IDS160B - Relativistic Astrophysics and Cosmology
Problem Set 5 1995 March

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1 Expansion of Earth's Orbit

It is alleged that the Universe is continuing to expand as it has for billions of years. If as you have been taught, it is space that is expanding, then why doesn't the Earth's orbit expand also? How about hydrogen and other atoms? The Earth's orbit is expanding. Your problem is to estimate the rate at which it is expanding and what are the relative level of the effects leading to its expansion.

We derived the Friedmann equations for the scale factor both from General Relativity and using Newtonian physics so this expansion is clearly going to be there.

Hint: Both derivations effectively treated the Universe and its contents as a fluid that contained tiny marker test particles. Make this a perturbation problem:

(a) If the Universe expands by a factor $1 + z$, by how much would the radius of the Earth's orbit expand taking the Earth to be a test particle? How much would its velocity change?

(b) To first order the total mass-energy inside of the Earth's orbit is fixed. What will be the new equilibrium radius of the Earth's orbit, if it was adiabatically adjusted?

(c) What decrease is there in the total mass-energy inside the Earth's orbit as a function of time?

(d) At what rate does the Earth's orbit expand due to the cosmological component?

(e) At what rate does the Earth's orbit expand due to the local effect?

2 Relics from the Early Universe

The rough criterion for a particle species either to be in equilibrium or to decouple is when the relevant interaction rate, Γ, per particle is significantly larger or smaller than the Hubble expansion rate, H.

We can understand this quantitatively by considering the rate of change of particles. If N is the number of particles of a species of mass m in a comoving volume V (note the number density $n \equiv N/V$) and Γ is the rate at which particles are created and destroyed, then one expects that

$$\frac{dN}{dt} = \Gamma(N - N_{eq})$$

where N_{eq} is the equilibrium number which depends upon the Boltzmann factor and the number densities of the particles into which and from which the particle species are converted. In general, one would expect that these particles will be in number equilibrium with the photons, if they are lighter than the species we are considering.

Show that this implies the following relation for the number density

$$\frac{dn}{dt} + 3Hn = \Gamma(n - n_{eq})$$

What does the term involving $H \equiv \dot{a}/a$ represent?

2.1 For your education:

This relationship can be derived by setting up the full Boltzmann equation

$$\mathbf{L}[f] = \mathbf{C}[f]$$

where $\mathbf{L}$ is the Liouville operator and $\mathbf{C}$ is the collision (interaction) operator.

The Liouville operator in covariant, relativistic form is

$$\mathbf{L} = p^{\alpha}\frac{\partial}{\partial x^{\alpha}} - \Gamma^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma}\frac{\partial}{\partial p^{\alpha}}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ is the affine connection (a combination of $g_{\mu\nu}$ derivatives previously seen and in Problem 5). For the Robertson-Walker metric the Liouville operator becomes

$$L_{RW} = E\frac{\partial f}{\partial t} - H|\mathbf{p}|^2\frac{\partial f}{\partial E}$$

where f is the homogeneous and isotropic phase space density $f = f(|\mathbf{p}|, t) = f(E, t)$. This is related to the number density n above according to the formula

$$n(t) = \frac{g}{(2\pi)^3}\int d^3p\ f(E, t).$$

The collision operator relates the process of various particles converting to others. In this particular case one would focus on those reactions that create and destroy particles of the species of interest. The full treatment invovles integrating over species and phase space as well as the appropriate matrix elements.

Two important simplifying assumptions are usually made that involve time reversal invariance and the simplification to Maxwell-Boltzmann statistics.

2.2 Equilibrium Equation in Temperature

Since the interaction term usually depends on temperature (energy) rather than time, it is useful to introduce a new independent variable

$$x \equiv \frac{mc^2}{kT}$$

where m is any convenient mass scale which in most cases is the mass of the particle species of interest.

During the radiation-dominated epoch x and t are related, since T and t are:

$$t = \frac{1}{4\sqrt{g_*}}\sqrt{\frac{45}{\pi^3}}\frac{\hbar m_{Pl}}{(kT)^2} = \frac{0.301}{\sqrt{g_*}}\frac{\hbar m_{Pl}}{(kT)^2} = \frac{0.301}{\sqrt{g_*}}\frac{\hbar m_{Pl}}{m^2}x^2$$

where the statistical weight g_* or relative number of degrees of freedom is the sum over bosons and fermions weighted by the effective energy density or entropy s

$$g_* \equiv \sum_{i=bosons} (\frac{T_i}{T})^4 g_i + \frac{7}{8}\sum_{i=fermions} (\frac{T_i}{T})^4 g_i$$

$$g_{totals} \equiv \sum_{i=bosons} (\frac{T_i}{T})^3 g_i + \frac{7}{8}\sum_{i=fermions} (\frac{T_i}{T})^3 g_i$$

For the case of annihilations your derived equation can be recast to

$$\frac{dn}{dt} + 3Hn = - < \sigma_A|v| > (n^2 - n_{eq}^2)$$

where σ_A is the cross section for the species to annihilate and we assume that the number density of particles and antiparticles is the same n so that we get an n^2 in the equation. Note that $\Gamma_A \equiv n_{eq} < \sigma_A|v| >$.

