DISCRETE-ANGLE RADIATIVE TRANSFER: IMPROVING ON DIFFUSION THEORY?

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Abstract: The somewhat idiosyncratic formalism of Discrete-Angle (DA) radiative transfer theory in cloud-like optical media is packaged for the Intercomparison of 3D Radiation Codes (I3RC) initiative. Preliminary results for the square-wave cloud model are presented and discussed. It is argued that, in time, DA methods will likely compete with diffusion methods in the applications.

1. Background and Outline

The concept of discrete ordinates [*Chandrasekhar*, 1950] is motivated by purely numerical considerations in the face of a computational problem formulated in a continuous angle-space, hence a discretization in $N_\varphi \times N_\mu$ for azimuthal and zenith angles respectively. In contrast, Discrete-Angle (DA) radiative transfer is a linear transport formalism with its own internal logic based on an inherently discrete angle-space. It can be viewed as a direct descendent of the *Schuster*'s [1905] original 2-stream theory in 1D to higher dimensions (primarily 2D and 3D). It is also a cousin of the Dodecahedron Approach to Radiative Transfer or "DART" [*Whitney*, 1974].

To obtain DA radiative transfer from its continuous-angle counterpart, we make a special choice of phase function composed of Dirac δ's and this induces a similar decomposition of the radiance field. Being the "coefficients" of an angular δ-decomposition of radiance, DA radiation fields are actually angularly-integrated radiances, a.k.a. radiative fluxes. Phase function symmetry considerations limit the choices of δ-functions to directions perpendicular to the sides of regular polygons in 2D and to the facets of regular polyhedrons (Platonic solids) in 3D [*Lovejoy et al.*, 1990; *Davis*, 1992]. If spatial discretization is also used —and this is a natural requirement in numerical applications— then the choices are further limited by a space-filling requirement: 3, 4 and 6 directions in 2D (for triangles, squares and hexagons respectively); 4, 6 and 8 in 3D (for tetrahedrons, cubes and octahedrons respectively).

The orthogonal nature of the 4-fold symmetry in 2D and 6-fold symmetry in 3D enables spatial discretization on Cartesian grids with different constants in different directions, as required in I3RC. As far as we know, these orthogonal DA systems where used sporadically in physical chemistry [*Chu and Churchill*, 1955] and furnace design [*Siddal and Selçuk*, 1979] before being independently introduced into atmospheric radiation by *Mosher* [1979] and *Cogley* [1981]. However, they were put into their broader context —and extensively exploited numerically— by the McGill/Metéo-France group in the late 1980's and early 1990's [*Lovejoy et al.*, 1990; *Gabriel et al.*, 1990; *Davis et al.*, 1990, 1991, 1993] who were strongly influenced by contemporary work in statistical physics: cellular automata, lattice-Boltzmann gases, renormalization theory, diffusion on fractals, etc. The relatively low memory overhead in 2D ($4\times N_x \times N_z$) and customized acceleration and pre-processing techniques in both Monte-Carlo and grid-based approaches enabled detailed DA solutions for extremely variable (singular log-normal multifractal) extinction fields on 1024×1024 grids as early as 1990, which is probably still somewhat of a record. These computations by *Davis et al.* [1991, 1993] on a Cray-II supercomputer were for isotropic ($g = 0$) and conservative ($\varpi_0 = 1$) scattering. Since then, the Clermont-Ferrand group [*Borde and Isaka*, 1996] used a somewhat different solution method for the same orthogonal 2D DA system to explore absorption effects in similar cloud models.

In the next section, we "reverse" generalize 3D radiative transfer to 2D —photons in Flatland, not just uniformity in the y-direction— and even 1D where photons travel on a line. We argue that the 2D problem is in same class of physical complexity as the 3D case but simpler to formulate mathematically. The quantum leap in complexity, both mathematical and physical, occurs going from 1D to 2+ dimensions. Section 3 presents DA radiative transfer formalism, its connections to the independent-pixel and diffusion approximations, and numerical solutions (including some preliminary results on the I3RC square-wave cloud). In section 4, we summarize and list the unresolved issues that must be addressed before DA theory can be viably competed against diffusion methods in the applications where efficiency is more important than accuracy.

