots, n$.

2. The covariance between aggregate consumption Brownian motion $B_{c,i}$ and stock price jump Y_j is given by $cov(B_{c,i}(T), Y_j) = \gamma \rho_{cy} \sqrt{T}$ for each $i, j = 1, 2, \cdots, n$.

3. The covariance between stock price Brownian motion B_i and aggregate consumption jump $Y_{c,j}$ is given by $cov(B_i(T), Y_{c,j}) = \gamma_c \rho_{sy_c} \sqrt{T}$ for each $i, j = 1, 2, \cdots, n$.

4. The covariance between stock price Brownian motion B_i and stock price jump Y_j is given by $cov(B_i(T), Y_j) = \gamma \rho_{sy} \sqrt{T}$ for each $i, j = 1, 2, \cdots, n$.

5. The covariance between aggregate consumption jump $Y_{c,i}$ and stock price jump Y_j is given by $cov(Y_{c,i}, Y_j) = v_{sc}$ for each $i, j = 1, 2, \cdots, n$.

Then, the mean and covariance matrix of the underlying random variables, $\mathbf{B}_c(T), \mathbf{B}(T), Y_{c,i}$ and Y_i to aggregate consumption and stock price are given by:[3]

$$\begin{bmatrix} \mathbf{B}_c(T) \\ \mathbf{B}(T) \\ Y_{c,i} \\ Y_i \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \alpha_c \\ \alpha \end{bmatrix}, \begin{bmatrix} \Sigma_c T & \Sigma_{cs} T & \gamma_c \rho_{cy_c} \sqrt{T}\mathbf{1} & \gamma \rho_{cy} \sqrt{T}\mathbf{1} \\ & \Sigma_s T & \gamma_c \rho_{sy_c} \sqrt{T}\mathbf{1} & \gamma \rho_{sy} \sqrt{T}\mathbf{1} \\ & & \gamma_c^2 & v_{sc} \\ & & & \gamma^2 \end{bmatrix} \right), \tag{7}$$

where $Y_{c,i} \sim N(\alpha_c, \gamma_c)$, $Y_i \sim N(\alpha, \gamma)$, $\Sigma_{cs}T$ is the covariance matrix between the n–dimensional Brownian motions $\mathbf{B}_c(T)$ and $\mathbf{B}(T)$, Σ_c and Σ_s are the covariance matrices of $\mathbf{B}_c(T)$ and $\mathbf{B}(T)$ respectively, $\mathbf{1}$ is the $1 \times n$ vector with every coordinate 1, and the covariance matrix is a symmetric matrix.

In the case of 1–dimensional Brownian motions, we have

$$a_c(T) = \ln(C_0) + \mu_c T - \frac{\sigma_c^2}{2} T, \qquad \beta_c = \sigma_c;$$

$$a(T) = \ln(S_0) + \mu T - \frac{\sigma^2}{2} T, \qquad \beta = \sigma.$$

Therefore the aggregate consumption and stock price in (5) and (6) are given by:

$$\begin{aligned} C_T &= exp\left(ln(C_0) + \mu_c T - \frac{\sigma_c^2}{2} T + \sigma_c B_c(T) + \sum_{i=1}^{N(T)} Y_{c,i} \right), & (8) \\ S_T &= exp\left(ln(S_0) + \mu T - \frac{\sigma^2}{2} T + \sigma B(T) + \sum_{i=1}^{N(T)} Y_i \right), & (9) \end{aligned}$$

where C_0 is the current level of aggregate consumption, μ_c is the instantaneous expected growth rate of consumption conditional on the fact that the Poisson event does not occur, σ_c is the volatility of consumption, $B_c(T) \sim N(0, T)$ is the 1-dimensional consumption Brownian motion, S_0 is the current stock price, μ is the instantaneous expected stock return conditional on the fact that the

[3]The technical appendix A shows that the Brownian motion and the IID jump sizes normally distributed are in general correlated, and that the covariance structure summarized in equation (7) is robust and reasonable.

Poisson event does not occur, σ is the stock volatility, $B(T) \sim N(0,T)$ is the stock price Brownian motion. Note that we have

$$\begin{aligned} cov(\sigma_c B_c(T), Y_{c,i}) &= \sigma_c cov(B_c(T), Y_{c,i}) \\ &= \sigma_c \gamma_c \rho_{cy_c} \sqrt{T} = \sigma_{cy_c} \sqrt{T}, \end{aligned} \tag{10}$$

for every $i = 1, 2, \cdots, n$. Similarly,

$$cov(\sigma_c B_c(T), Y_i) = \sigma_{cy} \sqrt{T}, \quad cov(\sigma B(T), Y_{c,i}) = \sigma_{sy_c} \sqrt{T}, \quad cov(\sigma B(T), Y_i) = \sigma_{sy} \sqrt{T}. \tag{11}$$

Then, the mean and covariance structure of the underlying uncertainty to aggregate consumption and stock price in the 1–dimensional Brownian motions case are given by:

$$\begin{bmatrix} \sigma_c B_c(T) \\ \sigma B(T) \\ Y_{c,i} \\ Y_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ \alpha_c \\ \alpha \end{bmatrix}, \begin{bmatrix} \sigma_c^2 T & \sigma_{cs} T & \sigma_{cy_c} \sqrt{T} & \sigma_{cy} \sqrt{T} \\ & \sigma^2 T & \sigma_{sy_c} \sqrt{T} & \sigma_{sy} \sqrt{T} \\ & & \gamma_c^2 & v_{cs} \\ & & & \gamma^2 \end{bmatrix} \right), \tag{12}$$

where the covariance matrix is reduced from (7) by the specifications of equations (10) and (11). The last sections will investigate important special cases of our general model by restricting in different ways the structure of covariances.

The novelty of these assumptions consists of the structure of correlations between jump sizes and Brownian motions, which brings a completely new dimension to option pricing under jump-diffusions. All previous research has assumed that jumps sizes are independent of the level of the Brownian motions. The next result yields the process followed by the pricing kernel obtained in equation (4).

Lemma 1. (The pricing kernel) *Assume that the representative agent has a power utility function of consumption given by equation (3) and that aggregate consumption is given by equation (5). Then the pricing kernel is given by:*

$$\psi(C) = \rho^T \exp(-b(a_c(T) - a_c(0)) - b\beta_c \cdot \mathbf{B}_c(T) - bY_c). \tag{13}$$

When $B_c(T)$ is 1–dimensional, we have

$$\psi(C) = exp\left(ln(\rho)T - b\mu_c T + b\frac{\sigma_c^2}{2}T - b\sigma_c B_c(T) + \sum_{i=1}^{N(T)}(-bY_{c,i})\right). \tag{14}$$

Proof: See Appendix B.

The general pricing kernel follows a jump–diffusion process with the same intensity and jumping times of aggregate consumption and stock price which were defined by equations (5) and (6) or equations (8) and (9). The role of the pricing kernel given in equations (13) or (14) is to discount the risky payoffs taking into account their risk and the time value of money. The use of a pricing kernel to discount assets avoids arbitrage opportunities to arise in the economy.

In Lemma 2, we obtain the interest rate of this economy by evaluating the expectation of the pricing kernel, and equating the result to the bond price. Since the consumption Brownian motion is correlated with consumption jump, we can not use the traditional method of proof given in jump–diffusion models which is based in the independence of $\mathbf{B}_c(T)$, N(T), and $Y_{c,i}$. Instead of breaking the expectation of the pricing kernel into the product of two expectations, we evaluate only one expectation and use the law of iterated expectations and the definition of the power series of the exponential function to obtain the following expression for the interest rate.

Lemma 2. (The interest rate) *Assume that the representative agent has a power utility function of consumption given by equation (3) and that aggregate consumption follows the jump–diffusion process given by equation (5) or (8). Let the current price of the riskless bond be $B_0 = e^{-rT}$, where r is the riskless interest rate. Then the equilibrium interest rate is given by:*

$$r = -\ln\rho + b\frac{a_c(T) - a_c(0)}{T} - \frac{b^2}{2}\beta_c\Sigma_c\beta_c^T - \lambda(e^{-b\alpha_c + \frac{1}{2}b^2\gamma_c^2 + b^2\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{cy_c}\sqrt{T}} - 1), \tag{15}$$

where $\langle\cdot,\cdot\rangle$ is the dot product between vectors in R^n and $\mathbf{1}$ in the dot product stands for the vector with every coordinate 1.[4] *Furthermore, if $B_c(T)$ is 1–dimensional,*

$$r = -ln(\rho) + b\mu_c - \frac{1}{2}b\sigma_c^2 - \frac{1}{2}b^2\sigma_c^2 - \lambda\left[e^{-b\alpha_c + b^2\frac{\gamma_c^2}{2} + b^2\sigma_{cy_c}\sqrt{T}} - 1\right]. \tag{16}$$

[4] Therefore $\langle\beta_c,\mathbf{1}\rangle = \beta_{c,1} + \beta_{c,2} + \cdots + \beta_{c,n}$.

Proof: See Appendix B.

As we can see the equilibrium interest rate given by equation (15) or (16) differs from previous equilibrium interest rate relations, obtained in jump–diffusion models with systematic jump risk, because now the interest rate also depends on the covariance of the pricing kernel Brownian motions with the pricing kernel jump size $b^2\langle\beta_c, \mathbf{1}\rangle\gamma_c\rho_{cy_c}\sqrt{T}$, or $b^2\sigma_{cy_c}\sqrt{T}$ for the 1–dimensional case. Equations (15) and (16) show that, in general, the term structure of interest rates is not flat in our economy. It is flat when the correlation of the diffusive pricing kernel and the jumps of the pricing kernel is zero or when μ_c or σ_c are deterministic functions of other parameters including T such that the effect of T on r cancels out.[5] We use the equilibrium interest rate given by equation (15) or (16) to eliminate some preference parameters from the pricing kernel given by equation (13) or (14). Hence, after using equation (15) into (13), we rewrite the pricing kernel as

$$\psi(C) = exp\left(-rT - b^2\frac{\beta_c\Sigma_c\beta_c^T}{2}T - \lambda T\left(e^{-b\alpha_c+b^2\frac{\gamma_c^2}{2}+b^2\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{cy_c}\sqrt{T}} - 1\right) - b\beta_c\cdot\mathbf{B}_c(T) + \sum_{i=1}^{N(T)}(-bY_{c,i})\right), \tag{17}$$

and the pricing kernel for the 1–dimensional Brownian motion case as

$$\psi(C) = exp\left(-rT - b^2\frac{\sigma_c^2}{2}T - \lambda T\left(e^{-b\alpha_c+b^2\frac{\gamma_c^2}{2}+b^2\sigma_{cy_c}\sqrt{T}} - 1\right) - b\sigma_c B_c(T) + \sum_{i=1}^{N(T)}(-bY_{c,i})\right). \tag{18}$$

The next result evaluates the stock price in equilibrium to obtain our general consumption capital asset pricing model (CCAPM) under jump–diffusion. This relation is obtained in closed–form as we can see in the Proposition.

Proposition 1. (The consumption capital asset pricing model under jump-diffusion) *Assume that the representative agent has a power utility function of consumption given by equation*

[5] We could write $r_T = -ln(\rho) + b\mu_c - \frac{1}{2}b\sigma_c^2 - \frac{1}{2}b^2\sigma_c^2 - \lambda\left[e^{-b\alpha_c+b^2\frac{\gamma_c^2}{2}+b^2\sigma_{cy_c}\sqrt{T}} - 1\right]$ instead of (16) to stress that the interest rate depends on T. When $T = 0$ then $r_0 = -ln(\rho) + b\mu_c - \frac{1}{2}b\sigma_c^2 - \frac{1}{2}b^2\sigma_c^2 - \lambda\left[e^{-b\alpha_c+b^2\frac{\gamma_c^2}{2}} - 1\right]$. Then $r_T = r_0 - \lambda e^{-b\alpha_c+b^2\frac{\gamma_c^2}{2}}\left[e^{b^2\sigma_{cy_c}\sqrt{T}} - 1\right]$. This expression is useful to explain that the interest rate does not explode when T increases. Indeed if $\sigma_{cy_c} > 0$ then there is a $T*$ where $r_{T*} = 0$ and if $\sigma_{cy_c} < 0$ then lim $r_T = r_0 + \lambda e^{-b\alpha_c+b^2\frac{\gamma_c^2}{2}}$ when $T \to \infty$. In the paper, we write r instead of r_T to simplify the notation.

(3), and that aggregate consumption and the stock price follow the jump-diffusion processes given by equations (5) and (6) or equations (8) and (9) with the covariance structure given by formula (7) or (12). Then the equilibrium expected stock return is given, in an implicit form, by the riskfree interest rate plus the stock risk premium:

$$r = \frac{a(T)-a(0)}{T} + \frac{1}{2}\beta\Sigma_s\beta^T - b\beta_c\Sigma_{cs}\beta^T \tag{19}$$

$$+\lambda e^{-b\alpha_c+\frac{1}{2}b^2\gamma_c^2+b^2\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{cyc}\sqrt{T}}\left(e^{\alpha+\frac{1}{2}\gamma^2+(-b\langle\beta_c,\mathbf{1}\rangle\gamma\rho_{cy}-b\langle\beta,\mathbf{1}\rangle\gamma_c\rho_{syc}+\langle\beta,\mathbf{1}\rangle\gamma\rho_{sy})\sqrt{T}-bv_{sc}}-1\right).$$

When both $B_c(T)$ and $B(T)$ are 1–dimensional, the formula is given by

$$r = \mu - b\sigma_{cs} + \lambda\left[e^{-b\alpha_c+\frac{b^2}{2}\gamma_c^2+b^2\sigma_{cyc}\sqrt{T}+\alpha+\frac{\gamma^2}{2}+\sigma_{sy}\sqrt{T}-b\sigma_{syc}\sqrt{T}-b\sigma_{cy}\sqrt{T}-bv_{sc}}-1\right] \tag{20}$$

$$-\lambda\left[e^{-b\alpha_c+\frac{b^2}{2}\gamma_c^2+b^2\sigma_{cyc}\sqrt{T}}-1\right].$$

Proof: See Appendix B.

Equations (19) and (20) tell us implicitly that the expected stock return is equal to the risk–free rate of return plus its risk premium. They can be seen as our general consumption capital asset pricing model (CCAPM). There are two main novelties in the equilibrium relations (19) or (20). First, the expected stock return takes into account the covariance between stock price jumps and the stock price Brownian motions. That is, the stock price level might affect the jumps of the stock price. As it can be seen, the expected stock return is composed of diffusive return $\frac{a(T)-a(0)}{T}+\frac{1}{2}\beta\Sigma_s\beta^T$ in (19) or μ in (20), jump return $\alpha+\frac{\gamma^2}{2}$, and the covariance between diffusive return and jump return $\langle\beta,\mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T}$ in (19) or $\sigma_{sy}\sqrt{T}$ in (20). Second, the equity risk premium takes into account all potential interactions between jumps and diffusions. The equity risk premium is composed of diffusive risk premium and jump risk premium, and those interactions lead to new ways of defining the diffusive equity risk premium and the jump equity risk premium. The diffusive risk premium depends both on the covariances between the stock price diffusions and the pricing kernel diffusions $b\beta\Sigma_{cs}\beta^T$ in (19) or $b\sigma_{cs}$ in (20) and on the covariances between the stock price diffusions and the jump of the pricing kernel $b\langle\beta, \mathbf{1}\rangle\gamma_c\rho_{sy_c}\sqrt{T}$ in (19) or $b\sigma_{sy_c}\sqrt{T}$ in (20). The equity jump risk premium depends both on the covariances between the jump of the stock price and the jump of the pricing kernel bv_{sc} and on the covariances between the jump of the stock price and the diffusions of the pricing kernel $b\langle\beta_c, \mathbf{1}\rangle\gamma\rho_{cy}\sqrt{T}$ in (19) or $b\sigma_{cy}\sqrt{T}$ in (20). The intensity of the Poisson process λ also affects the relation to take into account all the random jumps. The expected growth of the pricing kernel due to jumps $-b\alpha_c + \frac{b^2}{2}\gamma_c^2 + b^2\langle\beta_c, \mathbf{1}\rangle\gamma_c\rho_{cy_c}\sqrt{T}$ in (19) or $-b\alpha_c + \frac{b^2}{2}\gamma_c^2 + b^2\sigma_{cy_c}\sqrt{T}$ in (20) adds and subtracts in a nonlinear way to the equilibrium relationship, and therefore also affects the relation given by equation (19) or (20). It should also be noted that if the LHS of (19) or (20) does not explode, as we showed in footnote 4, then the RHS of (19) or (20) does not explode either.

The next result derives our general jump–diffusion option pricing model extending the technique introduced by Rubinstein (1976) of pricing by substitution in equilibrium. This is done in two steps. First, we derive the equilibrium price of the call that depends on several parameters including $\frac{a(T)-a(0)}{T} + \frac{1}{2}\beta\Sigma_s\beta^T$ or μ in the 1-dimensional case, and $b\beta\Sigma_{cs}\beta^T$ or $b\sigma_{cs}$ in the 1-dimensional case. Second, we use the equilibrium relation given by equation (19) or (20) to eliminate these two parameters from the option pricing formula. While this second step does not affect equilibrium option prices, it allows us to obtain interesting corollaries in the last sections.