This equation can be recast into another form that makes it clearer what is happening

$$\frac{x}{n_{eq}V}\frac{d(nV)}{dx} = -\frac{\Gamma_A}{H}[(\frac{n}{n_{eq}})^2 - 1].$$

The rate of change of particles is set by the ratio of the annihilation rate to the expansion rate. When the annihilation rate drops below the expansion rate, the number of particles is essentially frozen in. This generally happens rapidly since the annihilation rate usually drops much more rapidly with temperature (i.e. expansion factor) than the expansion rate. The full solution of the equlibrium equations is very complicated and in general has no closed form. However, it is possible to get approximate solutions that are often quite accurate. If the numbers are critical, then one resorts to numerical calculations.

The equilibrium values in the extreme relativistic ($x << 3$) and non-relativistic case ($x >> 3$) regimes in terms of the entropy s are

$$n = \frac{45\zeta(3)}{2\pi^4}\frac{g_{eff}}{g_{totals}}s = 0.278\frac{g_{eff}}{g_{totals}}s \qquad (x << 3)$$

$$n = \frac{45}{2\pi^4}(\frac{\pi}{8})^{1/2}\frac{g_{eff}}{g_{totals}}sx^{3/2}e^{-x} = 0.145\frac{g_{eff}}{g_{totals}}sx^{3/2}e^{-x} \qquad (x >> 3)$$

where $g_{eff} = g$ for bosons and $g_{eff} = 3g/4$ for fermions.

2.3 Hot Relics

This is the relativistic regime freeze out case. Assuming that the expansion remains isentropic (constant entropy per comoving volume), the abundance today is

$$n_0 = s_0 n_{eq} = 825[g_{eff}/g_{total-f}]\mathrm{cm}^{-3}$$

If the entropy per comoving volume of the Universe increases by a factor of $f_{0f} = s_0/s_f$ (where 0 and f refer to the present and the freeze out time respectively), then the present abundance is diminished by that factor. The next problem gives and example of entropy production during the electron-positron annihilation.

$$n_0 = s_0 n_{eq} = 825/f_{0f}[g_{eff}/g_{total-f}]cm^{-3}$$

A species that decouples when it is relativistic is often called hot. The present relic mass density contributed by a hot relic of mass m is simply

$$\rho_{hr} = n_0 m \quad \text{or} \quad \Omega_{hr}h^2 = 7.83 \times 10^{-2}\frac{g_{eff}}{g_{total-f}}\frac{mc^2}{1eV}$$

Assuming $\Omega_0 h^2 \leq 1$, what is the cosmological bound on the mass of such hot relics? Compute this for a neutrino assuming that the ratio $g_{eff}/g_{total-f} = 0.14$.

This ratio could be about 0.014 for a species (assume $g_{eff} = 1.5$) that decouples much earlier ($kT \geq 300$ GeV). What would be the limit then?

2.4 Cold Relics

The cold case is some what more complicated. In general the not only is the Boltzmann suppression factor in play but also one must consider the energy dependence of the cross-section. On general theoretical grounds the annihilation cross section should have a velocity dependence $\sigma_a|v| \propto v^{2n} \propto T^{-n}$ proportional to a power of the velocity. ($n = 0$ corresponds to s-wave annihilation, $n = 1$ corresponds to p-wave annihilation, etc. or some mixed sum.) Explictly, the cross section as a function of x is

$$< \sigma_A|v| >= \sigma_0 x^{-n}$$

and

$$n_0 = 1.13 \times 10^4 \frac{(n+1)x_f^{n+1}}{(g_{total}/g^{1/2})m_{Pl}m\sigma_0}\ \mathrm{cm}^{-3} = \frac{3.79(n+1)(g^{1/2}/g_{total})x_f}{mm_{Pl} < \sigma_A|v| >}s$$

showing that the weaker (smaller) the annihilation cross-section the greater the relic abundance and the greater the contribution to the present density Ω_0.

$$\Omega_{cr}h^2 = 1.07 \times 10^9 \frac{(n+1)x_f^{n+1}\mathrm{GeV}^{-1}}{(g_{total}/g^{1/2})m_{Pl}\sigma_0}$$

Consider a baryon symmetric universe and calculate the surviving relic abundance of nucleons and antinucleons. Use an annihilation cross section of $<\sigma_A|v|>\cong m_\pi^2 \approx 20mb$ where $m_\pi = 135$ MeV.