2. Radiative Transfer in d = 1,2,3 Spatial Dimensions

When we contemplate the steady-state radiative transfer equation (RTE) in coordinate-free notations, i.e.,

$$\Omega\bullet\nabla I = -\sigma(\boldsymbol{r})I(\boldsymbol{r},\Omega) + \varpi_0\sigma(\boldsymbol{r})\int p(\Omega'\bullet\Omega)I(\boldsymbol{r},\Omega)\mathrm{d}\Omega' + S(\boldsymbol{r},\Omega), \tag{1}$$

we naturally think of 3 spatial dimensions [e.g., *Chandrasekhar*, 1950]. Radiance I has the sanctioned SI units of W/m^2/st and the "volume" source term S is in W/m^3/st (and possibly also per unit of frequency or wavelength in both cases), the scattering phase-function p is expressed in st^{-1}, and extinction is in m^{-1}; the arguments of I and S are $\boldsymbol{r} = (x,y,z)^{\mathrm{T}}$, with superscript "T" meaning transpose, while $\Omega = (\cos\varphi\sin\theta,\sin\varphi\sin\theta,\cos\theta)^{\mathrm{T}}$, hence $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. However, there is no reason to not consider the simpler situation where $\boldsymbol{r} = (x,z)^{\mathrm{T}}$ and $\Omega = (\sin\theta,\cos\theta)^{\mathrm{T}}$, hence $\mathrm{d}\Omega = \mathrm{d}\theta$; in this inherently 2D case however, I will in W/m/rad, the "area" source term S is in W/m^2/rad, p is in rad^{-1}, but extinction is still in m^{-1} and its inverse is still the (local) photon mean-free-path (MFP).

Having covered the dimensionalities d = 3 and d = 2, can we go a step lower? For d = 1, we can always set $\boldsymbol{r} = z\hat{\boldsymbol{z}}$ (where $\hat{\boldsymbol{z}}$ is a vertical unit-vector) but then the associated angle space degenerates to {+,−} for "up" and "down." The RTE in (1) still has meaning as long as $\int(\cdot)\mathrm{d}\Omega$ is identified with $\sum(\cdot)$. Finally, the physical units of I are now in principle just W, W/m for S, and p is reduced to a pair of non-dimensional coupling constants for forward- and back-scattering.

Of course, the common understanding of "1D" radiative transfer is actually in 3D but with uniform properties in the (x,y)-plane, so the physical units are always the usual ones. The popular "2-flux" incarnation of 1D theory [e.g., *Meador and Weaver*, 1980] is however mathematically equivalent to the d = 1 situation described above, only with an extra m^{-2} in the units of I (generally denoted F) and S to restore them to the usual units of flux, specifically though the (x,y)-plane.

In 1D theory (either understanding), everything becomes analytically tractable because of the standard change of variables $\mathrm{d}\tau(z) = -\sigma(z)\mathrm{d}z$ for an infinitesimal increment in optical depth, usually increasing downwards (i.e., with the overall flow of solar radiation). In 2D and beyond ($d \geq 2$), there is little one can do analytically with the RTE in (1). The diffusion equation is easier to manipulate, and some general insights can indeed be gained [e.g., *Davis et al.*, 1998]. These insights show that complexities —for instance, radiative flow channeling around dense areas and into tenuous ones— do not change qualitatively as d is incremented from 2 to 3.

3. Discrete-Angle Radiative Transfer

3.1 1st-Order Formulation

For simplicity, we will assume d = 2 spatial dimensions (orthogonal unit-vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{z}}$) in most of the following. Generalization from 2D to 3D (d = 3) is straightforward. Take

$$I(\boldsymbol{r},\Omega) = F_{+\boldsymbol{x}}(\boldsymbol{r})\delta(\Omega-\hat{\boldsymbol{x}}) + F_{-\boldsymbol{x}}(\boldsymbol{r})\delta(\Omega+\hat{\boldsymbol{x}}) + F_{+\boldsymbol{z}}(\boldsymbol{r})\delta(\Omega-\hat{\boldsymbol{z}}) + F_{-\boldsymbol{z}}(\boldsymbol{r})\delta(\Omega+\hat{\boldsymbol{z}}), \tag{2}$$

