Improved Bounds on Maximum Size Binary Radar Arrays *

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Abstract

The maximum size of binary radar arrays (matrices) with only 8 or fewer rows has previously been determined. We determine the maximum size of radar arrays containing 9-16 rows, and for those containing 17 rows we narrow the maximum size down to two values. We also give improved upper and lower asymptotic bounds on the maximum size of radar arrays, which narrow the gap between the existing upper and lower asymptotic bounds by more than 25%.

Key words: radar arrays, binary matrices, frequency hopping.

I Introduction

A radar may transmit a sequence of tones whose frequency "hops" in time. The echo returning to the radar from a moving object is shifted in both time and frequency, according to the range and velocity, respectively, of the object. One design goal for such a radar is to construct a frequency hopping pattern that results in the minimum ambiguity in the range and velocity of the object upon evaluation of the returned signal [1]. The frequency hopping pattern may be described by an $N \times M$ binary array (matrix) with exactly one "1" per column. A "1" in the (i, j)th position indicates that the ith frequency tone is transmitted in the jth time slot. When the velocity is not important, such as for slowly moving objects, the returned echo pattern will correspond to a binary array shifted in time, i.e., shifted horizontally. The distance of the object is determined by the horizontal shift that maximizes the correlation between the transmitted signal and returned echo. Thus, a good design for the binary array is one which has a large number of columns and

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yet in which the horizontal autocorrelation is nearly zero at every shift except the null-shift. The more columns a binary array has, the more difficult it becomes to satisfy such an autocorrelation requirement.

A *radar array* is an $N \times M$ binary matrix, such that every column contains exactly one "1", and such that the horizontal autocorrelation function can only take on the values 0, 1, and M [2]. That is, the 1's of a horizontally (time) shifted version of the array overlap 1's of the unshifted array at most one time. Let $G(N)$ be the maximum value for which an $N \times G(N)$ radar array exists. The radar array problem is to determine $G(N)$. This is currently an unsolved problem, although some bounds have been obtained in the past by several researchers.

We may regard a radar array as an $N \times M$ grid with one "dot" per column, expressed by a vector $(r_1, \ldots, r_M)$, where for each i, the integer r_i indicates which row contains the dot of the ith column. Whenever $r_i = r_j$, the distance $|i-j|$ is called the *spacing* of the pair (i,j). Note that a binary matrix with exactly one "1" per column is a radar array if and only if each positive spacing appears at most once.[1]

The radar array problem was first introduced by Golomb and Taylor [1] in 1982. They provided the following bounds on radar array sizes.

Proposition 1 *[1]* $2 \leq \frac{G(N)}{N} \leq 3$, *for all* N.

A computational approach was taken by Robinson [2] in 1985, and radar arrays with $N \leq 25$ rows were designed.

Table I lists the best known upper and lower bounds on $G(N)$ for all $N \leq 20$ (listed under the headings "old lower bound" and "old upper bound"). It can be seen that for $N \leq 8$ the values of $G(N)$ are known exactly whereas for $9 \leq N \leq 20$ the value of $G(N)$ has not been known exactly. In this paper we determine the exact value of $G(N)$ for $9 \leq N \leq 16$, and we obtain lower bounds that are within 1 of the best known upper bound for $N = 17$.

The precise asymptotic behavior of $G(N)$ is not presently known. Robinson noted that in the finite set of cases he examined, $G(N)/N$ was never smaller than 2.5. Asymptotic bounds extending Robinson's observation were proved in 1988 by Blockhuis and Tiersma as stated in the following proposition.

Proposition 2 *[3]* $\frac{5}{2} \leq \limsup_{N\to\infty} \frac{G(N)}{N} \leq \frac{80}{27}$.

In 1994 this asymptotic result was further improved by Zhang and Tu as given in the following proposition.

Proposition 3 *[4]* $\frac{306}{113} \leq \limsup_{N\to\infty} \frac{G(N)}{N} \leq \frac{9+\sqrt{5}}{4}$.

This lower bound relied on the following lemma.