Theorem 1. (The general jump–diffusion option pricing model) *Assume that the representative agent has a power utility function of consumption given by equation (3), and that aggregate consumption and the stock price follow the jump-diffusion processes given by equations (5) and (6) with covariance structure (7) or equations (8) and (9) with the covariance structure given by formula (12). Then the general jump–diffusion option pricing model is given by:*

$$P_c = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n e^{-\lambda' T}}{n!} \left(S_0 N(d_1(n)) - K e^{-r_n T} N(d_2(n)) \right), \tag{21}$$

where $d_1(n), d_2(n), r_n$ and $\lambda^{'}$ are given by the following:

$$\begin{aligned}
d_1(n) &= \frac{ln\left(\frac{S_0}{K}\right) + (r_n + \frac{\sigma_n^2}{2})T}{\sigma_n\sqrt{T}}, \qquad d_2(n) = d_1(n) - \sigma_n\sqrt{T}, \\
\sigma_n^2 T &= Max\left\{n\gamma^2 + 2n\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T} + \beta\Sigma_s\beta^T T, 0\right\}, \\
r_n T &= rT + \lambda T\left(\beta_2 - 1\right) - \lambda T\left(\beta_1 - 1\right) + nln\left(\frac{\beta_1}{\beta_2}\right), \\
\lambda^{'} &= \lambda\beta_1, \\
\beta_1 &= e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2 + b^2\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{cyc}\sqrt{T} + \alpha + \frac{\gamma^2}{2} + \langle\beta,\mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T} - b\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{syc}\sqrt{T} - b\langle\beta_c,\mathbf{1}\rangle\gamma\rho_{cy}\sqrt{T} - bv_{sc}}, \\
\beta_2 &= e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2 + b^2\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{cyc}\sqrt{T}}.
\end{aligned}$$

If both $B_c(T)$ and $B(T)$ are 1–dimensional, then the jump–diffusion option pricing model is given by:

$$P_c = \sum_{n=0}^{\infty}\frac{(\lambda^{'}T)^n e^{-\lambda^{'}T}}{n!}\left(S_0 N(d_1(n)) - Ke^{-r_n T}N(d_2(n))\right), \tag{22}$$

with the following data

$$\begin{aligned}
d_1(n) &= \frac{ln\left(\frac{S_0}{K}\right) + (r_n + \frac{\sigma_n^2}{2})T}{\sigma_n\sqrt{T}}, \qquad d_2(n) = d_1(n) - \sigma_n\sqrt{T}, \\
\lambda^{'} &= \lambda\beta_1, \\
\sigma_n^2 &= Max\left\{n\gamma^2/T + 2n\sigma_{sy}/\sqrt{T} + \sigma^2, 0\right\}, \\
r_n &= r + \lambda\left(\beta_2 - 1\right) - \lambda\left(\beta_1 - 1\right) + \frac{n}{T}ln\left(\frac{\beta_1}{\beta_2}\right), \\
\beta_1 &= e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2 + b^2\sigma_{cyc}\sqrt{T} + \alpha + \frac{\gamma^2}{2} + \sigma_{sy}\sqrt{T} - b\sigma_{syc}\sqrt{T} - b\sigma_{cy}\sqrt{T} - bv_{sc}}, \\
\beta_2 &= e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2 + b^2\sigma_{cyc}\sqrt{T}}.
\end{aligned}$$

Proof: See Appendix B.

Equations (21) and (22) are our general option pricing formulas. They are of the Merton's (1976) type. Like the Black-Scholes (1973) model, equation (22) depends on S_0, K, T, r, and σ. Like the jump-diffusion model of Merton (1976), the equation also depends on λ, α, and γ. Like the jump–diffusion models with systematic jump risk of Naik and Lee (1990), Ahn (1992), and

Amin and Ng (1993), the equation also depends on $b\alpha_c$, $b^2\gamma_c$, and bv_{cs}. Equation (22) has four new parameters that do not affect any other existing option pricing formula σ_{sy}, $b\sigma_{sy_c}$, $b\sigma_{cy}$, and $b^2\sigma_{cy_c}$. These parameters play potentially interesting roles as we will see in the next sections.

Equation (21) extends (22) to multivariate Brownian motions which are correlated among themselves and with aggregate consumption jump and stock price jump. In the Black-Scholes case, the extension from univariate to multivariate Brownian motions is a trivial extension, since the resulting option pricing equation is the original Black–Scholes (1973) model with the volatility given by

$$\sigma = \sqrt{\sum_{i=1}^{n} \sigma_i^2 + \sum_{i \neq j}^{n} \sigma_{ij}},$$

for $i, j = 1, 2, \cdots, n$. However, our multivariate formula (21) is not the trivial extension of (22). We can see this by considering the special case of equation (21) with $\Sigma_s = 0$. This is a new multivariate jump-diffusion option pricing model that has no univariate counterpart. In the univariate case , if $\Sigma_s = 0$ i.e. if $\sigma = 0$ then the price of the call in the Black-Scholes world is the stock price minus the present value of the strike price, where the discount rate is the riskless rate. In the Merton's (1976) model and in our univariate option pricing model given by equation (22) if $\sigma = 0$ then we have pure jump option pricing models as in Cox and Ross (1976). A multivariate extension of Merton's (1976) with $\Sigma_s = 0$ also leads to a pure jump option pricing model. In our option pricing model (21) when $\Sigma_s = 0$, we still have a multivariate jump-diffusion option pricing model since the diffusion process β affects the covariances between the jump and the Brownian motions. As we highlighted, there is no univariate jump-diffusion option pricing model analogous to this multivariate jump-diffusion option pricing model with $\Sigma_s = 0$.

Option pricing models (21) and (22) share an important property that is not present in any existing jump-diffusion option pricing model, and that results from the fact that the price level and price jumps are correlated in the model, i.e. ρ_{sy} might be different from zero. The option pricing models (21) and (22) are able to generate increasing, decreasing, and non-monotone term structures of implied volatilities of at-the-money options depending on the value taken by ρ_{sy}. This novel aspect of the model is important since, as noted by Das and Sundaram (1999) and reported by Bakshi, Cao, and Chen (1997), traditional jump-diffusion option pricing models are only able to generate increasing term structures of implied volatilities of at-the-money options, while in practice we also observe decreasing and non-monotone term structures. If ρ_{sy} is either zero (Merton's case) or positive, then $\sigma_n^2 T$ increases with the time to maturity T. This results in higher option prices and higher implied volatilities for at-the-money options when the time to expiration T increases. If ρ_{sy} is negative then, when the time to maturity T increases, $\sigma_n^2 T$ might either decrease or display a U shaped pattern across maturities, which leads to decreasing or non-monotone term structures of implied volatilities of at-the-money options. This is what we see in our empirical analysis that we report in the next section. A negative ρ_{sy} means that higher prices are associated with jumps down, and lower prices are associated with jumps up.

The role of a negative ρ_{sy} in generating $\sigma_n^2 T$ with decreasing or U shaped patterns across maturities can be seen in the following way. If the correlation between the level of the stock price and the price jump is negative ($\rho_{sy} < 0$) then there is a diffusion vector $\beta \neq \mathbf{0}$ with $\beta \Sigma_s \beta^T > 0$ positive definite and $n\langle \beta, \mathbf{1} \rangle^2 \rho_{sy}^2 - \beta \Sigma_s \beta^T \geq 0$, such that for those n there are special times

$$
\begin{aligned}
T(n,0) &= \left(\frac{-2n\langle \beta, \mathbf{1} \rangle \gamma \rho_{sy} - \sqrt{4n^2 \langle \beta, \mathbf{1} \rangle^2 \gamma^2 \rho_{sy}^2 - 4n\gamma^2 \beta \Sigma_s \beta^T}}{2\beta \Sigma_s \beta^T} \right)^2 \\
T(n,1) &= \left(\frac{-2n\langle \beta, \mathbf{1} \rangle \gamma \rho_{sy} + \sqrt{4n^2 \langle \beta, \mathbf{1} \rangle^2 \gamma^2 \rho_{sy}^2 - 4n\gamma^2 \beta \Sigma_s \beta^T}}{2\beta \Sigma_s \beta^T} \right)^2
\end{aligned}
\tag{23}
$$

which are the solutions of

$$\sigma_n^2 T = n\gamma^2 + 2n\langle \beta, \mathbf{1} \rangle \gamma \rho_{sy} \sqrt{T} + \beta \Sigma_s \beta^T T = 0.$$

Since $\sigma_n^2 T \geq 0$ and $\sigma_n^2 T = n\gamma^2 + 2n\langle \beta, \mathbf{1} \rangle \gamma \rho_{sy} \sqrt{T} + \beta \Sigma_s \beta^T T$ for all real σ_n and nonnegative $T \geq 0$, we have

$$\sigma_n^2 T = \max\{n\gamma^2 + 2n\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T} + \beta\Sigma_s\beta^T T, 0\}.$$

When $T(n,0) \leq T \leq T(n,1)$, we have $n\gamma^2 + 2n\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T} + \beta\Sigma_s\beta^T T \leq 0$, so $\sigma_n^2 T = 0$ on this interval of times, i.e. $\sigma_n = 0$ on this interval. Therefore we have

$$\sigma_n^2 = \begin{cases} n\gamma^2/T + 2n\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}/\sqrt{T} + \beta\Sigma_s\beta^T & \text{if } 0 < T \leq T(n,0) \\ 0 & \text{if } T(n,0) \leq T \leq T(n,1) \\ n\gamma^2/T + 2n\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}/\sqrt{T} + \beta\Sigma_s\beta^T & \text{if } T(n,1) \leq T. \end{cases} \tag{24}$$

Equation (24) tell us that, for a given n, if $\rho_{sy} < 0$ then σ_n first decreases with T until the special time $T(n,0)$, then σ_n becomes zero between the special time $T(n,0)$ and the special time $T(n,1)$, and then σ_n increases after the special time $T(n,1)$, where $T(n,0)$ and $T(n,1)$ are given by (23). As we see in (23), $T(n,0)$ and $T(n,1)$ depend on several parameters including ρ_{sy}.

If ρ_{sy} is negative and small enough, then σ_n has a U-shaped pattern across maturities but never reaches zero for all exchange traded options. This statement can be proved in the following way. From theorem 1, we have:

$$\sigma_n^2 = \frac{n}{T}\gamma^2 + 2\frac{n}{\sqrt{T}}\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy} + \beta\Sigma_s\beta^T. \tag{25}$$

In order to see the periods of time when σ_n decreases, it reaches a stationary point or date, and increases we obtain from (25):

$$\frac{d(\sigma_n^2)}{dT} = -\frac{n\gamma}{T^{3/2}}\left(\frac{\gamma}{\sqrt{T}} + \langle\beta, \mathbf{1}\rangle\rho_{sy}\right). \tag{26}$$

Note that $\frac{d(\sigma_n^2)}{dT} = 0$ when $\sqrt{T} = -\frac{\gamma}{\langle\beta,\mathbf{1}\rangle\rho_{sy}}$, and define $\sqrt{T^*} = -\frac{\gamma}{\langle\beta,\mathbf{1}\rangle\rho_{sy}}$ where T^* is the stationary date.[6] As we can see from (26) if $T < T^*$ then σ_n^2 is decreasing and if $T > T^*$ then σ_n^2 is

[6] If $\rho_{sy} \geq 0$ then $\frac{d(\sigma_n^2)}{dT} < 0$ always, meaning that neither is there a stationary date nor that $\sigma_n^2 T$ can display a U shaped pattern across maturities. This explains the numerical findings reported by Das and Sundaram (1999) for the term structure of implied volatilities of at-the-money options generated by the jump–diffusion model of Merton.

increasing. Furthermore, from the definition of stationary date T^*, we see that if ρ_{sy} is a negative number then $\sqrt{T^*}$ is positive and large enough. Evaluating σ_n^2 at T^* yields:

$$\sigma_n^{2*} = -\frac{n}{T^*}\gamma^2 + \beta\Sigma_s\beta^T, \tag{27}$$

which is positive since T^* is large enough.

4. Empirical results

In this section we discuss the sample data and the empirical tests of the theoretical model presented in section 3. Then we see how the estimated parameter values yield expected returns, risk premia, and volatility surfaces.

4.1 Methodology and data

We examine the time series of implied parameters produced by fitting the nested equilibrium option pricing models of (22) to a non-overlapping sample of S&P 500 index option quotes. We obtain estimates of the covariance of the diffusive pricing kernel with price jumps $b\sigma_{cy}$, the covariance of the jumps of the pricing kernel with the diffusive price $b\sigma_{sy_c}$, the covariance between diffusive price with price jumps σ_{sy}, the covariance of the jumps of the pricing kernel with price jumps bv_{sc}, the covariance of diffusive pricing kernel with jumps in the pricing kernel $b^2\sigma_{cy_c}$ as implied parameters of the general jump diffusion equilibrium option pricing model. We show that these five covariances are statistically different from zero, and therefore important for asset and option pricing.

We obtain implied parameters values by minimizing the option pricing errors between the quote mid-point and model values. Because the distribution of pricing errors is unknown the sampling distribution is unknown. Therefore we can not use standard parametric hypothesis tests that rely on known distributions to determine if these covariances are different from zero. Broadie, Chernov, and Johannes (2007) handle this issue using a nonparametric bootstrapping procedure. Our strategy for attacking this problem is different and based on sampling design. We construct a nonoverlapping sample of index option quotes with the same time to expiration in order to eliminate the problems that arise with serial and cross-sectional correlation in the option pricing errors. We use nonparametric tests to draw conclusions about the statistical significance of the implied parameters.

For a given observation date, the sample is constructed from quotes for S&P 500 index options that expire in the next calendar month. During the sample period, January 1996 through April 2006, the CBOE S&P 500 index options traded with a third Friday of the month last day of trading/expiration cycle. Cash settlement for these European options takes place Saturday morning following the third Friday of each month. The first sample observation consists of quotes from Friday, January 19th, 1996 for index options with expiration date Saturday, February 17th, 1996, the second observation of quotes from Friday February 16th , 1996 for index options expiring Saturday March 16th, 1996, and so on. All quotes were obtained from OptionMetrics. Index dividend yield and risk free rate for each observation date were also obtained from OptionMetrics. The risk free rate for a given observation date is interpolated from that dates LIBOR term structure and dividend yield is estimated from the put call parity relationship as described in the OptionMetrics Reference Manual. Screens were applied to eliminate options from the sample with price quotes less than \$3/8, quotes that violate lower option pricing boundaries, and options with no trading volume or no open interest.

The sample produced in this fashion contains 124 observation dates and quotes from 6,430 index options. The number of index option quotes on a given observation date ranges from 22 on July 20th, 2001 to 142 on March 17th 2006. Table I contains the average quote mid point and bid/ask spread for "moneyness" categories, where "moneyness" is defined as the ratio of index value S_0 divided by the strike price K. The sample contains more puts than calls with twice as many quotes for deep out-of-the-money puts, $S_0/K > 1.06$, than for at-the-money calls, $0.97 < S_0/K < 1.00$.

Serially independent implied parameter values are generated with a minimization routine that calibrates option pricing models to the bid/ask midpoints of call and put quotes on a given observation date. Implied parameter values are generated separately in this fashion for each observation date j. Subject to parameter restrictions imposed for a given option pricing model, implied parameter values minimize the sum of squared error objective function for the N_j option quotes of a given observation date:

$$fval_j = \sum_{i=1}^{N_j} \left(\hat{P}_i - 0.5 \left(bid_i + ask_i \right) \right)^2$$

where $\hat{P}_i$ is the model value for option i, which is given by the nested equilibrium option pricing models of (22).

4.2. Estimates

In the proof of Theorem 1 of this paper, we substitute the equilibrium equity price in the equilibrium price of the option to eliminate the diffusive expected return μ and the diffusive risk premium $b\sigma_{sc}$ from option prices. Since all assets (i.e. the bond, equity, and options) are in equilibrium there are no arbitrage opportunities in the system, and the call-put parity holds. Since we price by substitution, the implied parameter values estimated from market option prices are also the parameter values of the actual or objective distributions obtained from the process of aggregate consumption (8), the process of the stock price (9), and the mean and covariance structure of the underlying uncertainty to aggregate consumption and stock price (12). This contrasts with the option pricing approach of other authors (e.g. Broadie, Chernov and Johannes (2007), Eraker (2004), and Pan (2002)) who change measure to value options. As a result of their methodological approach, their risk-neutral parameters obtained from option market prices are a combination of actual parameters, and it becomes impossible to recover actual parameter values from option market prices for some parameters. Therefore, somewhat in an arbitrary way, they estimate the risk premium as the difference between the actual and the risk neutral parameter values. Since we are able to recover actual parameter values from option prices and we have the underlying CCAPM of the economy (20), we are able to plug into an approximation of the CCAPM (20) the implied parameter values from option market prices in order to estimate the risk premia.[7]

Equation (20) is exact and holds for any arbitrary period of time. However, it gives the expected equity return as an implicit nonlinear function of the riskless return and risk premia. In order to provide estimates of expected returns and risk premia we approximate equation (20) in the next proposition. Since these are approximations, it should be noted that the results of the next Proposition only hold for short holding periods of time.