Now consider the case of a heavy neutrino. I.e. not a hot relic but one heavy enough to decouple in the non-relativistic regime. Take for a cross-section the typical weak interaction cross-section of roughly

$$\sigma_0 \cong \frac{5}{2\pi} G_F^2 m^2$$

$$\approx \frac{5}{2\pi} m^2 (1.166 \times 10^{-5})^2 \mathrm{GeV}^{-4} \times 0.389\ \mathrm{GeV}^2\ \mathrm{mb} \cong 0.421 \times 10^{-10} (\frac{mc^2}{1\ GeV})^2\ \mathrm{mb}$$

What mass heavy neutrino would give $\Omega_\nu \sim 1$?

3 Cosmic Neutrino Temperature

If between the time the relic cosmic neutrinos decouple and the present, electrons and positrons annihilate each other to negligible levels leaving their energy primarily as photons, what is the ratio of the thermodynamic temperature of the relic neutrinos compared to the cosmic background photons?

Remember to take into account the difference in effective statistical weight g_{eff} and energy density for fermions and bosons.

If roughly 56, spin one half species annihilate from the time of relic gravity wave decoupling, what is the approximate thermodynamic temperature of the relic gravity wave background?

4 Topological Defect Relics

It is possible that there are topological defects left of an earlier high energy state of the vacuum. These might be seeds for large scale structure formation and have other significant cosmological effects.

4.1 Hedgehogs or Global Monopoles

These are essentially residual points of the higher energy vacuum state. GUT magnetic monopoles interact primarily magnetically and that is so weak that they over close the universe, leading in some estimates to a total lifetime of the universe of 13 years. This crisis lead Alan Guth to the idea of inflation.

However, the broken symmetry might make Hedgehogs or Global Monopoles that have strong interactions that grow linearly with distance. Every hedgehog would have a string of energy connecting it to a matching anti-hedgehog. The strong interaction between hedgehogs and anti-hedgehogs leads to a high annihilation rate and leaves only a few per Hubble volume (scale $d \sim c/H$) at every epoch. The field energy in hedgehogs is spread unevenly across the Hubble length with a hedgehog core moving towards annihilation with its anti- at close to the speed of light. This means that hedgehogs produce density fluctuations ($\delta M/M \sim E_{GUT}^2/m_{Pl}^2$), on the scale of the Hubble length independent of the redshift. This is roughly the level needed to make galaxies, clusters, etc. and is an example of a scale-invariant spectrum of density fluctuations that are produced by nearly all topological defects.

4.2 Cosmic Strings

These are essentially one dimensional regions of the higher energy vacuum state. Compute the bending of light by a cosmic string of mass per unit length μ. How does this depend upon impact parameter?

The metric line element in cyclindrical coordinates for a straight cosmic string on the z-axis is

$$ds^2 = dt^2 - dr^2 - (1 - 8G\mu)d\theta^2 - dz^2$$

where the mass per unit length is $\mu = \pi r_0^2 \rho_0$ under the condition $G\mu << 1$ This shape for the line element can be derived from the cyclindrical symmetry and mirror/translational symmetry. Compute the circumference of a circle of radius r around the string at fixed values of z and t. Does this match the geometry of a conical space? That is, a sheet with an angle $\Phi = 8\pi G\mu$ cut from it and the edges identified to make a conical surface?

A static string has no active gravitational mass, because the tension along the string just cancels the energy per unit length. However, if a string is not straight as would be expected due to the random walk of its production in causally disconnected regions, the tension will cause the string to want to straighten. Since the tension and mass per unit length match, the string will try to straighten at relativistic speeds.

What would happen if such a string passed through you as an unbiased observer? Hint: We can assume that the cosmic string is moving at a speed comparable to the velocity of light. The passage of the string should causes no immediate rearrangement of material in you, because space is flat with no tidal forces, outside the very narrow coherence width of the string. However, the deficit angle leaves the two sides of the observer approaching at a speed approximately $v\Phi$. For $G\mu \sim 10^{-6}$ corresponding to a GUT energy scale of about 10^{16} GeV, this corresponding to a cut closure velocity of about 1 kilometer per second.