and

$$\varpi_0 p(\theta_{\mathrm{s}}) = f\delta(\theta_{\mathrm{s}}) + b\delta(\theta_{\mathrm{s}}-\pi) + s\delta(\theta_{\mathrm{s}}-\pi/2) + s\delta(\theta_{\mathrm{s}}+\pi/2), \tag{3}$$

where f, b, and s are the (unconditional) probabilities of scattering respectively in the forward, backward and (one of the) side directions. This means that $f+b+2s$ adds up to $1-a \leq 1$ where a is an optional probability of absorption at each scattering event. Conceptually, the sparse sampling of angle space described in Eqs. (2–3) is at the exact opposite of the smoothing operations implicit in a low-order spherical-harmonic truncation such as the P_1- and P_3-theories in neutron transport. This makes it all the more remarkable that we can show (cf. §3.3) that diffusion/P_1 theory is actually contained in the class of transport theories considered here.

The effect of using phase functions that are sums of angular δ's is to convert the integro-differential RTE into a relatively simple system of coupled 1st-order PDEs. For instance, with Eqs. (2–3) in mind, we define the formal 4-vector of fluxes:

$$\mathfrak{F}(\boldsymbol{r}) = \left(F_{+\boldsymbol{x}}(\boldsymbol{r}),F_{-\boldsymbol{x}}(\boldsymbol{r}),F_{+\boldsymbol{z}}(\boldsymbol{r}),F_{-\boldsymbol{z}}(\boldsymbol{r})\right)^{\mathrm{T}}, \tag{4}$$

representing fluxes in the $\pm x$ and $\pm z$ directions respectively in, say, the "square-wave" or "MMCR" cloud models (I3RC cases #1–2, that are actually 2D). Defining

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \text{ and } P = \begin{pmatrix} f & b & s & s \\ b & f & s & s \\ s & s & f & b \\ s & s & b & f \end{pmatrix}, \tag{5a}$$

we now need to solve

$$\left([A_x \partial_x + A_z \partial_z] + \sigma(\boldsymbol{r})[I - P] \right) \mathfrak{F}(\boldsymbol{r}) = \mathfrak{S}(\boldsymbol{r}) \tag{5b}$$

where $\mathfrak{S}(\boldsymbol{r})$ is an optional source term that looks like $\mathfrak{F}(\boldsymbol{r})$ in (4), and $I–P$ is a full matrix that describes at once extinction and scattering. Note how it couples the various fluxes in $\mathfrak{F}(\boldsymbol{r})$. Natural (unitary incoming flux F_0) BCs for the solar problem are easily expressed:

$$F_{-z}(x, L_z) = 1, \tag{6a}$$

$$F_{+z}(x, 0) = 0, \tag{6b}$$

for all x, recalling that with these BCs we can set $\mathfrak{S}(\boldsymbol{r}) \equiv 0$ in (5b); we adopt cyclical BCs along the lateral x-boundaries:

$$F_{\pm x}(L_x, z) = F_{\pm x}(0, z), \tag{6c}$$

for all z. Figure 1 illustrates schematically the propagation of a DA "photon" in a DA Monte Carlo scheme in 2D (there is no trace in the y-direction, perpendicular to the paper). Note that only the side- and some of the back-scatterings are apparent; indeed, a forward-scattering never leaves a visible signature on the trace, and quite possibly for two or more back-scatters with no side-scattering in between them as well.