Proposition 2. (Expected return and risk premia) *Assume that the CCAPM given by equation (20) holds. Then the following relationships hold up to the linear order:*

$$E(R_S(T)) = rT + EJRP(T) + EDRP(T) \tag{28}$$

where:

$$E(R_S(T)) = \mu T + \lambda T[e^{\alpha + 0.5\gamma^2 + \sigma_{sy}\sqrt{T}} - 1], \tag{29}$$

$$EJRP(T) = -\lambda T[e^{-bv_{sc} - b\sigma_{cy}\sqrt{T}} - 1], \tag{30}$$

$$EDRP(T) = b\sigma_{cs}T - \lambda T[e^{-b\sigma_{syc}\sqrt{T}} - 1], \tag{31}$$

and $E(R_S(T))$ is the expected equity return during the period T, $EJRP(T)$ is the equity jump risk premium during the period T, and $EDRP(T)$ is the equity diffusion risk premium during the period T.

[7] In the Black-Scholes (1973) model the difference between the actual expected diffusive return μ and the risk-neutral expected diffusive return r is the diffusive risk premium $b\sigma_{sc}$. Therefore the diffusive risk premium is exactly given by the difference between the actual and the risk-neutral parameter values. When systematic jump risk is introduced the relation between risk premia, actual, and risk-neutral parameters is no longer a simple linear equation as we demonstrate in equation (20).

Proof: See Appendix B.

The proposition shows that, in approximated terms, the expected stock return is equal to the risk free rate plus the jump risk premium plus the diffusive risk premium. It should be highlighted that even in these approximations the expected stock return, the jump risk premium, and the diffusive risk premium are nonlinear functions of time. Hence, in general, the annual expected stock return and the annual equity risk premia are not 12 times the monthly expected stock returns and the monthly equity risk premia.

Medians for implied parameter values and goodness of fit measure produced by calibrating the general jump-diffusion (GJD) option pricing model (22) are presented in Table II.[8] Statistical significance of implied parameter values is assessed with the large sample Wilcoxon signed rank test.[9] The z-statistic for this test, always negative in sign, is reported in parentheses below sample medians. If the z-statistic for this test is below its critical value then the null hypothesis that the median is zero is rejected. In order to verify if a parameter value is either positive or negative, we need to assess the sum of ranks for positive and negative sample values.

For the implied covariance between diffusive price level and price jumps, σ_{sy}, the null hypothesis of zero median is rejected in favor of the alternative hypothesis since the z-statistic is -8.797, and therefore below its critical value at the 1 percent significance level. For this implied covariance the sum of ranks for positive values is 347.50 and the sum of ranks for negative values is 7,402.50.[10] Since the sum of ranks for negative sample values is greater than the sum of ranks for positive sample values, we conclude that the median of the sampling distribution of σ_{sy} is negative. Therefore we conclude that the correlation between diffusive price level and price jumps is negative, which

[8]Constraints imposed on the optimization when calibrating the GJD model: $\lambda \geq 0$, $\sigma > 0$, $\gamma \geq 0$, $\gamma_c \geq 0$, $b > 0$, $|v_{cs}/(\gamma_c \gamma)| \leq 1$, and $|\rho_i| \leq 1$ where $i = sy, sy_c$.

[9]Implied parameter values for parameters constrained in the calibration process are necessarily different from zero. Test statistics for constrained parameter values are reported for comparison purposes only.

[10]The sums of ranks are not reported in the tables, but they are available upon request.

is also reported in table II. This is important for three reasons: first, we see that when market prices are high then jumps down are more likely and when market prices are low then jumps up are more likely; second, we demonstrate in (29) that this covariance is a significant component of the expected return of equity; third, we illustrate that nonmonotonic term structures of Black-Scholes implied volatilities result from a negative value of σ_{sy}. Figure 1 illustrates the time series of implied covariance, σ_{sy}. The preponderance of negative implied values reinforces the conclusion that this covariance is negative.

Equation (29) approximates the expected return on equity, $E(R_S(T))$. Using the values of table II and $T = 1$, we see that if we do not take into account the covariance between diffusive return and price jumps we erroneously overestimate the expected return of equity as $E(R_S(1)) = \mu - 0.00699$ when it is really $E(R_S(1)) = \mu - 0.02525$ since $\sigma_{sy} = -0.0103$. By other words, if we do not take into account the covariance between diffusive return and price jumps we overestimate the annual expected return of equity, $E(R_S(1))$ by approximately 1.83 percent.

The equation of the expected return on equity, $E(R_S(T))$, shows that this expected return on equity is a nonlinear function of time or holding period of the stock. Using the values of table II we estimate the monthly expected returns on equity as $E(R_S((t+1)/12)) - E(R_S(t/12))$, where $t = 0, 1, ..., 11$ and $E(R_S(0/12)) = 0$ by convention. Figure 2 reports the twelve monthly non-diffusive expected returns for the first year of investment.[11] As we can see from the figure, the monthly non-diffusive expected returns are not constant which contrasts with the traditional case of independent price jumps and diffusive returns $\rho_{sy} = 0$. Figure 2 shows that the weight of monthly non-diffusive expected returns on annual non-diffusive expected returns is smaller for the nearest months and larger for the furthest months.

Table III gives the Black-Scholes implied volatility surface for the GJD option pricing model

[11]The non-diffusive expected return is given by $\lambda T \left[e^{\alpha + 0.5\gamma^2 + \sigma_{sy}\sqrt{T}} - 1 \right]$.

(22) using the implied parameter values of table II, $S_0 = 100$, and $r = 2$ percent.[12] These values generate a nonmonotonic term structure of implied volatilities of at-the-money options consistent with the observed in other studies (e.g. Bakshi, Cao, and Chen (1997)). This effect is also illustrated in Figure 3, where Black-Scholes implied volatilities of at-the-money options resulting from the GJD option pricing model (22) are plotted against Black-Scholes implied volatilities of at-the-money options resulting from the Merton's (1976) jump-diffusion model. We plug into both option pricing models the implied parameter values of table II in order to find the Black-Scholes implied volatilities.

Table II presents median estimates of the five covariances implicit in the GJD option pricing model (22). These five covariances are all statistically different from zero at the 1 percent significance level. While the covariance between diffusive return and price jumps is negative as we reported, all the other four are positive. Covariances between aggregate consumption and equity prices are important determinants of the jump and diffusive risk premiums.

Equation (30) approximates the equity jump risk premium, $EJRP(T)$. Using the values of table II and $T = 1$, we see that the estimate of the annual EJRP(1) is 12.1 percent. If we do not take into account the jump risk premium that arises with the covariance between the diffusive pricing kernel and equity jumps, $b\sigma_{cy}$, then we would estimate the annual EJRP(1) as 6.8 percent, meaning that we would erroneously underestimate the annual EJRP(1) by approximately 5.3 percent. Pan (2002) estimates an annual mean price jump risk premium of 18.4 percent in the context of an option pricing model with price jumps and stochastic volatility. Eraker (2004) estimates an annual mean price jump risk premium of 6 percent using an option pricing model with stochastic volatility in price and jumps in both prices and volatility. Broadie, Chernov and Johannes (2007) in a similar model estimate an annual mean price jump risk premium between 2 and 4 percent. Santa-Clara

[12] We assume $S_0 = 100$, and $r = 2$ percent to maintain consistency of the analysis of this table with tables presented in the next sessions.

and Yan (2006) in a model with stochastic jump intensity estimate an annual mean price jump risk of 6.9 percent.

In the studies of Pan (2002), Eraker (2004), Broadie, Chernov and Johannes (2007), and Santa-Clara and Yan (2006) the annual equity jump risk premium is implicitly 12 times higher than the monthly equity jump risk premium since they do not take into account any correlation between Brownian motions and jumps. In their studies equity jump risk premia are implicitly a linear function of time. This contrasts with our model where the equity jump risk premium depends on the covariance of the diffusive pricing kernel with equity jumps $b\sigma_{cy}\sqrt{T}$, which makes the equity jump risk premium a nonlinear function of the holding period of the stock which coincides with the time to expiration of the option. Using the values of table II we estimate the monthly jump risk premia as $EJRP((t+1)/12) - EJRP(t/12)$, where $t = 0, 1, ..., 11$ and $EJRP(0/12) = 0$ by convention. Figure 4 reports the twelve monthly jump risk premia for the first year of investment. As we can see from the figure, the monthly jump risk premia are not constant which contrasts with the traditional case of independent price jumps and diffusive consumption $\rho_{cy} = 0$. The annual equity jump risk premium $EJRP(1)$ is 17.43 times the first monthly equity jump risk premium and 9.98 times the last monthly equity jump risk premium. From the figure we conclude that the weight of monthly jump risk premia on the annual jump risk premium is smaller for the nearest months and larger for the furthest months.

Equation (31) approximates the equity diffusion risk premium, $EDRP(T)$. Using the values of table II and $T = 1$, we see that when we take into account the covariance between diffusive return and jumps in the pricing kernel the annual equity diffusive risk premium $EDRP(1)$ increases by 4.9 percent. If $T = 1/12$ we see that when we take into account the covariance between diffusive return and jumps in the pricing kernel the first monthly equity diffusive risk premium $EDRP(1/12)$ increases by 0.12 percent. The estimates of the monthly diffusion risk premia are given by $EDRP((t+1)/12) - EDRP(t/12)$, where $t = 0, 1, ..., 11$ and $EDRP(0/12) = 0$ by convention. Our equity diffusive risk premium is also nonlinear on T. Figure 5 shows how monthly diffusive risk premia change due to the fact that ρ_{sy_c} is different from zero. The figure shows that the non-linear part of the monthly diffusive risk premiums increases during the year.[13]

We explain the slope of the implied volatility surface with respect to the option strike price as the result of the covariance between the diffusive pricing kernel and price jumps and the covariance between the diffusive equity and jumps in the pricing kernel. Our explanation of the sneer effect observed in market prices contrasts with other authors who tried to explain the sneer e.g. as a result of market imperfections such as transaction costs (Dennis (2001)), market illiquidity (Pena, Rubio and Serna (1999)), or as a result of stochastic volatility and jump-diffusions (Das and Sundaram (1999)).

Throughout our sample period, January 1996 through April 2006, the Black-Scholes implied volatilities exhibit a pronounced sneer. Figure 6 illustrates the Black-Scholes implied volatilities computed from option quote mid points for three pricing dates in our sample: January 19, 1996, January 19, 2001 and January 20, 2006. Many other studies document this pattern of implied volatilities in equity index option markets (e.g. Dumas, Fleming and Whaley (1998)).

Figures 7 and 8 present Black-Scholes implied volatilities of one-month options generated by the GJD option pricing model (22) evaluated for the implied parameter values of Table II, $S_0 = 100$, and $r = 2$ percent. Figure 7 shows that the shapes of the sneers observed in the marketplace can be reproduced when the covariance between diffusive return and jumps in consumption is $\sigma_{sy_c} = 0.0038$ obtained from Table II. Figure 7 also illustrates the impact of alternative values of the covariance between diffusive return and jumps in consumption on Black-Scholes implied volatility sneers. As we can see from figure 7, the ability to reproduce the sneers of figure 6 vanishes if we assume that the diffusive return and jumps in the pricing kernel are uncorrelated. Figure 8 shows that the shapes of the sneers observed in the marketplace can be reproduced when the covariance between

[13]The non-linear part of the equity diffusive risk premium is given by $-\lambda T[e^{-b\sigma_{sy_c}\sqrt{T}} - 1]$.

the diffusive consumption and equity jumps is $\sigma_{cy} = 0.0048$ obtained from Table II. Figure 8 also illustrates the impact of alternative values of the covariance between the diffusive consumption and equity jumps on Black-Scholes implied volatility sneers. As we can see from figure 8, the ability to reproduce the sneers of figure 6 vanishes if we assume that the diffusive pricing kernel and equity jumps are uncorrelated.

The median value of the jump intensity, λ, is consistent with less than two jumps per calendar year. The median values of λ, α, and γ indicate that an equity jump of -10 percent or less is expected to occur once every 3.03 years, and that an equity jump of -20 percent or less is expected to occur once every 24.76 years. Based on their estimates of average jump intensity and jump size, Santa-Clara and Yan (2006) report similar implications for market crashes, and state that "the stock market should experience market crashes with a magnitude of 9.8 percent once every 1.26 years". Although our estimate of expected time to market crash is more than the double the expected time to market crash estimated by Santa-Clara and Yan (2006), the implied parameter values of the GJD option pricing model (22) seem to embody market participant's expectations of ex-ante risk exposures encompassing risks in excess of those revealed by the historic record of index returns. Our results are consistent with the existence of the "peso problem" as examined for example by Brown, Goetzmann, and Ross (1995) and Veronesi (2004).

Bliss and Panigirtzoglou (2003) report a range for the estimated values of the power utility function's coefficient of proportional risk aversion for an investor with one-month horizon. The median value of the implied coefficient, b, reported in Table 2 is within the range of their estimated parameter values. Although larger than their estimated median, the degree of proportional risk aversion implied by fitting the GJD model to our sample is not by itself large enough to account for the equity risk premium puzzle in the sense of Mehra and Prescott (1985, 2003).

The average sum of squared pricing error, "fval" in Table II, from fitting the GJD model to the sample is consistent with an average absolute pricing error, \$0.117. For the sample of 6,430 options, 45.38% of GJD model values are within bid/ask spread.

5. A jumping stock in diffusion economies

This section specializes our general jump-diffusion option pricing model by restricting in two different directions the covariance structure of the underlying uncertainty to aggregate consumption and stock price given in formula (12). Throughout the section we assume that the representative agent has a power utility function of consumption, that aggregate consumption follows a geometric Brownian motion, and that the stock price follows a jump-diffusion process. The first subsection presents the pricing kernel that discounts all assets throughout the section. The second subsection gives new sufficient conditions for the Merton (1976) jump-diffusion model to hold in equilibrium. The third subsection obtains a new option pricing model assuming that price jumps are correlated with aggregate consumption.

5.1. The pricing kernel with diffusive consumption

This subsection presents the stochastic process followed by the pricing kernel, and the equilibrium interest rate of the economies studied in the next two subsections.

The section assumes that aggregate consumption and the stock price follow respectively a geometric Brownian motion and a jump-diffusion process given by the following two equations:

$$C_T = exp\left(ln(C_0) + \mu_c T - \frac{\sigma_c^2}{2}T + \sigma_c B_c(T)\right), \tag{32}$$

$$S_T = exp\left(ln(S_0) + \mu T - \frac{\sigma^2}{2}T + \sigma B(T) + \sum_{i=1}^{N(T)} Y_i\right). \tag{33}$$

Equation (32) means that aggregate consumption follows a geometric Brownian motion, and has a lognormal distribution at the end of each period. Equation (33) is identical to equation (6) of Merton (1976), and implies that the stock price follows a jump-diffusion process, where the jumps are IID lognormal variates.

The next result yields the process followed by the pricing kernel of the economies where we are going to derive option prices. The result is obtained as a special case of Lemma 1 when aggregate consumption does not jump.

Corollary 1. (The pricing kernel) *Assume that the representative agent has a power utility function of consumption given by equation (3), and that aggregate consumption is given by equation (32). Then the pricing kernel is given by:*

$$\psi(C) = exp\left(ln(\rho)T - b\mu_c T + b\frac{\sigma_c^2}{2}T - b\sigma_c B_c(T)\right). \tag{34}$$

It can easily be seen that the pricing kernel given by equation (34) has a standard lognormal distribution:

$$\psi(C) \sim \Lambda\left(ln(\rho)T - b\mu_c T + b\frac{\sigma_c^2}{2}T, b^2\sigma_c^2 T\right), \tag{35}$$

where $E(ln(\psi(C))) = ln(\rho)T - b\mu_c T + b\frac{\sigma_c^2}{2}T$ and $Var(ln(\psi(C))) = b^2\sigma_c T$. It is well known that the pricing kernel is lognormally distributed when aggregate consumption is lognormal and the investor has a power utility (See e.g. Cochrane (2003)). We use this pricing kernel to obtain two distinct option pricing models, one with idiosyncratic jump risk and the other with systematic jump risk. Naik and Lee (1990), Ahn (1992), and Amin and Ng (1993) assume that consumption follows a jump-diffusion process in economies where investors have log or power utility functions to study jump-diffusion option pricing models with systematic jump risk.

The equilibrium interest rate is obtained as a special case of Lemma 2 when aggregate consumption does not jump. The result is presented in the next corollary.