[1]A radar array has the property that for any horizontal shift of the array, at most one dot in the shifted array overlaps a dot in the unshifted array. A *Costas array* is a special case of a radar array in which this property holds for any horizontal shift followed by any vertical shift. The vertical shifts of the Costas array model doppler shifts resulting from moving targets.

Lemma 1 *[4] If there exists an $N \times M$ radar array whose rows each contain 2 or 3 dots, and one of the rows is a 2-dot row with a spacing of 1, then for all integers i, there exists a $(((6N-1)7^i+1)/6) \times (7^i M)$ radar array.*

There is a gap of 0.101 between the bounds in Proposition 3. In the present paper, we provide improved asymptotic upper and lower bounds which reduce the gap to 0.074. Our asymptotic lower bound exploits our discovery of a 17×46 radar array and our upper bound is based on a counting argument.

II Radar Array Search Algorithm

In this section we give an algorithm to search for radar arrays of a given size. The number of $N \times M$ arrays that have 1 dot per column is N^M, a prohibitively large number for exhaustive computer searches. Thus, effective heuristic searching techniques are important if they produce new radar arrays. Most radar arrays known to be optimal have the following characteristics:

1. All rows have either 2 dots or 3 dots.
2. The spacings present in the radar array are all the integers in the range 1 to $2M - 3N$.

We construct an efficient search algorithm that restricts attention to radar arrays meeting the conditions above.

For each i, let $P(i)$ be an integer that indicates the column of the leftmost dot of spacing i, if spacing i is present in the array, and otherwise let $P(i) = 0$. The positions $P(1), \ldots, P(2M-3N)$ are said to form a *radar subarray* if dots may be placed in an empty $N \times M$ array such that no row has more than 3 dots, no column has more than one dot, no spacing is repeated, and for all $P(i) \neq 0$, the leftmost dot of spacing i is in column $P(i)$. This condition can be easily checked. The following algorithm finds all radar arrays meeting conditions 1 and 2 above.

Step 1. Initialize an empty $N \times M$ array, let $s := 2M - 3N$, and let $P(i) = 0$ for all i.

Step 2. If $s = 2M - 3N + 1$, output "No Solution" and halt.
If $s = 0$, output "Solution found" and halt.

Step 3. $temp := \min\{j > P(s) \colon P(1), \ldots, P(s-1), j, P(s+1), \ldots, P(2M-3N)$ is a radar subarray$\}$.

Step 4. If the minimum in Step 3 existed, then:

(a) $P(s) := temp$

(b) $s := s - 1$

Else

(c) $P(s) := 0$

(d) $s := s + 1$

Step 5. Go to Step 2.

One feature of the algorithm is that it gives a definite yes or no answer for whether a radar array of the given type exists. To see that the algorithm always terminates, note that $P(2M-3N), P(2M-3N-1), \ldots, P(1)$ can be viewed as digits of a number written in base M, where $P(1)$ is the least significant digit. Let P_t denote the value of this number after the tth execution of Step ($4a$). Since $s \leq 2M-3N$, Step ($4d$) can be executed at most $2M-3N$ times between executions of Step ($4b$). When Step ($4a$) is executed, the value of P_t increases by at least M^{s-1}, which is strictly greater than the sum of all the decreases occurring in executions of Step ($4c$) which followed the previous execution of Step ($4a$). Thus, $P_t > P_{t-1}$ and the algorithm eventually halts since P_t is bounded above by $P_t < M^{(2M-3N)}$.

Table I summarizes the improvements made in the best known bounds on small radar arrays, using the algorithm above, or minor modifications of it. We have eliminated the gap between upper and lower bounds for all $N \leq 16$, and reduced the gap to one for N in the range $17 \leq N \leq 19$.

The upper bounds in Table I were obtained by modifying the algorithm into an efficient exhaustive search. In any $N \times M$ radar array, let x_i denote the number of rows which contain exactly i dots, where $i = 1, 2, 3, \ldots$. Then $\sum_i x_i = N$, $\sum_i i x_i = M$, and the number of distinct spacings present in the radar array is $\sum_i \binom{i}{2} x_i \leq M-1$. These three conditions constrain $(x_1, x_2, \ldots)$. Let T denote the set of spacings not represented in the radar array. The definition of radar subarray is modified to allow for more than three dots per row, according to $(x_1, x_2, \ldots)$. Step ($4b$) is modified to "do $s := s-1$ while ($s \in T$)," and Step ($4d$) is modified analogously to "do $s := s+1$ while ($s \in T$)." An exhaustive search over permissible $(x_1, x_2, \ldots)$ and T is conducted.