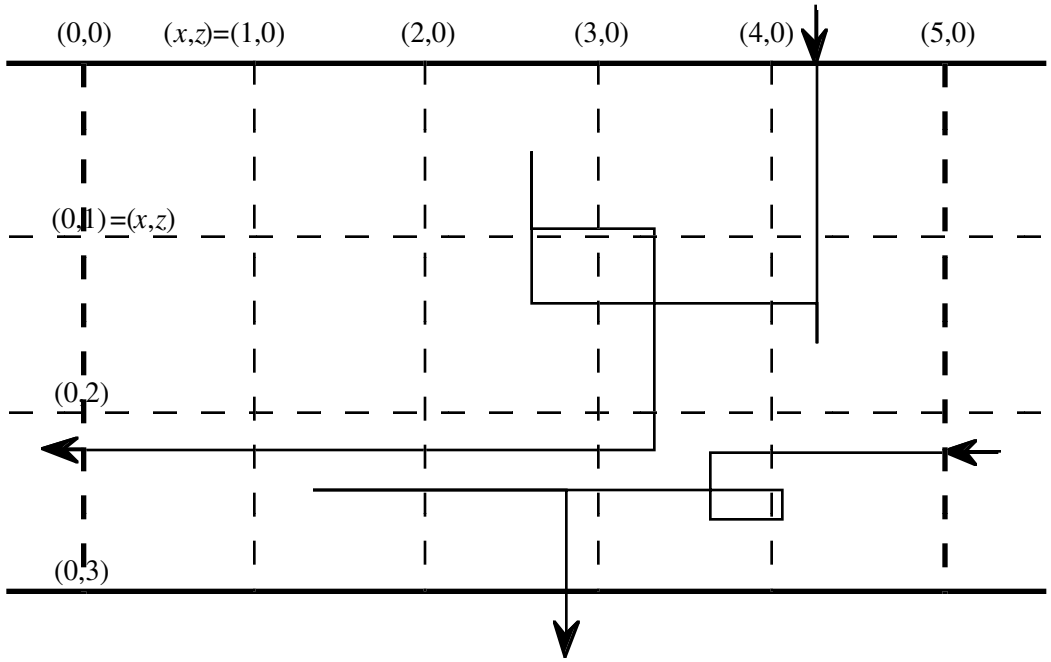

Figure 1: *Schematic of 4-Beam DA Radiative Transfer in "Flatland" (i.e., 2D space).* The medium is characterized on a regular $N_z \times N_x = 3 \times 5$ Cartesian grid ($\ell_z = \ell_x = 1$) with cyclical BCs in x. In fact, this particular DA photon is "recycled" at the $x = 5$ boundary, having encountered the $x = 0$ boundary. This particular history eventually ends in transmission.

The elements of the scattering/absorption matrix P in orthogonal DA systems can be related to two familiar quantities,

$$\varpi_0 = f + b + 2(d-1)s, \tag{7}$$

and

$$\varpi_0 g = f - b, \tag{8}$$

and one less familiar,

$$\varpi_0 h = f + b - 2(d-1)s. \tag{9}$$

Considering (3), the multipliers of ϖ_0 in Eqs. (7–8) are respectively the 0th- and 1st-order moments of $\cos\theta_s$ with respect to $p(\theta_s)$ in this DA model. In Eq. (9), h combines twice the 2nd-order minus once the 0th-order moments. This 3-way orthogonality is reminiscent of the first three Legendre coefficients of $p(\theta_s)$, not a coincidence as we will see further on. The new relation in (9) enables us in principle to make different DA models for, say, Henyey–Greenstein and C1 phase functions in $d = 3$ with identical values of g by using s. In the former case, s is between b and f; in the latter case, it is less than both b and f.

As in (7–9) for the phase function, the DA fields in (4) can be combined into two familiar quantities:

$$J = (F_{+z}+F_{-z}) + (F_{+x}+F_{-x}), \tag{10}$$

and

$$\boldsymbol{F} = \begin{pmatrix} F_x \\ F_z \end{pmatrix} = \begin{pmatrix} F_{+x}-F_{-x} \\ F_{+z}-F_{-z} \end{pmatrix}, \tag{11}$$

and a less familiar one,

$$K = (F_{+z}+F_{-z}) - (F_{+x}+F_{-x}), \tag{12}$$

that cannot be obtained by linear combination from J and the components of $\boldsymbol{F}$. *Davis et al.* [1991] called this new quantity the "non-diffusive" component. This is justified because they were able to relate K to the traceless (non-isotropic) part of the normalized radiation pressure (momentum transport) tensor, $\int\Omega\Omega I(\boldsymbol{r},\Omega)\mathrm{d}\Omega$, which vanishes identically in the photon diffusion limit where $I(\boldsymbol{r},\Omega)$ is in the form $(J+3\Omega\bullet\boldsymbol{F})/4\pi$.