Corollary 2. (The interest rate) *Assume that the representative agent has a power utility function of consumption given by equation (3), and that aggregate consumption follows the geometric Brownian motion given by equation (32). Let the current price of the riskless bond be $B_0 = e^{-rT}$, where r is the riskless interest rate. Then the equilibrium interest rate is given by:*

$$r = -ln(\rho) + b\mu_c - \frac{1}{2}b\sigma_c^2 - \frac{1}{2}b^2\sigma_c^2. \tag{36}$$

5.2. The Merton model in a diffusion economy

This subsection obtains new sufficient conditions for the jump-diffusion option pricing model of Merton (1976) to hold in an equilibrium economy by assuming that the representative agent has a power utility function, aggregate consumption follows a geometric Brownian motion, and that the stock price follows a jump-diffusion process. In this subsection, we also assume that aggregate consumption and stock price are correlated with $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$. There is no other source of correlation in the economy. As highlighted by Merton (1976), his option pricing model assumes that jumps are idiosyncratic, i.e. nonsystematic. In our economy, this assumption means that the jumps of the stock are not correlated with the process followed by aggregate consumption.

First, we present the underlying equilibrium relation of the economy which is obtained as a special case of Proposition 1 when aggregate consumption does not jump, and the only source of covariance in the economy is the covariance between diffusive consumption and diffusive price σ_{cs}.

Corollary 3. (The expected stock return in equilibrium) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the geometric Brownian motion given by equation (32), and that the stock price follows the jump-diffusion process given by equation (33). Assume that aggregate consumption and stock price are correlated with $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$, and that there is no other source of correlation in the economy. Then the expected stock return in equilibrium is given by:*

$$\mu + \lambda \left[e^{\alpha + \frac{\gamma^2}{2}} - 1 \right] = r + b\sigma_{cs}. \tag{37}$$

Equation (37) has a simple interpretation. It tells us that the expected stock return in equilibrium is equal to the riskless return plus a risk premium. The expected stock return is given by the expected stock return due to the diffusion μ and the expected stock return due to the jumps $\lambda \left[e^{\alpha + \frac{\gamma^2}{2}} - 1 \right]$. The risk premium of the stock is given by the instantaneous covariance between the diffusions of the stock and the pricing kernel, $b\sigma_{cs}$. The risk premium of the stock is positive if the stock and consumption are positively correlated since the coefficient of proportional risk aversion, b, is positive. There is no risk premium associated with the jumps since the price jumps are not correlated with aggregate consumption. This equation (37) can be seen as a version of the CCAPM under the jump-diffusion process.

Corollary 4. (The jump–diffusion model of Merton (1976) as a RNVR) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the geometric Brownian motion given by equation (32), and that the stock price follows the jump–diffusion process given by equation (33). Assume that aggregate consumption and stock price are correlated with* $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$*, and that there is no other source of correlation in the economy. Then the jump–diffusion option pricing model of Merton (1976) holds.*

Corollary 4 is an important result, and it shows that the Merton (1976) model can be obtained in an economy where the pricing kernel is lognormal. This Corollary is obtained as a special case of Theorem 1 when aggregate consumption does not jump, and the only source of covariance in the economy is σ_{cs}. In their jump-diffusion model with systematic jump-risk, Naik and Lee (1990, p. 504) argue that their "formulas price the risk in option cash-flows ...by restricting investor's preferences. This is the reason why the risk aversion parameter enters the option pricing formulas". Corollary 4 provides a counterexample to this argument by showing that even with restrictions in preferences the preference-free model of Merton (1976) might hold.

The Black-Scholes (1973) formula was obtained as a risk-neutral valuation relationship (RNVR) by Rubinstein (1976), Brennan (1979), Stapleton and Subrahmanyam (1990), Camara (2003) and many others assuming that the representative agent has a power utility function, that aggregate consumption has a lognormal distribution, and that the stock price has a lognormal distribution. Camara (2003) also shows that the Black-Scholes (1973) model holds with HARA utility when aggregate consumption has a displaced lognormal distribution. We highlight that the Merton (1976) model also holds when we replace power utility with HARA utility and consumption lognormally distributed with consumption displaced lognormally distributed, which provides a range of new sufficient conditions for this model. All these equilibrium economies, where the Black-Scholes model was derived, had a lognormal pricing kernel. Therefore, corollary 4 shows that different option pricing formulas (i.e. the Black-Scholes (1973) model and the jump-diffusion option pricing model of Merton (1976)) can be obtained with the same pricing kernel (i.e. a lognormal pricing kernel).

5.3. Rewarded stock price jumps in a diffusion economy

This subsection extends the jump-diffusion option pricing model of Merton (1976) in an equilibrium economy by assuming that the representative agent has a power utility function, aggregate consumption follows a geometric Brownian motion, and that the stock price follows a jump-diffusion process. While in the previous subsection it was assumed that the only source of correlation between aggregate consumption and stock price was the correlation between the Brownian motions of aggregate consumption and stock price, in this section we also assume that price jump is correlated with aggregate consumption. In this way, jump risk becomes systematic risk since it is correlated with the pricing kernel. Jump risk is priced or rewarded in equilibrium, and this jump risk premium affects the price of the assets in equilibrium. This analysis is in sharp contrast with previous jump-diffusion option pricing models with systematic jump risk, where it is always assumed that systematic jump risk arises with a simultaneous jump in consumption and the stock price. In our economy, aggregate consumption does not jump. The literature on jump-diffusion option pricing models with systematic jump risk includes Naik and Lee (1990), Ahn (1992), and Amin and Ng (1993).

The subsection assumes that aggregate consumption and the stock price follow respectively a geometric Brownian motion and a jump–diffusion process given by equations (32) and (33). In this subsection, we also assume that aggregate consumption and stock price are correlated with

$Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$, and $Cov[\sigma_c B_c(T), Y_i] = \rho_{c,y} \gamma \sigma_c \sqrt{T} = \sigma_{cy} \sqrt{T}$ where $\sigma_{cy} \sqrt{T}$ denotes the covariance between the aggregate consumption diffusion and the stock price jump. There is no other source of correlation in the economy. Under these assumptions, the results of Corollaries 1 and 2 are still valid in this economy. However, the results of Corollaries 3 and 4 are no longer valid. First, we extend Corollary 3, which is obtained as a special case of Proposition 1.

Corollary 5. (The expected stock return in equilibrium) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the geometric Brownian motion given by equation (32), and that the stock price follows the jump-diffusion process given by equation (33). Assume that aggregate consumption and stock price are correlated with $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$, $Cov[\sigma_c B_c(T), Y_i] = \sigma_{cy} \sqrt{T}$, and that there is no other source of correlation in the economy. Then the equilibrium expected stock return is given, in an implicit form, by the riskfree interest rate plus the stock risk premium:*

$$\mu - b\sigma_{cs} + \lambda \left[e^{\alpha + \frac{\gamma^2}{2} - b\sigma_{cy}\sqrt{T}} - 1 \right] = r. \tag{38}$$

Corollary 5 offers a new way to look at the systematic jump risk of the stock in an economy where both types of equity risk are systematic, diffusion risk and jump risk. Intuitively, equation (38) tells us that the expected stock return over the risk premium of the stock is equal to the risk-free rate of return. The expected stock return over the risk premium of the stock has two parts, a part associated with the diffusion of the stock and a part associated with the jumps of the stock. The part associated with the diffusion of the stock is $\mu - b\sigma_{cs}$. It is the expected stock return due to the diffusion over the diffusion risk premium. The part associated with the jumps of the stock is $\lambda \left[e^{\alpha + \frac{\gamma^2}{2} - b\sigma_{cy}\sqrt{T}} - 1 \right]$. We interpret this term as the expected return of the jumps over the risk premium of the jumps since the expected return of a single jump is $ln(E[e^{Y_i}]) = \alpha + \frac{\gamma^2}{2}$ and the risk premium of a single jump is $b\sigma_{cy}\sqrt{T}$. Since, in this economy, we have a compound Poisson process where the jump is correlated with a Brownian motion, we can not present the equilibrium relation of the economy in the traditional way where the expected return of the stock is given by the riskless return plus the risk premium. However, this is also the meaning of equation (38) which can be seen as a version of the CCAPM under jump-diffusion when consumption diffusion is correlated with stock price jump.

Corollary 6. (Option prices with 'visible' systematic jump risk premium) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the geometric Brownian motion given by equation (32), and that the stock price follows the jump-diffusion process given by equation (33). Assume that aggregate consumption and stock price are correlated with* $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$*,* $Cov[\sigma_c B_c(T), Y_i] = \sigma_{cy}\sqrt{T}$*, and that there is no other source of correlation in the economy. Then the jump–diffusion option pricing model with 'visible' systematic jump risk premium is given by:*

$$P_c = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n e^{-\lambda' T}}{n!} \left(S_0 N(d_1(n)) - K e^{-r_n T} N(d_2(n)) \right), \tag{39}$$

where:

$$\begin{aligned}
d_1(n) &= \frac{ln\left(\frac{S_0}{K}\right) + (r_n + \frac{\sigma_n^2}{2})T}{\sigma_n \sqrt{T}}, \qquad d_2(n) = d_1(n) - \sigma_n\sqrt{T}, \\
\lambda' &= \lambda e^{\alpha + \frac{\gamma^2}{2} - b\sigma_{cy}\sqrt{T}}, \\
\sigma_n^2 &= n\gamma^2/T + \sigma^2, \\
r_n &= r - \lambda\left[e^{\alpha + \frac{\gamma^2}{2} - b\sigma_{cy}\sqrt{T}} - 1\right] + n\left(\alpha + \frac{\gamma^2}{2} - b\sigma_{cy}\sqrt{T}\right)/T.
\end{aligned}$$

In order to write equation (39), first one obtains the price of the call in equilibrium. The equilibrium option pricing formula depends on several parameters including the expected return on the stock due to the diffusion μ, the risk premium associated with the diffusion of the stock $b\sigma_{cs}$, the expected return of a single jump $\alpha + \frac{\gamma^2}{2}$, and the risk premium of a single jump $b\sigma_{cy}$. Second, one uses the equilibrium expression of the underlying asset (i.e. equation (38) to eliminate two parameters, μ and $b\sigma_{cs}$ from the option pricing formula. The result is equation (39) which extends the jump-diffusion option pricing model of Merton (1976) to an economy with systematic diffusion risk and systematic jump risk.

Previous work on the systematic jump risk has assumed that there is a systematic jump risk when there are simultaneous jumps in aggregate consumption and stock price, and the jump sizes in aggregate consumption and the stock price are correlated. See e.g. Naik and Lee (1990), Ahn (1992), and Amin and Ng (1993). These important models deeply contrast with this application of our general model since in this economy aggregate consumption does not even jump.

To understand the isolated effect of the covariance between the diffusive pricing kernel and stock price jumps $b\sigma_{cy}$ on option prices, we provide a sample of call option prices given by equation (39) as a percentage of Merton's (1976) jump-diffusion option prices for a representative set of parameter values in Table IV. Option prices given by (39) are generated by assuming that $S_0 = 100$, $r = 2$ percent, $\sigma = 25$ percent, $\lambda = 2$, $\gamma = 15$ percent, $\alpha = -\frac{\gamma^2}{2} = -0.01125$, $b = 3.72$, $\sigma_c = 15$ percent, and $\rho_{cy} = 0.75$ (or $\rho_{cy} = 0$ in Merton's case).

From Table IV we notice that the systematic jump risk arising from the covariance between the diffusive pricing kernel and price jumps is more important for short-term deep-out-of-the-money options. The short-term deep-out-of-the-money call options become less expensive than under Merton's model because when ρ_{cy} is positive the intensity λ' decreases. The fact that the "number of jumps" decreases in the model makes short-term deep-out-of-the-money calls less attractive and cheap. The long term options become more expensive relatively to the Merton's (1976) model since the systematic jump risk increases with the square root of time. This maturity effect makes long term options more expensive than under the Merton's (1976) model. Option prices given by (39), and implicit in Table IV, generate a smile effect for the Black-Scholes implied volatilities.

6. Stock price diffusions in jumping economies

This section specializes our general jump-diffusion option pricing model by restricting in two other directions the covariance structure of the underlying uncertainty to aggregate consumption and stock price given in formula (12). Throughout the section we assume that aggregate consumption follows a jump-diffusion process, and that the stock price follows a geometric Brownian motion. The first subsection presents the pricing kernel that is going to be used to discount assets throughout the section. The second subsection gives new sufficient conditions for the Black-Scholes (1973) model to hold in equilibrium. The third subsection gives a new option pricing model assuming that the stock price diffusion is correlated with aggregate consumption jumps.

6.1. The pricing kernel with jump–diffusion consumption

This subsection presents the stochastic process of the pricing kernel, and the equilibrium interest rate of the two economies studied in this section.

The section assumes that aggregate consumption and the stock price follow respectively a jump-diffusion process and a geometric Brownian motion given by the following two equations:

$$C_T = exp\left(ln(C_0) + \mu_c T - \frac{\sigma_c^2}{2}T + \sigma_c B_c(T) + \sum_{i=1}^{N(T)} Y_{c,i}\right), \tag{40}$$

$$S_T = exp\left(ln(S_0) + \mu T - \frac{\sigma^2}{2}T + \sigma B(T)\right). \tag{41}$$

Equation (40) means that aggregate consumption follows a jump-diffusion process, which contrasts with all previous representative agent economies where the Black-Scholes model has been derived.[14] Equation (41) is identical to the assumption made by Black-Scholes (1973), and implies that the stock price follows a geometric Brownian motion, and therefore is lognormal at the end of each period. In our economy, this particular stock does not jump. We make the following remark to have a self-contained section.

Remark 1. (The pricing kernel) *In this section, assume that the representative agent has a power utility function of consumption given by equation (3) and that aggregate consumption is given by equation (40). Then the pricing kernel is given by equation (14) of Lemma 1.*

As it can be seen, the pricing kernel given by equation (14) is not lognormal, but instead

[14] See Camara (2003) for a review.

follows a jump-diffusion process. All previous equilibrium literature that obtained the Black-Scholes model made joint assumptions on the utility function of the representative agent and on aggregate consumption that always implied a lognormally distributed pricing kernel. See e.g. Rubinstein (1976), Brennan (1979), Stapleton and Subrahmanyam (1990), and Camara (2003). It should be stressed that the no-arbitrage literature on the Black-Scholes model also assumes a lognormal pricing kernel. See e.g. Cox and Huang (1989) and Duffie (2001). The pricing kernel of this section is clearly not lognormal. This contrasts our work with previous literature on sufficient conditions for the Black-Scholes model to hold.

The interest rate in equilibrium is obtained as a special case of Lemma 2 when the aggregate consumption Brownian motion $B_c(T)$ and aggregate consumption jumps $Y_{c,i}$ are independent.

Corollary 8. (The interest rate) *Assume that the representative agent has a power utility function of consumption given by equation (3) and that aggregate consumption follows the jump-diffusion process given by equation (40) with $B_c(T)$ independent of $Y_{c,i}$. Let the current price of the riskless bond be $B_0 = e^{-rT}$, where r is the riskless interest rate. Then the equilibrium interest rate is given by:*

$$r = -ln(\rho) + b\mu_c - \frac{1}{2}b\sigma_c^2 - \frac{1}{2}b^2\sigma_c^2 - \lambda\left[e^{-b\alpha_c + b^2\frac{\gamma_c^2}{2}} - 1\right]. \tag{42}$$

Equation (42) yields the equilibrium interest rate of the economy which, as we see, depends on the jump parameters of consumption.

6.2. The Black-Scholes model in an economy with jumps

This subsection obtains new sufficient conditions for the Black-Scholes model (1973) to hold in an equilibrium economy by assuming that the representative agent has a power utility function, aggregate consumption follows a jump-diffusion process, and that the stock price follows a geometric Brownian motion. In this subsection, we also assume that aggregate consumption and stock price are correlated with $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs}T$. There is no other source of correlation in the economy. We start by presenting the underlying equilibrium relation between expected return, risk premium and interest rate.

Corollary 9. (The expected stock return in equilibrium) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the jump-diffusion process given by equation (40), and that the stock price follows the geometric Brownian motion given by equation (41). Assume that aggregate consumption and stock price are correlated with* $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$*, and that there is no other source of correlation in the economy. Then the expected stock return in equilibrium is given by:*

$$\mu = r + b\sigma_{cs}. \tag{43}$$

Equation (43) is identical to the CCAPM obtained by Rubinstein (1976) in an economy where both aggregate consumption and the stock price are lognormally distributed. This equation tells us that the expected stock return in equilibrium is equal to the riskless return plus a risk premium. The risk premium of the stock is given by the instantaneous covariance between the diffusions of the stock and the pricing kernel, $b\sigma_{cs}$. The risk premium of the stock is positive if the stock and consumption are positively correlated since the coefficient of proportional risk aversion, b, is positive.

We now state formally that the Black-Scholes model holds in this economy.

Corollary 10. (The Black-Scholes model) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the jump-diffusion process given by equation (40), and that the stock price follows the geometric Brownian motion given by equation (41). Assume that aggregate consumption and stock price are correlated with* $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$*, and that there is no other source of correlation in the economy. Then the Black-Scholes (1973) model holds.*

Corollary 10 allows us to conclude that a given option pricing formula (i.e. the Black-Scholes valuation equation) can be obtained with different pricing kernels (i.e. a lognormal pricing kernel and a jump-diffusion pricing kernel). Therefore, we withdraw the important conclusion that a lognormal pricing kernel is not a necessary condition for the Black-Scholes model to hold. We could extend this result for when aggregate consumption has multiple independent compound Poisson processes uncorrelated with the Brownian motion. The jump-diffusion model of Merton (1976) would also hold under such assumptions if the only source of correlation in the economy arises with the Brownian motions of aggregate consumption and stock price. In the next subsection, we study what happens to the Black-Scholes (1973) model if stock price diffusion is correlated with aggregate consumption jumps.