For example, if a 10×27 radar array exists, the only possibilities are $(x_1, x_2, x_3, x_4) = (0,3,7,0)$, $(0,4,5,1)$, $(0,5,3,2)$, $(1,1,8,0)$, or $(1,2,6,1)$. With $(x_1, x_2, x_3, x_4) = (1,1,8,0)$, there are exactly 25 distinct spacings, i.e., only one missing spacing in the range 1 to 26. Thus, the algorithm is run with $T = \{26\}$, then it is run with $T = \{25\}$, then $T = \{24\}$, and so on. If no radar array is found, then configuration $(1,1,8,0)$ is eliminated. If all configurations are eliminated in a similar manner, then no radar array of that size exists. In this manner, a reduced complexity full search is accomplished.

Theorem 1 *There exist radar arrays of sizes* 9×24, 13×35, 16×43, *and* 17×46. *There do not exist radar arrays of sizes* 10×27, 11×30, 12×33, 14×38, 15×41, *or* 16×44.

III New asymptotic bounds

Theorem 2 $\frac{276}{101} \leq \limsup_{N\to\infty} \frac{G(N)}{N} \leq \frac{20+\sqrt{6}}{8}$.

Proof: To prove the lower bound note that every row of the 17×46 radar array in Table II has either 2 or 3 dots, and the last row has 2 dots and a spacing of 1. By Lemma 1,

$$\limsup_{N\to\infty} \frac{G(N)}{N} \geq \lim_{i\to\infty} \frac{6 \cdot 7^i \cdot 46}{(6 \cdot 17 - 1)7^i + 1} = \frac{276}{101}.$$

To prove the upper bound we let $\mathcal{A}$ be an arbitrary $N \times M$ radar array and we will count the spacings in $\mathcal{A}$. (Throughout the proof we take N even – this does not affect the result.) Let C_k be the subset of $\mathcal{A}$ consisting of the columns from locations 1 to k together with the columns from location $M-k+1$ to M. For any fixed $K \in \{1,2,\ldots,M\}$, let W_j be the subset of $\mathcal{A}$ consisting of the columns starting from location $\max\{1, j-K+1\}$ up to and including the column at location $\min\{M, j\}$, i.e., a window in $\mathcal{A}$ of length K columns or smaller. As j sweeps from 0 to $M+K$, the window W_j slides across $\mathcal{A}$. Let $S_k = \{M-k, M-k+1, \ldots, M-1\}$, the set of k largest possible spacings in $\mathcal{A}$. Let D_j be the number of spacings contained in W_j. If $K \geq 2N$, it follows (from Lemma 6 and Lemma 7 in [4], with $\rho = 3$ and $k = 1$, in their notation), that

$$\sum_{j=1}^{M+K} D_j \quad \leq \quad \frac{K(K-1)}{2} \tag{1}$$

$$\sum_{j=1}^{M+K} D_j \quad \geq \quad (2K-3N)M + 5N^2 - 3KN. \tag{2}$$

The bound in (2) can be strengthened by noting that its derivation does not count any spacings in windows W_j which contain N or fewer columns. Let r be the number of rows in $\mathcal{A}$ which have at least two spacings in S_N. In each of these r rows, there exists at least three dots, two of which are either in the first N columns of $\mathcal{A}$ (i.e., in W_N) or in the last N columns of $\mathcal{A}$ (i.e., in W_{M+K-N}).

Of the r rows, let l denote the number of rows which contain two or more dots in W_N. Then $D_N \geq l$. Furthermore, a left shift of the window W_N results in the window W_{N-1}, and this affects the bound by at most one, that is, $D_{N-1} \geq l-1$. Similarly, $D_{N-2} \geq l-2$, and in general $D_{N-i} \geq l-i$ for $1 \leq i \leq l$. Also, $D_i \geq l$ for all $i \in \{N+1, \ldots, N+l\}$.