The parallel we have here between low-order Legendre coefficients and Eqs. (7–9) as well as a multi-pole expansion and Eqs. (10–12) is not an accident. Indeed, the eigenvalues of the matrix $I–P$ in the DA transport equation (5) for 2D are

$$a = 1-\varpi_0, \tag{13}$$

$$b = 1-\varpi_0 g \text{ (twice)}; \tag{14}$$

$$c = 1-\varpi_0 h; \tag{15}$$

the associated eigenfunctions being J, $\boldsymbol{F}$, an K respectively. The physical limits on the parameters in (13–15) are respectively $a \in [0,1]$, $b \in [a,2-a]$, and $c \in [a,1]$. This suggests that the DA system in (5a–b) can be further simplified by using the quantities that appear in (10–12) and the parameterization of scattering/absorption processes in (13–15).

3.2 2nd-Order Formulation

Adding and subtracting subsequent rows of Eqs. (5a–b) two-by-two yield coupled PDEs for the quantities

$$F_{\boldsymbol{i}}^{(\pm)} = F_{+\boldsymbol{i}} \pm F_{-\boldsymbol{i}} \ (\boldsymbol{i} = \boldsymbol{x},z) \tag{16}$$

which appear in the various parentheses of Eqs. (10–12). Similar quantities $S_{\boldsymbol{i}}^{(\pm)}$ can be defined for the source-vector in (5b) but we can assume here symmetry in $\pm\boldsymbol{i}$ for the source term, so $S_{\boldsymbol{i}}^{(-)} \equiv 0$. The $F_{\boldsymbol{i}}^{(-)}$ quantities are simply the components of the usual flux vector and they verify Fick-type relations with their $F_{\boldsymbol{i}}^{(+)}$ counterparts:

$$F_{\boldsymbol{i}}^{(-)} = \frac{-1}{b\sigma(\boldsymbol{r})}\partial_i F_{\boldsymbol{i}}^{(+)}, \tag{17}$$

as follows from the pair subtractions. The pair additions yield

$$\partial_i F_{\boldsymbol{i}}^{(-)} = -c\sigma(\boldsymbol{r})F_{\boldsymbol{i}}^{(+)} + 2s\sigma(\boldsymbol{r})(J-F_{\boldsymbol{i}}^{(+)}) + S_{\boldsymbol{i}}^{(+)} \tag{18}$$

where $J-F_{\boldsymbol{x}}^{(+)}$ is just $F_{\boldsymbol{z}}^{(+)}$ and vice-versa; so this enters the coupling term between the x- and z-axis beams via side-scattering. By substitution of (17) into (18) and using (9) in (15) and (18), we are left with two coupled 2nd-order PDEs for the $F_{\boldsymbol{i}}^{(+)}$ quantities:

$$[-\partial_i \frac{1}{b\sigma(\boldsymbol{r})}\partial_i + c\sigma(\boldsymbol{r})]F_{\boldsymbol{i}}^{(+)} = (c-a)\sigma(\boldsymbol{r})(J-F_{\boldsymbol{i}}^{(+)}) + S_{\boldsymbol{i}}^{(+)} \tag{19}$$

To finish recasting the DA problem in 2D at 2nd-order, we need proper BCs. These BCs are easily obtained by first retrieving $F_{\pm i}$ from the $F_i^{(\pm)}$:

$$F_{\pm i} = \frac{1}{2}\,[F_i^{(+)} \pm F_i^{(-)}] = \frac{1}{2}\,[1 \mp \frac{1}{b\sigma(\boldsymbol{r})}\,\partial_i]F_i^{(+)}. \tag{20}$$

Then, looking back to the 1st-order BCs in (6a–c), we find mixed 2nd-order BCs for $i = z$ and again cyclic BCs for $i = x$.