6.3. Rewarded stock price diffusions in an economy with jumps

This subsection extends the Black-Scholes option pricing model (1973) in an equilibrium economy (see Rubinstein (1976)) by assuming that the representative agent has a power utility function, aggregate consumption follows a jump-diffusion process, and that the stock price follows a geometric Brownian motion. While in the previous subsection it was assumed that the only source of correlation between aggregate consumption and stock price was the correlation between the Brownian motions of aggregate consumption and stock price, in this subsection we also assume that the stock price diffusion is correlated with the jumps of aggregate consumption. In this way, the systematic diffusion risk of the stock has two distinct sources, one that arises from the frictions between price diffusion and the diffusion of the pricing kernel and another that arises from the frictions between price diffusion and the jumps of the pricing kernel. The rewarded diffusion risk of the stock is a nonlinear two factor model, where the factors arise from the frictions of the diffusive price with the diffusion and the jumps of the pricing kernel. This analysis is in sharp contrast with previous jump-diffusion literature since diffusion risk premium arising from the covariance between stock price diffusion and jumps in the pricing kernel was not, to the best of our knowledge, previously investigated.

The section assumes that aggregate consumption and the stock price follow respectively a jump-diffusion process and a geometric Brownian motion given by equations (40) and (41). In this subsection, we also assume that aggregate consumption and stock price are correlated with $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$, and $Cov[\sigma B(T), Y_{c,i}] = \rho_{s,y_c}\gamma_c\sigma\sqrt{T} = \sigma_{sy_c}\sqrt{T}$ where $\sigma_{sy_c}\sqrt{T}$ denotes the covariance between the aggregate consumption jump and the stock price diffusion. There is no other source of correlation in the economy. Under these assumptions, the results presented in Remark 1 and Corollary 8 are still valid in this economy. However, the results of Corollaries 9 and 10 are no longer valid. First, we extend Corollary 9.

Corollary 11. (The expected stock return in equilibrium) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the jump-diffusion process given by equation (40), and that the stock price follows the geometric Brownian motion given by equation (41). Assume that aggregate consumption and stock price are correlated with $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$, $Cov[Y_{c,i}, \sigma B(T)] = \rho_{sy_c}\gamma_c\sigma\sqrt{T} = \sigma_{sy_c}\sqrt{T}$, and that there is no other source of correlation in the economy. Then the expected stock return in equilibrium is given by:*

$$\mu = r + b\sigma_{cs} + \lambda\left[e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2} - 1\right] - \lambda\left[e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2 - b\sigma_{sy_c}\sqrt{T}} - 1\right]. \tag{44}$$

Corollary 11 offers a new way to look at the systematic diffusive risk of the stock. Equation (44) shows that the expected rate of return on the stock is equal to the risk-free rate of return plus the risk premium of the stock. The risk premium of the stock has two factors. The first factor results from the covariance between the stock diffusion and the diffusion of the pricing kernel, and is given by $b\sigma_{cs}$. The second factor results from the covariance between the stock diffusion and the jumps of the pricing kernel, and is given by $\lambda\left[e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2} - 1\right] - \lambda\left[e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2 - b\sigma_{sy_c}\sqrt{T}} - 1\right]$.

Corollary 12. (The diffusion option pricing model) *Assume that the representative agent has a power utility function of consumption given by equation (3), that aggregate consumption follows the jump-diffusion process given by equation (40), and that the stock price follows the* *geometric Brownian motion given by equation (41). Assume that aggregate consumption and stock price are correlated with* $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$*,* $Cov[Y_{c,i}, \sigma B(T)] = \sigma_{sy_c}\sqrt{T}$*, and that there is no other source of correlation in the economy. Then the diffusion option pricing model is given by:*

$$P_c = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n e^{-\lambda' T}}{n!} \left(S_0 N(d_1(n)) - K e^{-r_n T} N(d_2(n)) \right), \tag{45}$$

where:

$$\begin{aligned}
d_1(n) &= \frac{ln\left(\frac{S_0}{K}\right) + (r_n + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad d_2(n) = d_1(n) - \sigma\sqrt{T}, \\
\lambda' &= \lambda e^{-b\alpha_c + b^2 \frac{\gamma_c^2}{2} - b\sigma_{sy_c}\sqrt{T}}, \\
r_n &= r - bn\sigma_{sy_c}/\sqrt{T} + \lambda\left[e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2} - 1\right] - \lambda\left[e^{-b\alpha_c + \frac{b^2}{2}\gamma_c^2 - b\sigma_{sy_c}\sqrt{T}} - 1\right].
\end{aligned}$$

The option pricing formula (45) is of the Merton's (1976) type, and depends on preference and consumption parameters. It depends on the part of the risk premium of the stock that arises due to the frictions of the price diffusion with the jumps of the pricing kernel. We have used the expected rate of return on the stock obtained in equilibrium to eliminate the part of the risk premium of the stock that arises due to the frictions of the price diffusion with the diffusion of the pricing kernel. Hence, the option pricing formula (45) shows clearly that prices depend on the risk premium that arises from jumps in the pricing kernel.

To understand the isolated effect of the covariance between the diffusive stock price and the jumps of the pricing kernel $b\sigma_{sy_c}$ on option prices, we provide a sample of call option prices given by equation (45) as a percentage of Black-Scholes (1973) option prices for a representative set of parameter values in Table V. Option prices given by (45) are obtained by assuming that $S_0 = 100$, $r = 2$ percent, $\sigma = 25$ percent, $\lambda = 2$, $\gamma_c = 12.5$ percent, $\alpha_c = -\frac{\gamma_c^2}{2} = -0.00781$, $b = 3.72$, and $\rho_{sy_c} = 0.75$ (or $\rho_{sy_c} = 0$ in the Black-Scholes case). Since there are jumps in the pricing kernel, and those jumps are correlated with the stock price, the distribution implicit in the option pricing formula (45) becomes more leptokurtic than the distribution implicit in the Black-Scholes model.

Hence, options become more expensive under the option pricing model (45) than under the Black-Scholes model as we can see in Table V. Option prices given by (45) when $b\sigma_{sy_c} > 0$ generate a sneer, with implied volatilities decreasing with the strike price, since the distribution implicit in equation (45) becomes negatively skewed. While this factor contributes for the sneer observed in equity option markets, there might be other factors that also affect the sneer observed in market data. See e.g. Rubinstein (1994).[15]

7. Correlated jump–diffusion option pricing model

This section provides another interesting application of our general model by specializing yet into another direction the covariance structure of the underlying uncertainty to aggregate consumption and stock price given in formula (12). Suppose that aggregate consumption follows the geometric Brownian motion given by equation (32), and that the stock price follows the jump-diffusion process given by equation (33). Aggregate consumption and the stock price are only correlated via the Brownian motions, where $Cov[\sigma_c B_c(T), \sigma B(T)] = \sigma_{cs} T$. The stock price Brownian motion and stock price jumps are correlated where $Cov[\sigma B(T), Y_i] = \sigma_{sy}\sqrt{T}$. There are no further correlated variates in this economy. This is an extension of the Merton (1976) model for when the size of the jumps depends on the stock price level.

The pricing kernel and the interest rate are given by equation (34) and equation (36), respectively. The expected return of the stock in equilibrium is given by the following equation:

$$\mu + \lambda \left(e^{\alpha + \frac{\gamma^2}{2} + \sigma_{cy}\sqrt{T}} - 1 \right) = r + b\sigma_{cs}. \tag{46}$$

This equation tells us that the expected rate of return of the stock in equilibrium is equal to the riskless return plus the risk premium of the stock. The expected return of the stock has

[15] Option prices given by (45) when $b\sigma_{sy_c} < 0$ yield a sneer with implied volatilities increasing with the strike price, since the distribution implicit in equation (45) becomes positively skewed.

three parts, one due to the diffusion, other due to the jumps, and the third due to common shocks. However we can not disentangle the expected return due to the Brownian motion from the expected return due to the jumps since the term σ_{sy} represents precisely the expected return due to the covariance between the diffusion of the stock and the jump of the stock.[16] The risk premium of the stock represents the reward of the systematic risk of the stock which is diffusive in this particular application.

The price of the call option in this application of the general model is given by the following valuation equation:

$$P_c = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n e^{-\lambda' T}}{n!} \left(S_0 N(d_1(n)) - K e^{-r_n T} N(d_2(n)) \right), \tag{47}$$

where:

$$\begin{aligned}
d_1(n) &= \frac{ln\left(\frac{S_0}{K}\right) + (r_n + \frac{\sigma_n^2}{2})T}{\sigma_n \sqrt{T}}, \qquad d_2(n) = d_1(n) - \sigma_n \sqrt{T}, \\
\lambda' &= \lambda e^{\alpha + \frac{\gamma^2}{2} + \sigma_{sy}\sqrt{T}}, \\
\sigma_n^2 &= n\gamma^2/T + 2n\sigma_{sy}/\sqrt{T} + \sigma^2, \\
r_n &= r - \lambda\left(e^{\alpha + \frac{\gamma^2}{2} + \sigma_{sy}\sqrt{T}} - 1\right) + \frac{n}{T}\left(\alpha + \frac{\gamma^2}{2} + \sigma_{sy}\sqrt{T}\right).
\end{aligned}$$

This option pricing model differs from Merton (1976) since the covariance between the level of the stock price and the jumps of the stock price σ_{sy} affects three parameters, the "intensity" λ', the "volatility" σ_n, and the "riskless return" r_n. When this σ_{sy} is zero then Merton (1976) option pricing formula obtains. It is interesting to observe that equation (47) does not depend on preference parameters.

To understand better the isolated effect of the covariance between the diffusive stock price and stock price jumps σ_{sy} on option prices, we provide a sample of call option prices given by equation (47) as a percentage of Merton's (1976) jump-diffusion option prices for a representative

[16] Ait-Sahalia (2004) shows that it is possible to perfectly disentangle Brownian noise from jumps in an economy where the Brownian motion and the jumps are independent.

set of parameter values in Table VI. Option prices given by (47) are generated by assuming that $S_0 = 100$, $r = 2$ percent, $\sigma = 25$ percent, $\lambda = 2$, $\gamma = 15$ percent, $\alpha = -\frac{\gamma^2}{2} = -0.01125$, and $\rho_{sy} = -0.25$ (or $\rho_{sy} = 0$ in Merton's case). In this application of our model, since $\rho_{sy} < 0$, an higher level of prices is associated with price jumps down and a lower level of prices is associated with price jumps up. In the Merton's model the level of prices is independent of the size of the jumps, a characteristic of the model that seems counter-intuitive as the recent crash of the Chinese stock market demonstrates.

From Table VI we notice that the effect arising from the covariance between the diffusive price and price jumps is more important for out-of-the-money options, and long-term options. The short-term out-of-the-money call options are less expensive than under Merton's model because when ρ_{sy} is negative the intensity $\lambda^{'}$ decreases. The fact that the number of jumps decreases in the model makes short-term out-of-the-money calls less attractive and cheap. When $\rho_{sy} < 0$, long term options are less expensive relatively to the Merton's model since $\sigma_n^2 T$ tends to be smaller for the model (47) than it is for the Merton's model when the time to maturity T increases. This effect was explained in some detail for the multivariate case in section 3. Option prices given by (47) with $\rho_{sy} = -0.25$ generate a nonmonotonic term structure of implied volatilities of at-the-money options.

8. Conclusion

In this paper, we extend the analysis of Rubinstein (1976) on the Black-Scholes (1973) model to the Merton (1976) jump-diffusion option pricing model in several directions. For example, we assume that the pricing kernel follows a jump-diffusion process and the stock price follows a geometric Brownian motion, and we ask the following question: does the Black-Scholes (1973) model hold under such conditions? Surprisingly, our answer is that it depends. The Black-Scholes model holds if the diffusion of the pricing kernel and the diffusion of the stock price are correlated, and the jumps of the pricing kernel are independent of the diffusive stock price. However, the Black-Scholes model does not hold anymore if the jumps of the pricing kernel are correlated with the diffusive stock price. In this case, option prices are given by a new option pricing formula of the Merton's (1976) type. This result is surprising because the literature has been unanimous in linking the Black-Scholes (1973) model with a lognormal pricing kernel.

We invert the above argument to ask if the Merton (1976) model holds when the pricing kernel is lognormal, and the stock price follows a jump-diffusion process. Once again we reach the conclusion that the model holds if the diffusive pricing kernel and stock price jumps are independent, but that it does not hold anymore if they are correlated. In this last case, we get a new option pricing model with systematic jump risk in an economy where the pricing kernel does not jump.

The above models are special cases of our general jump-diffusion option pricing model that fully takes into account the covariance structure of the underlying uncertainty to the pricing kernel and the stock price. Our general option pricing formula is affected, among other parameters, by the covariance between the diffusive stock price and the stock price jumps, the covariance between the diffusive stock price and the pricing kernel jumps, the covariance between the stock price jumps and the diffusive pricing kernel, and the covariance between the diffusive pricing kernel and the jumps of the pricing kernel. These four covariances do not affect any other existing option pricing formula, and they are all statistically different from zero. The covariance between the diffusive price and price jumps is negative in the sample, and it plays a novel role in jump-diffusion models. It generates a non-monotonic term structure of implied volatilities of at-the-money options similar to the observed in the marketplace. This reflects the fact that when market prices are high then it is more probable to observe jumps down. This negative correlation between diffusive return and price jumps has a negative effect of 1.83 percent on the annual expected equity return.

Our general jump diffusion option pricing model has implicit an annual equity jump risk premium of 12.1 percent, but our monthly equity jump risk premiums are not constant. The covariance between the stock price jumps and the diffusive pricing kernel is responsible for 45 percent of this annual equity jump risk premium. The fact that we take into account the covariance between the diffusive stock price and the jumps of the pricing kernel increases the annual diffusive equity risk premium by 4.9 percent. We see these two new risk premium factors explaining the sneers that arise in options market prices.

Appendix A

In general, any two stochastic processes are correlated. This statement follows from Feller (1971, p. 82).[17] The uncorrelated stochastic processes are very special processes. The Brownian motion and the compound Poisson processes are correlated in general if the size of the jump has a continuous distribution. We first show that the Brownian motion and the compound Poisson process with i.i.d jump sizes normally distributed are indeed correlated. Then, when we fix a given time T, the Brownian motion at this time is correlated with the size of jumps, for every jump. We also show that equation (7) on page 7 is mathematically valid.

First we give an explicit construction to show that the correlation does indeed exist between the Brownian motion and the compound Poisson process, where the i.i.d jump sizes are normally distributed.

Let $B(t)$ be a Brownian motion. Define the maximal process $M(t) = \sup_{s \leq t} B(s)$. Let $T_1 = \inf\{s : M(s) = B(s)+1\}$. Using induction to define $T_{i+1} = \inf\{s > T_n : \sup_{T_i \leq u \leq s} B(u) = B(s)+1\}$, we can define

$$Y(t) = \sup_{T_i \leq u \leq t} B(u); \qquad T_i \leq t < T_{i+1}.$$

Now the process $Y(t)$ is an increasing process with jumps and the jump sizes are i.i.d with normal distributions. Then there is a way to manage the time-change Y by using the fact that $T_1, T_2 - T_1, \cdots, T_{i+1} - T_i, \cdots$ are i.i.d exponential random variables. The idea of the construction of the compound Poisson process $Y(t)$ follows Revesz (1981).

The $Y(t)$ is an integrable compound Poisson process with i.i.d for the size of the jump, where $Y(t)$ is given by $\sum_{i=1}^{N(t)} Y_i$ with the counting process $N(t)$ defined by T_i's and Y_i by $B(u)$'s in the construction. There exists a time-change such that one has the representation $Y(t) = B(T(t))$. Now

[17] The independent random variables have the positive diagonal covariance matrix. The covariance matrix of any non-degenerate probability distribution is positive definite (see Feller 1971, p. 82, line 12). The positive definite matrices are much more general than those positive diagonal matrices.

we define a time-shift compound Poisson process $Y^{'}(T(t)) = Y(t)$ for each t. The new compound Poisson process $Y^{'}$ is correlated with $B(T(t))$, and $Cov(B(T(t)), Y^{'}(T(t))) = T(t)$. Until now, we have shown that the Brownian motion $B(t)$ and the compound Poisson process $Y(t)$ are correlated if the jump sizes are normally distributed. This proves that the Brownian motion $B(t)$ and the compound Poisson process $Y(t)$ are correlated at all the time. This construction also follows from Khoshnevisan (1993).

We now give a second proof of the correlation between the Brownian motion $B(T)$ and the compound Poisson process $Y(T)$ where the size of jumps are i.i.d normal random variables by using the imbedding method in stochastic theory. Let Z be a compensated compound Poisson process. Then we can define

$$Z_n(t) = Z(nt)/(\gamma\sqrt{n}),$$

where n is an integer number and γ is the standard deviation of the jump size. The well-known result in probability is that the compound Poisson process Z_n converges to the Brownian motion in the weak convergence with the equipped Skorohod topology (Corollary 3.7 of Khoshnevisan (1993)). The explicit construction of $Z(t)$ is also give in section 2 of Khoshnevisan (1993) and the size of jumps for the process Z satisfies the i.i.d in Lemma 2.1 of Khoshnevisan (1993).