This same process would happen to the matter in the universe leaving a wake behind a moving string. It is possible to produce galaxies as a result of string wakes after matter-radiation decoupling at a redshift of about 1000. A Rough estimate is that the wake from a string with $G\mu = 10^{-6}$ moving speed about the speed of light will at a redshifit of about 1000 produce a density fluctuation on the scale of $1h^1$ Mpc a density enhancement per unit area of about $10^{11}h^{-1}$ $M_\odot$ Mpc^{-2}. This is about the right scale for a galaxy.

4.3 Domain Walls

A domain wall is a planar generalization of a cosmic string. It is essentially a two-dimensional state of higher energy density vacuum. It has stresses and energy densities concentrated in a plane and an energy-momentum tensor which is independent of boosts or translations in that plane. The metric solution to Einstein's equations for a domain wall lying in the $z = 0$ plane is

$$ds^2 = (1 - k|z|)^2 dt^2 - dz^2 - (1 - k|z|)^2 e^{2kt}(dx^2 + dy^2)$$

where the constant $k = 2\pi G\sigma$ is proportional to the energy per unit area σ. This value is set because the higher energy vacuum state has negative pressure and thus negative components of the stress-energy tensor like cosmic strings.

Despite that fact that space is flat, particle geodesics have some peculiar features. Show that a particle placed at rest a small distance to one side of the domain wall accelerates away from the wall and calculate the initial magnitude of the invariant acceleration.

Note that if not constrained, a small piece of wall material will follow the particle and chase it around the universe accelerating all the way. Consider the same test particle with mass m. Consider the domain wall material to be a particle. What mass would one need for the replacement particle, assuming the only interaction is gravitational, to get the same behaviour? Hint: remember the equivalence principle. Remember the domain wall is only interacting gravitationally. Would this make a good way to power a spaceship?

Though they make a scale-invariant spectrum of density perturbations, the level is too low to account for galaxy formation.

4.4 Textures

Textures are dyanamically unstable regions of the higher energy vacuum state. They do make a scale-invariant spectrum of density perturbations and could produce large scale structure formation. They do have a distinctive appearance as they quickly form collapsing spheres.

5 The Affine Connection - $\Gamma_{\alpha\beta\gamma}$

The Christoffel symbol

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha})$$

It is frequently useful to raise the first suffix of the Christoffel symbol

$$\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\lambda}\Gamma_{\lambda\beta\gamma}$$

A vector A^μ is located at a point P. If the space is curved, then there is no clear meaning to a parallel vector at a different point Q. Consider a spherical surface to get a easy idea of the ambiguity. If we consider a point P' close to P, then there is a parallel vector at P', with an uncertainty of second order, where the distance between points P and P' defines first order. Thus one can define a meaning to parallel displacement (or transport) of the vector a^μ by keeping it parallel to itself and keeping its length constant.

One can readily show that

$$dA_\beta = A^\alpha \Gamma_{\alpha\beta\gamma} dx^\gamma$$

and

$$dA^\beta = -A^\alpha \Gamma^\beta_{\alpha\gamma} dx^\gamma$$

Show that if the affine connection is zero - $\Gamma^\beta_{\alpha\gamma} = 0$, that space is flat (Minkowski) and that geodesics are straight lines.

Compute the Christoffel symbol (affine connection $\Gamma^\beta_{\alpha\gamma}$) for the two dimensional space defined by the surface of a sphere with radius r where $x^1 = \theta$ and $x^2 = \phi$ are the standard polar coordinates. Consider the transport of the vector (1, 0) from its initial position at the equator on the meridan $\phi = 0$. First transport the vector along the equator to the meridan $\phi = \pi/2$ and calculate the change in the vector. Next transport the vector along the meridan $\phi = \pi/2$ to the north pole. Calculate the change in the vector. Now tranport it back down along the $\phi = 0$ meridan back to the original location on the equator and compute the change from the original vector. It should be clear from this excerise that parallel transport along the equator and along a meridan is exactly what you would intuitively expect – just like parallel transport in a flat 2-D euclidean plane.

This way of looking at the affine connection is through what is called parallel transport. You will also note that the formula relating dA^β and dx^γ are just what one has for a covariant derivative.

We have already seen that the geodesic equation has the form

$$\frac{du^\beta}{d\tau} + \Gamma^\beta_{\alpha\gamma} u^\alpha \frac{dx^\gamma}{d\tau} = 0$$

or

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

Note that this follows immediately from the derivative relationship if we replace A^β with u^β and divide through by $d\tau$ (or for null geodesics replace τ by an arbitrary parameter to characterize the trajectory). The geodesic equation is simply parallel transport - or inertia. It will hold true as long as the equivalence principle is valid, e.g. gravity as opposed to electromagnetism.