In summary for 2D, we need to numerically solve

$$[-\partial_z \frac{1}{b\sigma(\boldsymbol{r})}\,\partial_z + c\sigma(\boldsymbol{r})]F_z^{(+)} = (c-a)\sigma(\boldsymbol{r})F_x^{(+)} \tag{21a}$$

$$[-\partial_x \frac{1}{b\sigma(\boldsymbol{r})}\,\partial_x + c\sigma(\boldsymbol{r})]F_x^{(+)} = (c-a)\sigma(\boldsymbol{r})F_z^{(+)} \tag{21b}$$

where we have made use of the fact that there is no need for an internal source term if we use the following BCs:

$$[1 - \frac{1}{b\sigma(\boldsymbol{r})}\,\partial_z]F_z^{(+)}\big|_{z=L_z} = 2, \tag{22a}$$

$$[1 + \frac{1}{b\sigma(\boldsymbol{r})}\,\partial_z]F_z^{(+)}\big|_{z=0} = 0, \tag{22b}$$

for all x (assuming zero surface reflectance); and

$$F_x^{(+)}(L_x,z) = F_x^{(+)}(0,z),\ \partial_x F_x^{(+)}\big|_{x=L_x} = \partial_x F_x^{(+)}\big|_{z=0}, \tag{22c}$$

for all z (cyclical continuity in the horizontal). The more interesting fields (required in I3RC – Phase 1 at least) are

$$R(x) = F_{+z}(x) = \frac{1}{2}\,[1 + \frac{1}{b\sigma(\boldsymbol{r})}\,\partial_z]F_z^{(+)}\big|_{z=L_z} = F_z^{(+)}(x,L_z) - 1, \tag{23a}$$

using (22a), for reflectance, and similarly

$$T(x) = F_{-z}(x) = \frac{1}{2}\,[1 - \frac{1}{b\sigma(\boldsymbol{r})}\,\partial_z]F_x^{(+)}\big|_{z=0} = F_z^{(+)}(x,0), \tag{23b}$$

for transmittance; the local flux convergence is

$$-\nabla\bullet\boldsymbol{F} = a\sigma(\boldsymbol{r})J(\boldsymbol{r}) = a\sigma(x,z)[F_x^{(+)}(x,z) + F_z^{(+)}(x,z)] \tag{23c}$$

which can be used to compute local heating rates in the medium and total column absorption.

3.3 Connections to the Independent-Pixel and Diffusion Approximations

An interesting feature of DA radiative transfer theory is that it opens a completely new route from the RTE to the diffusion approximation. This is done through a choice of phase function parameters, not a constraint on the radiation field. Indeed, consider the limit $c \to \infty$ (which is purely formal, cannot be reached physically), for given a and b (equivalently, ϖ_0 and g) in Eq. (21a) or (21b). The solvability condition then requires $F_i^{(+)} = J - F_i^{(+)}$ ($i = x,z$); in other words, we have $F_x^{(+)} = F_z^{(+)}$, so in (12) we find $K = F_z^{(+)} - F_x^{(+)} = 0$ the "non-diffusive" component vanishes. Furthermore, we then have $F_z^{(+)} = F_x^{(+)} = J/2$ from (10) and J obeys a bone fide diffusion equation since the remaining terms in Eqs. (21a) and (21b) can be combined into

$$[-\nabla\bullet D(\boldsymbol{r})\nabla + (1-\varpi_0)\sigma(\boldsymbol{r})]J = 0, \tag{24}$$

where we used (13), and the 2D diffusivity is properly given by

$$D(\boldsymbol{r}) = 1/[2b\sigma(\boldsymbol{r})] = 1/[2(1-\varpi_0 g)\sigma(\boldsymbol{r})]. \tag{25}$$

For more details, see *Davis et al*. [1991]. The important finding here is that one can go all the way from the standard (continuous angle) RTE in (1) to the diffusion equation in (24) only by manipulating phase function parameters, not the radiation field as in the better-known derivations.