Therefore we have a compound Poisson process $Y = Z_n$ for large enough n and

$$\begin{aligned} Cov(Y(T), B(T)) &= Cov(B(T) + Z_n(T) - B(T), B(T)) \\ &= Cov(B(T), B(T)) + Cov(Z_n(T) - B(T), B(T)) = T, \end{aligned}$$

and the proof is complete.

Now we show that when we fix the time T, the Brownian motion $B(T)$ at this time is correlated with the size of jumps Y_i, i.e., $Cov(B(T), Y_i) \neq 0$ for every i. Since the construction for each Y_i is identical with different time-change, we can show this by using the contradiction method. Suppose $Cov(B(T), Y_i) = 0$ for one of the size of jumps. Note that the Brownian motion $B(T)$ is uncorrelated with the counting process $N(t)$. From the i.i.d property, we have $Cov(B(T), Y_i) = 0$ for all i. Then using the law of iterated expectation for the compound Poisson process, we have

$$\begin{aligned} Cov(B(T), Y(T)) &= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} E\left(B(T) \cdot (\sum_{i=1}^{n} Y_i) | N(T) = n\right) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} E\left(B(T) \cdot (\sum_{i=1}^{n} Y_i)\right) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} \left(\sum_{i=1}^{n} Cov(B(T), Y_i)\right) \\ &= 0. \end{aligned}$$

which contradicts what we have proved, that is the correlation between the Brownian motion $B(t)$ and the compound Poisson process. Therefore we have shown that the size of jump Y_i and the Brownian motion $B(T)$ at time T are correlated.

Now we show that the covariance structure summarized in equation (7) is indeed correct. It is enough to show that $Cov(B(T), Y_i) = \gamma \rho_{sy} \sqrt{T}$ since the other covariance entries are obtained in the same way. This follows from the standard definition of covariance

$$Cov(B(T), Y_i) = \sigma_{B(T)} \sigma_{Y_i} \rho_{B(T)Y_i},$$

where $Cov(B(T), Y_i)$ is the covariance between the Brownian motion and the jump size, $\sigma_{B(T)}$ is the standard deviation of the Brownian motion $B(T)$, σ_{Y_i} is the standard deviation of the jump size Y_i, and $\rho_{B(T)Y_i}$ is the correlation between $B(T)$ and Y_i. Since $\sigma_{B(T)} = \sqrt{T}, \sigma_{Y_i} = \gamma, \rho_{B(T)Y_i} = \rho_{sy}$, our equation (7) on page 7 is mathematically valid.

Appendix B

Proof of Lemma 1: Note that $\psi(C) = \rho^T (\frac{C_T}{C_0})^{-b}$ for $\rho > 0$. The result follows from the definition of C_T and the pricing kernel. The 1–dimensional case follows from the identifications on $a_c(T) = \ln(C_0) + \mu_c T - \frac{\sigma_c^2}{2} T$ and $\beta_c = \sigma_c$. $\square$

Proof of Lemma 2: By equation (2) and $\phi(S_T) = 1$, $e^{-rT} = E(\psi(C)|\mathcal{F}_0) = E(\psi(C))$. By Lemma 1, we have

$$E(\psi(C)) = \rho^T e^{-b(a_c(T)-a_c(0))} \cdot E(e^{-b\beta_c \cdot \mathbf{B}_c(T) - bY_c}),$$

with $Y_c = \sum_{i=1}^{N(T)} Y_{c,i}$ and $Y_{c,i} \sim N(\alpha_c, \gamma_c)$ for $1 \le i \le n$.

$$\begin{aligned} E(e^{-b\beta_c \cdot \mathbf{B}_c(T) - bY_c}) &= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} E(e^{-b\beta_c \cdot \mathbf{B}_c(T) + \sum_{i=1}^{n}(-bY_{c,i})} | N(T) = n) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} e^{-b\alpha_c n + \frac{b^2}{2}\beta_c \Sigma_c \beta_c^T \cdot T + \frac{1}{2} b^2 \gamma_c^2 n + b^2 \langle \beta_c, \mathbf{1} \rangle \gamma_c \rho_{c y_c} \sqrt{T} n} \\ &= e^{-\lambda T + \frac{b^2}{2}\beta_c \Sigma_c \beta_c^T \cdot T} e^{\lambda T (e^{-b\alpha_c + \frac{1}{2} b^2 \gamma_c^2 + b^2 \langle \beta_c, \mathbf{1} \rangle \gamma_c \rho_{c y_c} \sqrt{T}} - 1)}, \end{aligned} \tag{48}$$

where the first equality follows from the law of iterated expectations for the Poisson process and the second from $E(e^{\xi \cdot X}) = e^{\xi \cdot E(X) + \frac{1}{2}\xi \Sigma_X \xi^T}$ for the normal distributions of X, and the last from the basic identity of the exponential series. Our equilibrium interest rate follows from the above identities. The 1–dimensional case follows from the identification $a_c(T) = \ln(C_0) + \mu_c T - \frac{\sigma_c^2}{2} T$ and $\beta_c = \sigma_c$. $\square$

Proof of Proposition 1: By using equations (2) and (4), we have $1 = E\left(\psi(C)\frac{S_T}{S_0}\right)$. By Lemma 1 and S_T,

$$\begin{aligned} \psi(C)\frac{S_T}{S_0} &= \rho^T \exp\left(-b(a_c(T) - a_c(0)) - b\beta_c \cdot \mathbf{B}_c(T) - bY_c\right) \cdot \exp(a(T) - a(0) + \beta \cdot \mathbf{B}(T) + Y) \\ &= \exp\left(\ln\rho \cdot T - b(a_c(T) - a_c(0)) + a(T) - a(0) - b\beta_c \cdot \mathbf{B}_c(T) + \beta \cdot \mathbf{B}(T) + (-b)Y_c + Y\right). \end{aligned}$$

Using the law of iterated expectations for the Poisson process, one gets

$$E\left(\psi(C)\frac{S_T}{S_0}\right) = \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} E\left(e^X | N(T) = n\right), \tag{49}$$

where $X = \ln\rho \cdot T - b(a_c(T) - a_c(0)) + a(T) - a(0) - b\beta_c \cdot \mathbf{B}_c(T) + \beta \cdot \mathbf{B}(T) + (-b)Y_c + Y_s$ from the normal distributions. Note that

$$E(X) = \ln\rho \cdot T - b(a_c(T) - a_c(0)) + a(T) - a(0) - b\alpha_c n + \alpha n,$$

$$Var(X) = b^2\beta_c\Sigma_c\beta_c^T T + \beta\Sigma_s\beta^T T + b^2\gamma_c^2 n + \gamma^2 n - 2b\beta_c\Sigma_{cs}\beta^T + 2b^2\langle\beta_c, \mathbf{1}\rangle\gamma_c\rho_{cy_c}\sqrt{T}n$$
$$-2b\langle\beta_c, \mathbf{1}\rangle\gamma\rho_{cy}\sqrt{T}n - 2b\langle\beta, \mathbf{1}\rangle\gamma_c\rho_{sy_c}\sqrt{T}n + 2\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T}n - 2bv_{sc}n.$$

By the same method of the proof of Lemma 2, we have, from (49),

$$\begin{aligned} E\left(\psi(C)\frac{S_T}{S_0}\right) &= \sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}e^{E(X)+\frac{1}{2}Var(X)} \\ &= \sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}e^{x+yn} \\ &= e^{x+\lambda T(e^y-1)}, \end{aligned}$$

where x and y are given by

$$x = \ln\rho \cdot T - b(a_c(T) - a_c(0)) + (a(T) - a(0)) + \frac{1}{2}b^2\beta_c\Sigma_c\beta_c^T T + \frac{1}{2}\beta\Sigma_s\beta^T T - b\beta_c\Sigma_{cs}\beta^T T; \quad (50)$$

$$y = -b\alpha_c + \alpha + \frac{1}{2}b^2\gamma_c^2 + \frac{1}{2}\gamma^2 + b^2\langle\beta_c, \mathbf{1}\rangle\gamma_c\rho_{cy_c}\sqrt{T} - b\langle\beta_c, \mathbf{1}\rangle\gamma\rho_{cy}\sqrt{T} - b\langle\beta, \mathbf{1}\rangle\gamma_c\rho_{sy_c}\sqrt{T} + \langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T} - bv_{sc}. \quad (51)$$

Hence $x + \lambda T(e^y - 1) = 0$ from $1 = e^{x+\lambda T(e^y-1)}$. By Lemma 2 and equations (50) and (51), the result (19) follows.

In the case of 1–dimensional $B_c(T)$ and $B(T)$, we have $a(T) = \mu T - \frac{\sigma^2}{2}T$, $\beta_c = \sigma_c, \beta = \sigma$ and

$$r = \mu - b\sigma_{cs} + \lambda e^{-b\alpha_c + \frac{1}{2}b^2\gamma_c^2 + b^2\sigma_{cy_c}\sqrt{T}}(e^{\alpha + \frac{\gamma^2}{2} - b\sigma_{cy}\sqrt{T} - b\sigma_{sy_c}\sqrt{T} + \sigma_{sy}\sqrt{T} - bv_{sc}} - 1).$$

The result (20) follows. □

Proof of Theorem 1: We write equation (2) for the call option, understanding that the expectation is taken conditional on the information available at the current time using the definition of the pricing kernel given by equation (4):

$$P_c = E\left[\rho^T\left(\frac{C_T}{C_0}\right)^{-b}(S_T - K)^+\right]. \quad (52)$$

By Lemma 1, $\psi(C) = \rho^T \exp(-b(\alpha_c(T) - \alpha_c(0)) - b\beta_c \cdot \mathbf{B}_c(T) - bY_c) = e^v$ for

$$\begin{aligned} v &= \ln \rho \cdot T - b(\alpha_c(T) - \alpha_c(0)) - b\beta_c \cdot \mathbf{B}_c(T) - bY_c \\ &= -rT - \frac{b^2}{2}\beta_c \Sigma_c \beta_c^T \cdot T - \lambda T \left(e^{-b\alpha_c + b^2 \frac{\gamma_c^2}{2} + b^2 \langle \beta_c, \mathbf{1} \rangle \gamma_c \rho_{cy_c} \sqrt{T}} - 1 \right) \\ &\quad -b\beta_c \cdot \mathbf{B}_c(T) + \sum_{i=1}^{n} (-bY_{c,i}), \end{aligned} \tag{53}$$

where the second identity follows from Lemma 2. By the definition of S_T, let $e^z = \frac{S_T}{S_0}$ for

$$z = a(T) - a(0) + \beta \cdot \mathbf{B}(T) + Y. \tag{54}$$

The first two moments of v and z are given by the followings:

$$\mu_v = -rT - \frac{b^2}{2}\beta_c \Sigma_c \beta_c^T T - \lambda T \left(e^{-b\alpha_c + b^2 \frac{\gamma_c^2}{2} + b^2 \langle \beta_c, \mathbf{1} \rangle \gamma_c \rho_{cy_c} \sqrt{T}} - 1 \right) - nb\alpha_c, \tag{55}$$

$$\sigma_v^2 = b^2 \beta_c \Sigma_c \beta_c^T T + b^2 n \gamma_c^2 + 2nb^2 \langle \beta_c, \mathbf{1} \rangle \gamma_c \rho_{cy_c} \sqrt{T}, \tag{56}$$

$$\mu_z = a(T) - a(0) + n\alpha, \tag{57}$$

$$\sigma_z^2 = \beta \Sigma_s \beta^T T + n\gamma^2 + 2n \langle \beta, \mathbf{1} \rangle \gamma \rho_{sy} \sqrt{T}, \tag{58}$$

$$\sigma_{vz} = -b\beta_c \Sigma_{cs} \beta^T T - nb \langle \beta_c, \mathbf{1} \rangle \gamma \rho_{cy} \sqrt{T} - nb \langle \beta, \mathbf{1} \rangle \gamma_c \rho_{sy_c} \sqrt{T} - nbv_{cs}. \tag{59}$$

Hence we have

$$\begin{aligned} P_c &= E\left(S_0 e^v \left(\frac{S_T}{S_0} - \frac{K}{S_0} \right)^+ \right) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} S_0 E\left[e^v \left(e^z - \frac{K}{S_0} \right)^+ \mid N(T) = n \right]. \end{aligned}$$

Note that

$$\begin{aligned} & E\left[e^v \left(e^z - \frac{K}{S_0} \right)^+ \mid N(T) = n \right] \\ = & \int_{-\infty}^{+\infty} \int_a^{+\infty} e^{v+z} f(v,z) dz dv - \frac{K}{S_0} \int_{-\infty}^{+\infty} \int_a^{+\infty} e^v f(v,z) dz dv \\ = & \, exp\left(\mu_v + \mu_z + \frac{1}{2}\sigma_v^2 + \frac{1}{2}\sigma_z^2 + \sigma_{vz} \right) N(d_1) - \frac{K}{S_0} exp\left(\mu_v + \frac{1}{2}\sigma_v^2 \right) N(d_2), \end{aligned} \tag{60}$$

where $d_1 = \frac{\mu_z+\sigma_{vz}+\frac{1}{2}\sigma_z^2-a}{\sigma_z}$, $d_2 = \frac{\mu_z+\sigma_{vz}-a}{\sigma_z}$, $a = ln(K/S_0)$, $f(v,z)$ is the p.d.f. of the normal variate, and $N(.)$ is the cumulative distribution function of a standard normal variate.

By Proposition 1, we get

$$(a(T)-a(0)) + \frac{1}{2}\beta\Sigma_s\beta^T T - b\beta_c\Sigma_{cs}\beta^T T - \lambda T(\beta_2 - 1) = rT - \lambda T(\beta_1 - 1), \tag{61}$$

where β_1 and β_2 are given by

$$\begin{aligned}
\beta_1 &= e^{-b\alpha_c+\frac{b^2}{2}\gamma_c^2+b^2\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{cyc}\sqrt{T}+\alpha+\frac{\gamma^2}{2}+\langle\beta,\mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T}-b\langle\beta,\mathbf{1}\rangle\gamma_c\rho_{syc}\sqrt{T}-b\langle\beta_c,\mathbf{1}\rangle\gamma\rho_{cy}\sqrt{T}-bv_{sc}} \\
\beta_2 &= e^{-b\alpha_c+\frac{b^2}{2}\gamma_c^2+b^2\langle\beta_c,\mathbf{1}\rangle\gamma_c\rho_{cyc}\sqrt{T}}.
\end{aligned}$$

Therefore, by a straightforward calculation from (55) to (60), we obtain

$$\begin{aligned}
\mu_v + \mu_z + \frac{1}{2}\sigma_v^2 + \frac{1}{2}\sigma_z^2 + \sigma_{vz} &= -\lambda T(\beta_1 - 1) + n\ln\beta_1, \\
\mu_v + \frac{1}{2}\sigma_v^2 &= -rT - \lambda T(\beta_2 - 1) + n\ln\beta_2,
\end{aligned}$$

$$\begin{aligned}
P_c &= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} \left\{ S_0 exp\left[-\lambda T(\beta_1 - 1) + nln\left(\beta_1\right)\right] N(d_1(n)) \right. \\
&- \left. Kexp\left[-rT - \lambda T(\beta_2 - 1) + nln(\beta_2)\right] N(d_2(n)) \right\}, \\
&= \sum_{n=0}^{\infty} \frac{(\lambda\beta_1 T)^n e^{-\lambda\beta_1 T}}{n!} \{S_0 N(d_1(n)) - Ke^{-r_n T} N(d_2(n))\} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda' T)^n e^{-\lambda' T}}{n!} \{S_0 N(d_1(n)) - Ke^{-r_n T} N(d_2(n))\},
\end{aligned}$$

where $d_1(n), d_2(n), r_n$ and λ' are given by

$$\begin{aligned}
d_1(n) &= \frac{ln\left(\frac{S_0}{K}\right) + (r_n + \frac{\sigma_n^2}{2})T}{\sigma_n\sqrt{T}}, \qquad d_2(n) = d_1(n) - \sigma_n\sqrt{T}, \\
\sigma_n^2 T &= \sigma_z^2 = n\gamma^2 + 2n\langle\beta,\mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T} + \beta\Sigma_s\beta^T T, \qquad (62) \\
r_n T &= rT + \lambda T\left(\beta_2 - 1\right) - \lambda T\left(\beta_1 - 1\right) + nln\left(\frac{\beta_1}{\beta_2}\right), \\
\lambda' &= \lambda\beta_1.
\end{aligned}$$

Notice that $\sigma_n^2 T$ is always nonnegative. The rhs of (62) could be negative. In that case $\sigma_n^2 T$ is defined to be the $Max\left(n\gamma^2 + 2n\langle\beta, \mathbf{1}\rangle\gamma\rho_{sy}\sqrt{T} + \beta\Sigma_s\beta^T T, 0\right)$. The desired result follows. If both $B_c(T)$ and $B(T)$ are 1-dimensional, then the formula (22) follows from the identification used before and $\sigma_c = \beta_c, \sigma = \beta$. $\square$

Proof of Proposition 2: First, consider the approximated expected return on equity:

$$E(R_S(T)) = \mu T + \lambda T\left[e^{\alpha+0.5\gamma^2+\sigma_{sy}\sqrt{T}} - 1\right].$$

By using the second order Taylor formula $e^x = 1 + x + x^2/2 + R_2(x)$, we have:

$$e^{\sigma_{sy}\sqrt{T}} = 1 + \sigma_{sy}\sqrt{T} + \frac{\sigma_{sy}^2 T}{2} + R_2(T^{3/2}).$$

$$\begin{aligned}
E(R_S(T)) &= \mu T + \lambda T\left[e^{\alpha+0.5\gamma^2}\cdot e^{\sigma_{sy}\sqrt{T}} - 1\right] \\
&= \mu T + \lambda T\left[e^{\alpha+0.5\gamma^2}\cdot(1 + \sigma_{sy}\sqrt{T} + \frac{\sigma_{sy}^2 T}{2} + R_2(T^{3/2})) - 1\right] \\
&= T\left[\mu + \lambda(e^{\alpha+0.5\gamma^2} - 1)\right] + \lambda e^{\alpha+0.5\gamma^2}\sigma_{sy}T^{3/2} + \frac{1}{2}\lambda e^{\alpha+0.5\gamma^2}\sigma_{sy}^2 T^2 + O(T^{5/2}),
\end{aligned}$$

where $O(T^{5/2})$ denotes for the order up to $T^{5/2}$.