It is also interesting to note that another phase function manipulation takes us back to the Independent Pixel Approximation (IPA) where the horizontal gradients (hence net horizontal fluxes) vanish identically. Indeed, simply by setting $s = 0$ (or $h = 1$, or $c = a$) in Eqs. (7–9), we find $F_{\pm x}(\boldsymbol{r}) \equiv 0$ from (5a–b) and (6a–c), i.e., the IPA prediction when a simple 2-steam model is in control of the columnar transport. So, as in (24–25), only parameters ϖ_0 and g are relevant.

Although DA photon transport may not seem very realistic at first inspection (cf. Fig. 1), the above results show that it is exactly one step above diffusion theory in a hierarchy of approximations starting at the IPA using 2-stream theory. At this first level of 3D radiative transfer modeling, absorption is parameterized by ϖ_0, scattering by g and possibly with an f-factor for the recommended [*Joseph et al.*, 1976] δ-rescaling of the phase function to account for the forward peak; it is customary to set $f = g^2$ which is not to be confused with the probability f of forward DA scattering in Eq. (3). This model can with or without μ_0 (solar angle) dependence, depending on how the source is specified, but there are no horizontal fluxes in spite of horizontal σ-gradients. 3D diffusion theory is at the next level with the same absorption (ϖ_0) and scattering parameters (g, maybe f) plus the ability to model horizontal fluxes in response to the horizontal gradients in σ; a priori, the solar boundary source is modeled as diffuse but this limitation can be removed by again using an internal source term instead (and μ_0-dependence restored). At the next level, we find 6-beam DA radiative transfer with all the same features as in 3D diffusion but three, rather than two, parameters to describe the scattering (f,g,h) however. To keep the above formulation as simple as possible, we did not incorporate the δ-rescaling which would affect h the same way as g, nor did we explain how one would distribute the light scattered out of a slant solar beam onto the 6 DA beams depending on the continuous angle phase function to be approximated. At the next level of approximation, one would find "P_3" and "S_4" schemes with yet another parameter to model the scattering, especially if they were worked out analytically to some extent in 3D. And so on, to higher-order P_L and S_N methods as implemented in DANTSYS [*Kornreich et al.*, this volume] and in other models.

3.4 Computational Considerations

Computational DA radiative transfer can be done in either of the two traditional ways of solving continuous angle problems: statistical Monte Carlo (MC) techniques, and deterministic methods using spatial discretization. The particular mathematical structure of orthogonal DA transfer enables a large degree of customization (hence acceleration) in both approaches. For instance, in MC there is no need to test for which cell boundary the photon trajectory is going to cross next and it is easy to cumulate extinction; in fact, cumulative extinction can be pre-computed throughout the grid, and even inverted so that the DA photon can be moved almost directly to its next scattering position. Of course the scattering computations are highly simplified here as well.

As simplified as they are in DA transfer, MC methods are always less efficient and accurate than grid-based alternatives if the full spatial solution is required, also when computer memory allocation becomes a problem and only a part of the solution is required. If memory is a problem due to grid size, then the angular simplicity of DA transfer gives it a definite advantage with respect to its continuous-angle counterpart in any dimensionality. The most straightforward spatial grid approach uses the 1st-order formulation in Eqs. (5–6) which leads to a large but sparse matrix problem easily solved by an iterative relaxation scheme. *Davis* [1992] implemented both (full-grid) MC and relaxation solvers for 2D and exercised them on a particularly challenging problem based on a 10-step log-normal singular multifractal cascade model leading to a 1024×1024 grid with extinction variability over 11 orders-of-magnitude; good convergence was obtained in both cases after significant amounts of Cray-II supercomputer time, as neither method was fully optimized. We also note that the structure of Eqs. (21a–b) can be accommodated by any reasonably general elliptical PDE system solver. This should yield a considerable speed-up in applications requiring efficient algorithms. Finally, the same "long-characteristics" procedure as used in diffusion-based modeling can make use a DA-based estimate of the source function to evaluate radiances at point and any angle.