Note that the first term $T\left[\mu + \lambda(e^{\alpha+0.5\gamma^2} - 1)\right]$ which is linear in T is the usual expect return on equity when the jump process is independent of the diffusive stock price.

Second, consider the approximated equity jump risk premium:

$$EJRP(T) = -\lambda T\left[e^{-bv_{sc}-b\sigma_{cy}\sqrt{T}} - 1\right].$$

By the same method of using the second order Taylor formula, we have:

$$e^{-b\sigma_{cy}\sqrt{T}} = 1 - b\sigma_{cy}\sqrt{T} + \frac{b^2\sigma_{cy}^2 T}{2} + R_2(T^{3/2}).$$

Thus the jump risk premium $EJRP(T)$ is given by:

$$EJRP(T) = -\lambda T(e^{-bv_{sc}} - 1) + \lambda e^{-bv_{sc}}b\sigma_{cy}T^{3/2} - \frac{\lambda b^2\sigma_{cy}^2 e^{-bv_{sc}}}{2}T^2 + O(T^{5/2}).$$

Note that the first term $-\lambda T(e^{-bv_{sc}} - 1)$, again linear in T, is the usual equity jump risk premium when $\sigma_{cy} = 0$.

Third, consider the approximated equity diffusive risk premium:

$$EDRP(T) = b\sigma_{cs}T - \lambda T\left[e^{-b\sigma_{syc}\sqrt{T}} - 1\right].$$

By the second order Taylor formula, we have:

$$EDRP(T) = b\sigma_{cs}T + \lambda b\sigma_{sy_c}T^{3/2} - \frac{\lambda b^2\sigma_{sy_c}^2}{2}T^2 + O(T^{5/2}).$$

Note that the linear term in T is the usual equity diffusive risk premium when there is no correlation between the diffusive equity and the jumps of the consumption process.

Fourth, let $X_1 = \alpha + 0.5\gamma^2 + \sigma_{sy}\sqrt{T}$, $X_2 = -b\alpha_c + 0.5b^2\gamma_c^2 + b^2\sigma_{cy_c}^2\sqrt{T}$, $X_3 = -bv_{sc} - b\sigma_{cy}\sqrt{T}$ and $X_4 = -b\sigma_{sy_c}\sqrt{T}$. With the notations defined, we have:

$$E(R_S(T)) = \mu T + \lambda T(e^{X_1} - 1), \tag{63}$$

$$EJRP(T) = -\lambda T(e^{X_3} - 1), \tag{64}$$

$$EDRP(T) = b\sigma_{cs}T - \lambda T(e^{X_4} - 1). \tag{65}$$

Then equation (20) can be rewritten as:

$$rT = \mu T - b\sigma_{cs}T + \lambda T(e^{X_1+X_2+X_3+X_4} - 1) - \lambda T(e^{X_2} - 1).$$

By the definition of r in (20) and (63), (64) and (65), we have:

$$E(R_S(T)) - rT - EJRP(T) - EDRP(T)$$

$$\begin{aligned}
&= \mu T + \lambda T(e^{X_1} - 1) - [\mu T - b\sigma_{cs}T + \lambda T(e^{X_1+X_2+X_3+X_4} - 1) \\
&\quad -\lambda T(e^{X_2} - 1)] + \lambda T(e^{X_3} - 1) - [b\sigma_{cs}T - \lambda T(e^{X_4} - 1)] \\
&= \lambda T[e^{X_1} + e^{X_2} + e^{X_3} + e^{X_4} - e^{X_1+X_2+X_3+X_4} - 3] \\
&= -\lambda T \cdot \{X_1X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4 + O(X^3)\},
\end{aligned}$$

where the last equality follows from the second order Taylor formula for the function $f(X_1, X_2, X_3, X_4) = e^{X_1} + e^{X_2} + e^{X_3} + e^{X_4} - e^{X_1+X_2+X_3+X_4} - 3$ around the point $(0,0,0,0)$[18], and $O(X^3)$ stands for any third order or above in variables X_1, X_2, X_3, X_4. Hence, up to the linear order, our result follows.

□

[18] Note that $f(0,0,0,0) = 0$, $Df(0,0,0,0) = (0,0,0,0)$ and $D^2f(0,0,0,0) = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$. Hence the second order Taylor formula for f at $(0,0,0,0)$ is

$$f(X_1, X_2, X_3, X_4) = -(X_1X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4) + O(X^3).$$

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Table I

Average quote midpoints and average bid ask spreads

The sample for this study is constructed from the market quotes of 6,430 one-month CBOE S&P 500 index options observed one-month prior to expiration. The sample period, January 1996 through April 2006, contains 124 dates on which the one-month option quotes are observed. The sample consists of both call and put option quotes and includes only quotes for options with positive trading volume, positive open interest and premiums greater than \$3/8. All quotes were obtained from OptionMetrics.

			Calls				
S_0/K	< 0.94	0.94-0.97	0.97-1.00	1.00-1.03	1.03-1.06	> 1.06	Total
midpoint	2.08	4.90	13.91	31.95	55.78	141.29	
bid/ask spread	0.47	0.60	1.07	1.64	1.81	1.89	
number of options	350	512	661	561	360	504	2948
			Puts				
S_0/K	< 0.94	0.94-0.97	0.97-1.00	1.00-1.03	1.03-1.06	> 1.06	Total
midpoint	128.11	54.04	28.89	14.40	7.50	2.78	
bid/ask spread	2.18	1.99	1.64	1.09	0.77	0.47	
number of options	134	255	603	608	504	1378	3482

Table II
Median implied parameter values
general jump-diffusion (GJD) option pricing model (22)

For each of the sample's 124 observations the structural parameters of the generalized jump diffusion model (22) are estimated by minimizing the sum of squared pricing errors between the mid point of the market quotes and model determined value for each option. Median values of the model's parameters are reported in this table. Statistical significance of the reported medians is tested with the Large sample Wilcoxon signed rank test. The z-statistic for this test is reported in parentheses below sample medians. Medians marked with ** are significantly different from zero at the one percent significance level.

Parameter	Median	Correlation	Median
λ	1.7885**		
	(-9.624)		
σ	0.1238**		
	(-9.663)		
γ	0.1066**		
	(-9.663)		
γ_c	0.0682**		
	(-9.663)		
α	-0.0096**		
	(-6.516)		
α_c	-0.0058**		
	(-5.366)		
σ_{sy}	-0.0103**	ρ_{sy}	-0.891**
	(-8.797)		(-9.334)
σ_{sy_c}	0.0038**	ρ_{sy_c}	0.507**
	(-6.389)		(-6.533)
σ_{cy}	0.0048**		
	(-6.021)		
σ_{cy_c}	0.0042**		
	(-5.645)		
v_{sc}	0.0059**	$v_{sc}/\gamma\gamma_c$	0.9623**
	(-9.552)		(-9.562)
b	6.5585**		
	(-9.624)		
fval	50.204		

Table III

Black-Scholes implied volatility surface generated by the general jump-diffusion (GJD) option pricing model (22)

This table contains the Black-Scholes implied volatility surface produced from the GJD option pricing model (22) evaluated for the implied parameter values of table II, $S_0 = 100$, and $r = 2$ percent. Option values from the general jump diffusion option pricing model generate a nonmonotonic term structure of implied volatilities of at-the-money options.

T	15 days	1 month	3 months	6 months	9 months	1 year
K						
109	19.23	15.38	15.66	16.43	16.63	16.53
106	16.19	15.10	16.08	16.46	16.53	15.92
103	14.99	15.50	16.58	16.48	16.27	16.72
100	15.88	16.68	17.14	16.43	15.67	17.23
97	19.33	18.87	17.75	16.29	15.80	17.26
94	24.97	21.88	18.38	16.14	16.54	17.21
91	30.14	24.85	19.02	16.20	16.65	17.98

General jump diffusion option values given by (22) are obtained by assuming that $S_0 = 100$, $r = 2$ percent, $\lambda = 1.7885$, $\sigma = 0.1238$, $\gamma = 0.1066$, $\gamma_c = 0.0682$, $\alpha = -0.0096$, $\alpha_c = -0.0058$, $\sigma_{sy} = -0.0103$, $\sigma_{sy_c} = 0.0038$, $\sigma_{sy_c} = 0.0038$, $\sigma_{cy} = 0.0048$, $\sigma_{cy_c} = 0.0042$, $v_{sc} = 0.0059$ and b = 6.5585.

Table IV

Option prices with 'visible' systematic jump risk premium (39)

as a percentage of Merton's jump-diffusion option prices

T	15 days	1 month	3 months	6 months	9 months	1 year
K						
109	92.58	94.62	97.88	100.04	101.48	102.64
106	96.32	97.32	99.10	100.63	101.77	102.76
103	98.86	99.11	99.99	101.06	101.98	102.82
100	100.01	100.09	100.58	101.36	102.10	102.83
97	100.34	100.50	100.92	101.53	102.15	102.79
94	100.34	100.58	101.06	101.59	102.14	102.70
91	100.24	100.50	101.06	101.57	102.06	102.57

Option prices given by (39) are generated by assuming that $S_0 = 100$, $r = 2$ percent, $\sigma = 25$ percent, $\lambda = 2$, $\gamma = 15$ percent, $\alpha = -\frac{\gamma^2}{2} = -0.01125$, $b = 3.72$, $\sigma_c = 15$ percent, and $\rho_{cy} = 0.75$ (or $\rho_{cy} = 0$ in Merton's case).

Table V
Diffusion option prices (45)
as a percentage of Black-Scholes option prices

T	15 days	1 month	3 months	6 months	9 months	1 year
K						
109	102.41	103.38	106.49	110.33	113.69	116.74
106	101.64	102.47	105.24	108.77	111.92	114.80
103	101.02	101.71	104.13	107.35	110.27	112.98
100	100.56	101.10	103.17	106.07	108.76	111.29
97	100.26	100.65	102.36	104.93	107.39	109.74
94	100.10	100.35	101.69	103.92	106.15	108.31
91	100.03	100.16	101.16	103.06	105.04	107.01

Option prices given by (45) are obtained by assuming that $S_0 = 100$, $r = 2$ percent, $\sigma = 25$ percent, $\lambda = 2$, $\gamma_c = 12.5$ percent, $\alpha_c = -\frac{\gamma_c^2}{2} = -0.00781$, $b = 3.72$, and $\rho_{sy_c} = 0.75$ (or $\rho_{sy_c} = 0$ in the Black-Scholes case).

Table VI
Correlated jump-diffusion option prices (47)
as a percentage of Merton's jump-diffusion option prices

T	15 days	1 month	3 months	6 months	9 months	1 year
K						
109	88.46	88.51	85.85	82.12	79.14	76.60
106	93.34	92.40	88.59	84.54	81.45	78.86
103	96.61	95.15	90.93	86.76	83.64	81.03
100	98.31	96.94	92.89	88.79	85.70	83.12
97	99.12	98.06	94.49	90.62	87.63	85.10
94	99.50	98.75	95.79	92.25	89.41	86.98
91	99.69	99.18	96.83	93.68	91.04	88.73

Option prices given by (47) are generated by assuming that $S_0 = 100$, $r = 2$ percent, $\sigma = 25$ percent, $\lambda = 2$, $\gamma = 15$ percent, $\alpha = -\frac{\gamma^2}{2} = -0.01125$, and $\rho_{sy} = -0.25$ (or $\rho_{sy} = 0$ in Merton's case).

Figure 1

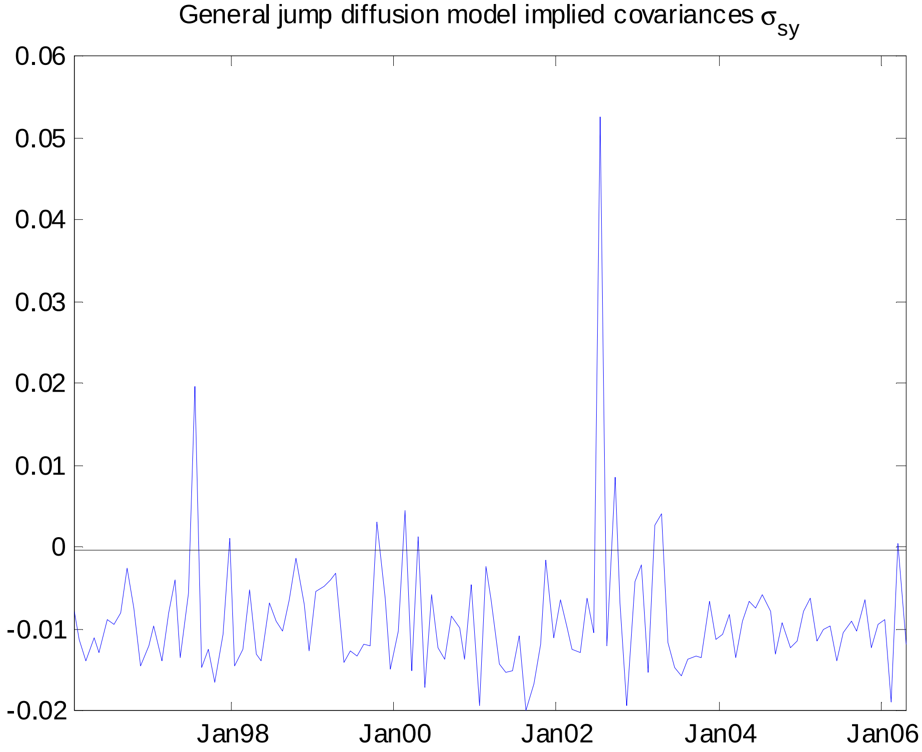
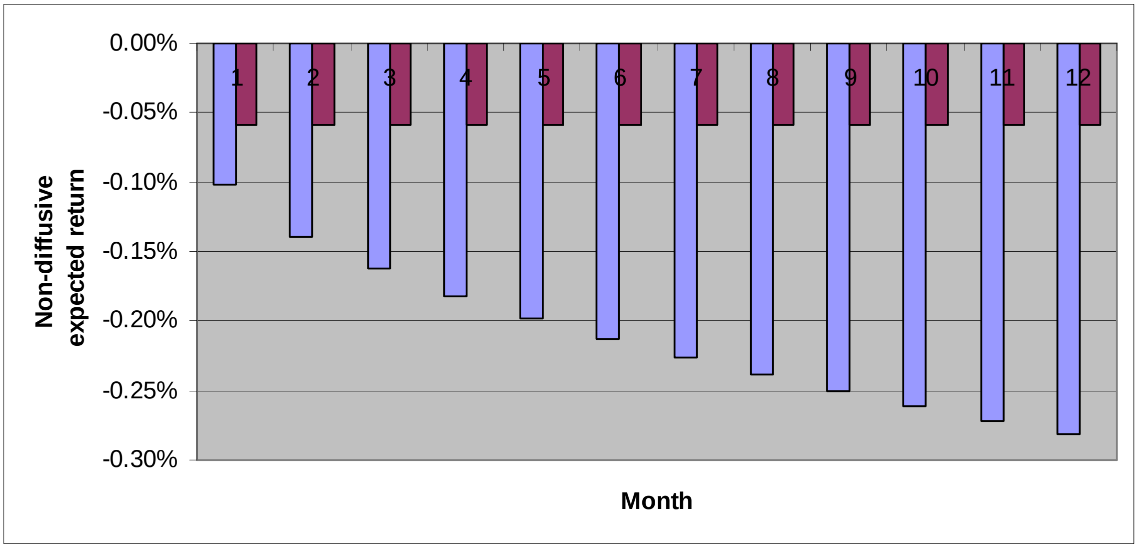


Figure 1 presents the time series of implied covariances, σ_{sy}, estimated by minimizing the sum of squared pricing errors resulting from fitting the general jump diffusion option pricing model, equation (22) to the CBOE S&P 500 index option quotes of the sample.

Figure 2

Non- Diffusive Expected Return



Monthly non-diffusive expected returns when σ_{sy} = -1.03 percent.

Monthly non-diffusive expected returns when diffusive returns and price jumps are independent.

Figure 2 shows monthly non-diffusive expected returns using the values of Table II. The monthly non-diffusive expected returns decrease during the year since σ_{sy} = -1.03 percent. The figure also shows the traditional case, where price jumps and diffusive price are independent, that leads to constant monthly non-diffusive expected returns.

Figure 3

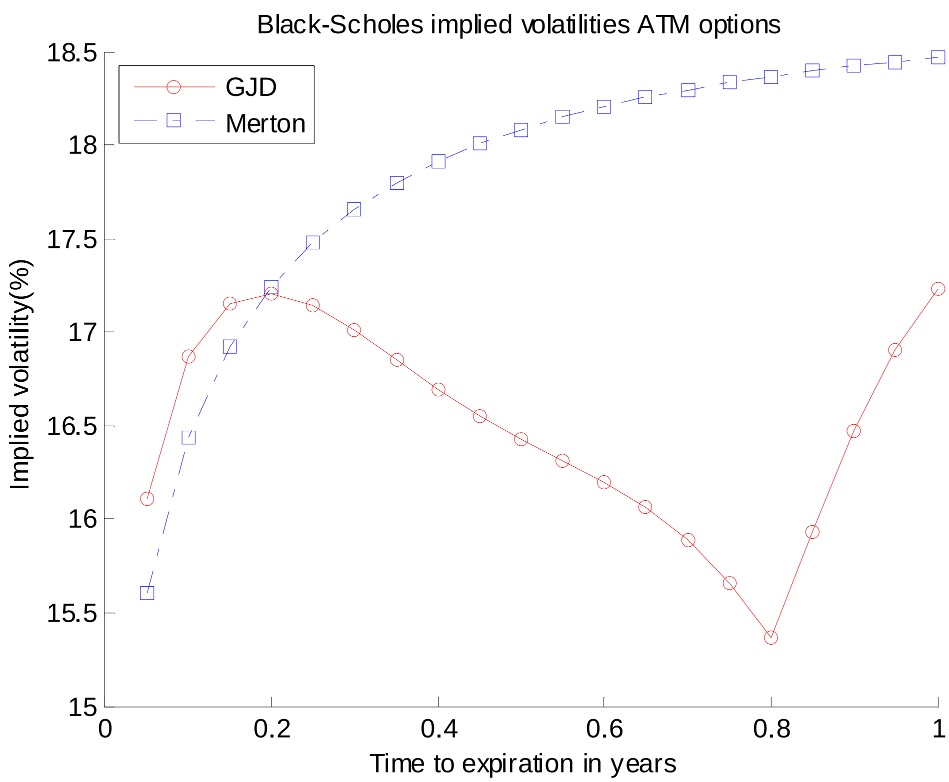
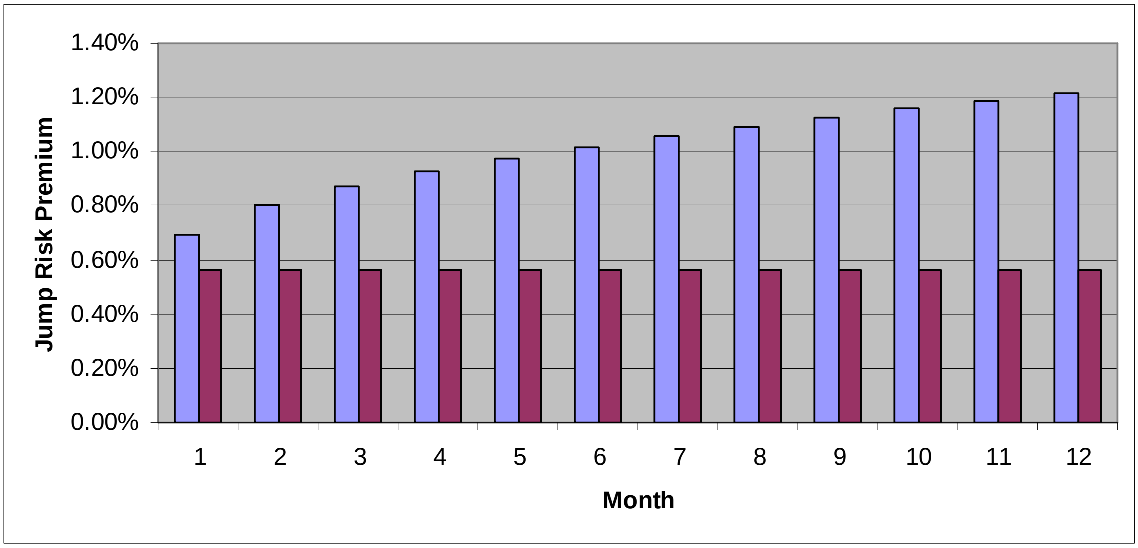


Figure 3 presents Black-Scholes implied volatilities for at-the-money options produced from the GJD option pricing model (22) and Merton's option pricing model evaluated for the implied parameter values of table II, $S_0 = 100$, and r = 2 percent. Since the covariance between diffusive price level and price jumps, σ_{sy}, is negative, option values from the general jump diffusion option pricing model generate a nonmonotonic term structure of implied volatilities of at-the-money options.

Figure 4

Equity Jump Risk Premium



Monthly equity jump risk premiums using values of Table II.

Monthly equity jump risk premiums when price jumps and the diffusive pricing kernel are independent

Figure 4 shows monthly equity jump risk premiums using the values of Table II. The monthly equity jump risk premiums increase during the year since the covariance between the diffusive pricing kernel and price jumps is positive. The figure also shows the traditional case, where price jumps and the diffusive pricing kernel are independent, which leads to constant monthly equity jump risk premiums.

Figure 5

The Non-linear part of the Diffusive Risk Premium

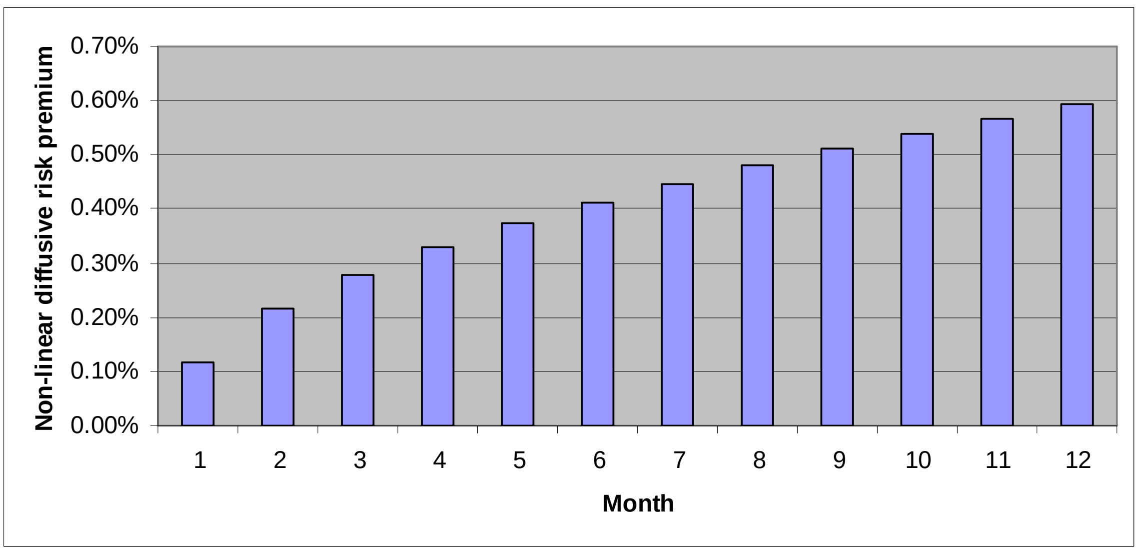


Figure 5 shows the monthly non-linear equity diffusive risk premiums using the values of Table II. The monthly non-linear equity diffusive risk premiums increase during the year since the covariance between the diffusive price and jumps in the pricing kernel is positive. The traditional case assumes that the covariance between the diffusive price and jumps in the pricing kernel is zero, and therefore adds nothing to the equity diffusive risk premium.

Figure 6

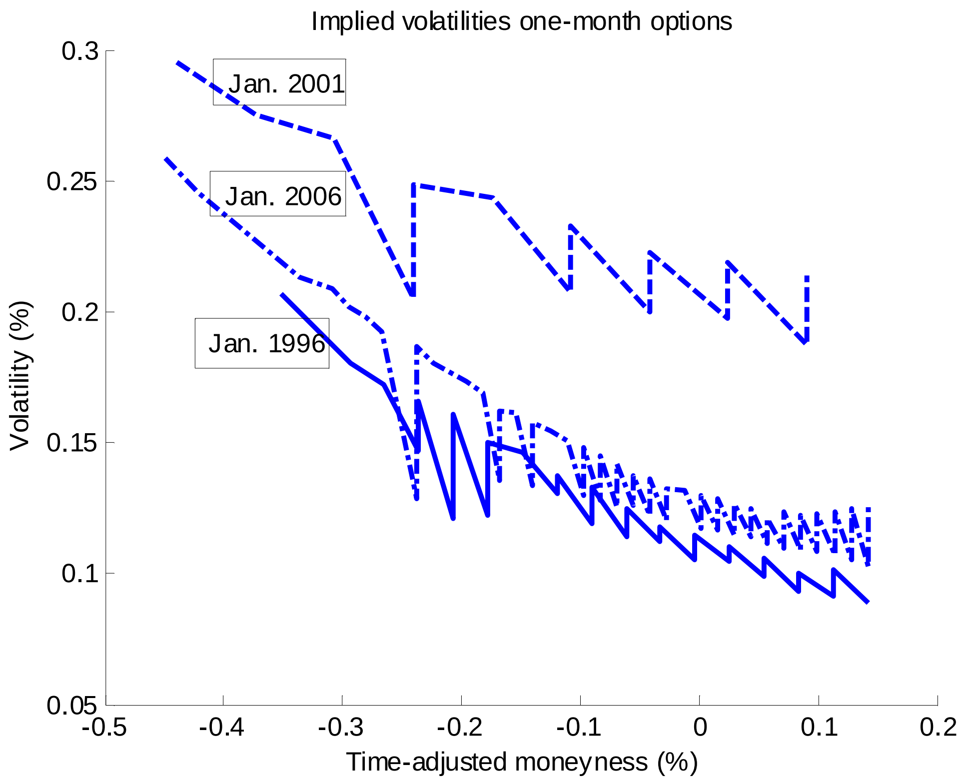


Figure 6 presents Black-Scholes implied volatilities computed from one-month CBOE S&P 500 index call and put quote midpoints on January 19th 1996, January 19th 2001, January 20th 2006. Due to difference in the index level on these dates, the horizontal axis is scaled for consistency of presentation, where time adjusted moneyness is defined as $\left(\frac{K}{S_0 * e^{-d*T}} - 1\right) \div \sqrt{T}$. Negative values of time adjusted moneyness correspond to in-the-money calls and out-of-the-money puts; positive values to out-of-the-money calls and in-the-money puts.

Figure 7

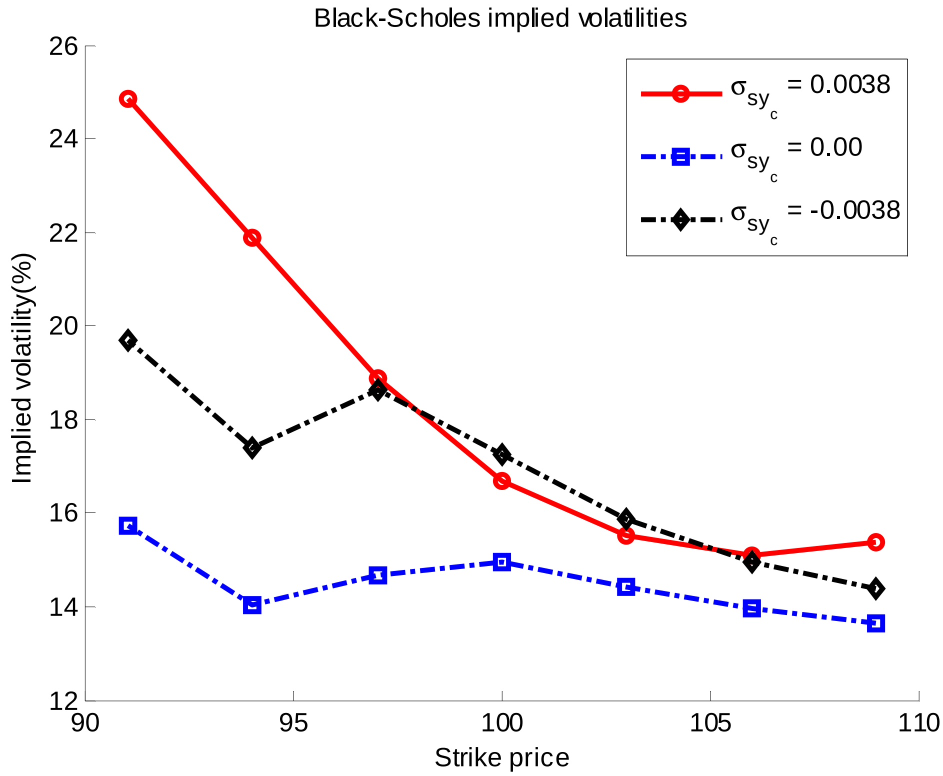


Figure 7 presents Black-Scholes implied volatilities of one-month options produced from the GJD option pricing model (22) evaluated for the implied parameter values of Table II, $S_0 = 100$, and r = 2 percent. The figure shows that a covariance between diffusive return and jumps in consumption of 0.0038, obtained from Table II, generates a sneer with a shape observed in the market and reported in figure 6. Figure 7 also illustrates the impact on Black-Scholes implied volatility sneers of alternative values of the covariance between diffusive return and jumps in consumption holding all other parameter values constant at their values indicated in Table II.

Figure 8

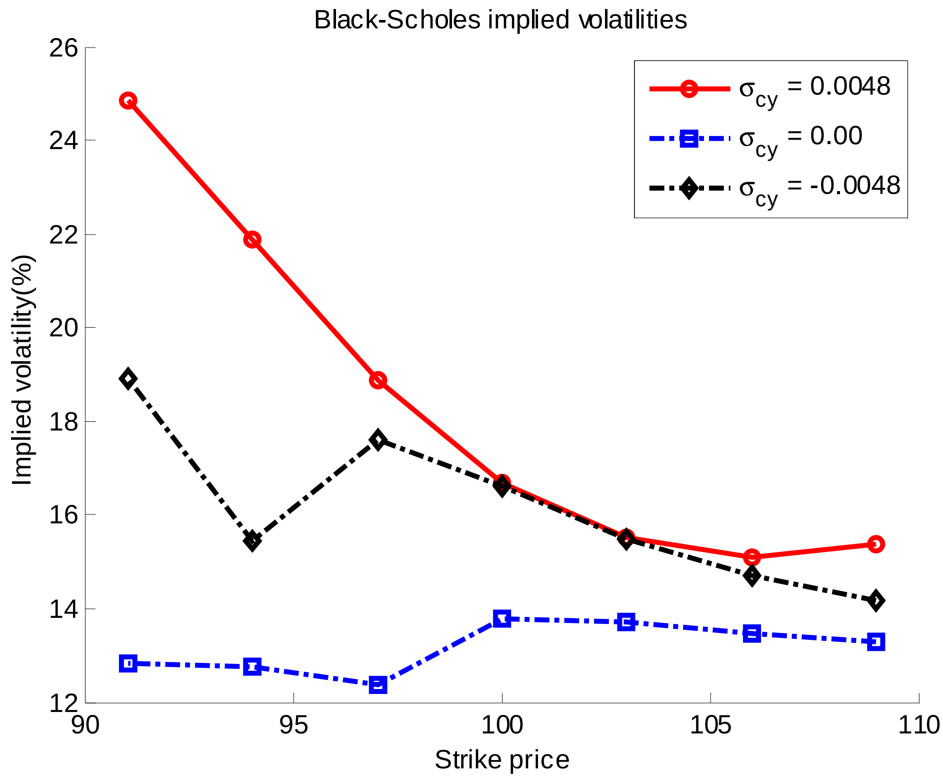


Figure 8 presents Black-Scholes implied volatilities of one-month options produced from the GJD option pricing model (22) evaluated for the implied parameter values of Table II, $S_0 = 100$, and r = 2 percent. The figure shows that a covariance between the diffusive consumption and equity jumps of 0.0048, obtained from Table II, generates a sneer with a shape observed in the market and reported in figure 6. Figure 8 also illustrates the impact on Black-Scholes implied volatility sneers of alternative values of the covariance between the diffusive consumption and equity jumps in consumption holding all other parameter values constant at their values indicated in Table II.