Figure 2 shows preliminary 6-beam MC results for the DA reflectance $R(x)$, transmittance $T(x)$, absorptance $A(x)$, and z-integrated horizontal flux-divergence $H(x)$ fields of the square-wave cloud model for $\varpi_0 = 0.99$ under overhead sun (so no internal source term is used to start the 10^8

photon histories). Note that the right-hand scale for H is expanded 10-fold with respect to the left-hand scale for A,R,T; this is also the only field that shows much horizontal variability. The observed ultra-accuracy of the IPA prediction is traceable to the spurious angular "resonance" between the illumination and the internal boundaries on the one hand, and to the strongly forward-scattering DA phase function in Eq. (3) ($f/\varpi_0 = 0.865$, $b/\varpi_0 = 0.015$, $s/\varpi_0 = 0.030$) on the other hand. Indeed, it takes a photon about $1/(1-\varpi_0 g) \approx 7$ scatterings (mostly forward here) to almost surely populate the diffuse field but, by the time this happens (about 7 MFPs further on), the DA photon has generally been transmitted, reflected or absorbed without hardly ever leaving the column it started in. Recall that these columns are 2 (left) and 18 (right) MFPs tall (as well as wide), so if a DA photon is lucky enough to be side-scattered from the dense side it will generally not be stopped until it hits the next dense "crest" in the square-wave structure. *Kinne* [this volume] obtains similar artifacts even though he uses a completely different numerical approach based on discretization of the 1st-order DA equations in §3.1 and relaxation of the associated sparse matrix problem. This underscores the need to properly δ-rescale the DA radiative transfer equations to remove this adverse effect of the (exact) forward scattering.

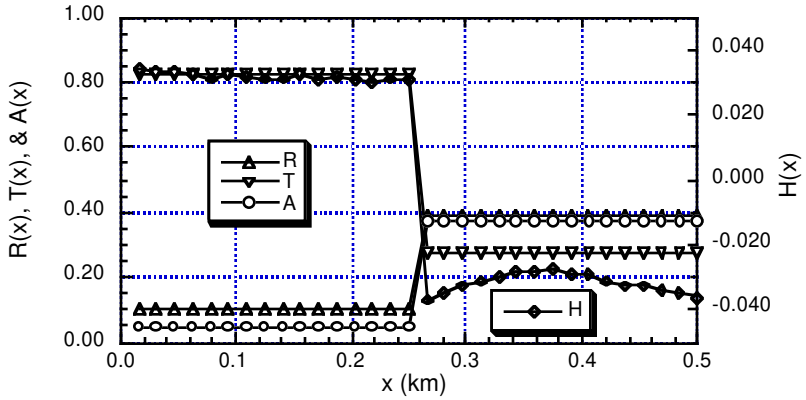

Figure 2: *DA Solutions for the Square-Wave Cloud Model (I3RC case #1).* See main text.

The redeeming quality of the MC results in Fig. 2 is the efficiency with which they can be obtained, especially in more involved cases with large grids. Indeed, all the column- and row-integrals of the extinction field can be pre-computed, and inverted. This way, the random walking DA photon is informed about exactly where to go at each step with a large look-up table, much the same way scattering angle computation is usually accelerated in standard (continuous-angle) MC.

4. Summary and Future Work

We have presented DA radiative transfer theory in enough technical detail to put it into the broader context of approximations to 3D radiative transfer which is of interest to a large portion of the I3RC community. Approximate-but-efficient 3D radiative transfer is also likely to find many applications in dynamical cloud modeling projects where the goal is to reproduce the observed 3D structure of clouds but current radiation schemes are still 1D, leading to significant biases in the local heating rates. Diffusion and DA radiative transfer theories are the likely candidates for these applications, and we have shown here that DA theory contains diffusion theory in a certain asymptotic limit of its natural parameter space.

We have discussed previous, current, and future numerical implementations of DA radiative transfer using relaxation of sparse matrix systems, Monte Carlo simulation, and elliptical PDE solvers. The preliminary results obtained for the I3RC square-wave cloud model are disappointing in a rather interesting way. Tasks for the near future include a detailed study of how to best use δ-rescaling of the DA phase function, also how to best distribute the sources of DA radiance given the solar illumination geometry and the (continuous-angle) phase function to be modeled.

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