The analysis in [4] uses the looser bound $D_i \geq \max\{0, i-N\}$ for $i \leq 2N$, and thus undercounts $\sum_{j=1}^{2N} D_j$ by at least $1+2+\cdots+(l-1)+l+(l-1)+(l-2)+\cdots+2+1 = \sum_{i=1}^{l}(2i-1)$. A similar undercount occurs in windows W_j for j ranging from $M+K-2N$ to $M+K$, yielding a total of at least

$$2\sum_{i=1}^{\lfloor r/2 \rfloor}(2i-1) = 2\left\lfloor \frac{r}{2} \right\rfloor^2 \tag{3}$$

uncounted spacings in (2). Hence, (2) is strengthened to

$$\sum_{j=1}^{M+K} D_j \geq (2K-3N)M + 5N^2 - 3KN + 2\left\lfloor \frac{r}{2} \right\rfloor^2. \tag{4}$$

Together with (1), this implies

$$M \leq \frac{K^2 - K - 10N^2 + 6KN - 4\lfloor r/2 \rfloor^2}{4K - 6N}. \tag{5}$$

The inequality in (5) can be tightened by carefully choosing K. We choose $K = \left\lfloor \frac{3N}{2} + \frac{1}{2}\sqrt{5N^2 - 4r^2} \right\rfloor$ (recall that $r \leq N$, and thus $K \geq 2N$ as required) and simplify to obtain

$$\limsup_{N\to\infty} \frac{M}{N} \le \limsup_{N\to\infty} \frac{9+\sqrt{5-4(\frac{r}{N})^2}}{4}. \tag{6}$$

This is a generalization of the result in [4] which implicitly assumes $r = 0$.

Next, we develop another upper bound for M. The analysis from here to (13) applies to any $N \times M$ array $\hat{\mathcal{A}}$ (not necessarily a radar array) for which every column contains exactly one dot and for which every row contains three or fewer dots. It is later extended to arbitrary $N \times M$ radar arrays.

Let P be the set of spacings present in the $N \times M$ radar array. Let $T_k = |S_k \setminus P|$, i.e., the total number of spacings in S_k not represented in the radar array. Let $r_{i,j}$ be the number of rows which have exactly i spacings in S_N and exactly j dots in $C_{N/2}$, i.e., in the collection of the first and last $N/2$ columns. It is easy to see that $r_{1,3} = 0$. Each of the two dots defining any spacing in $S_{N/2}$ must be in $C_{N/2}$; at least one dot of a spacing in S_N must be in $C_{N/2}$. Counting the columns in $C_{N/2}$ gives

$$N = \sum_{i=0}^{2}\sum_{j=1}^{3} j r_{i,j} \ge r_{1,1} + 2r_{1,2} + r_{2,1} + 2r_{2,2} + 3r_{2,3}. \tag{7}$$

Counting the spacings in $S_{N/2}$ which are represented in $\hat{\mathcal{A}}$ at least once gives

$$|P \cap S_{N/2}| = |S_{N/2}| - T_{N/2} \le r_{1,2} + r_{2,2} + 2r_{2,3}. \tag{8}$$

Since $|S_{N/2}| = N/2$ and $T_N \ge T_{N/2}$, this gives

$$T_N \ge N/2 - r_{1,2} - r_{2,2} - 2r_{2,3}. \tag{9}$$

Counting the spacings in S_N which are represented in the array at least once gives

$$|P \cap S_N| = |S_N| - T_N = \sum_{i=0}^{2}\sum_{j=1}^{3} i r_{i,j}. \tag{10}$$

Since $|S_N| = N$, this gives

$$T_N \ge N - r_{1,1} - r_{1,2} - 2r_{2,1} - 2r_{2,2} - 2r_{2,3}. \tag{11}$$

Adding (7), (9), and (11) gives

$$T_N \ge N/4 - (r_{2,1} + r_{2,2} + r_{2,3})/2 \tag{12}$$

and using $r = r_{2,1} + r_{2,2} + r_{2,3}$ and $T_{M-1} \ge T_N$ gives

$$T_{M-1} \ge N/4 - r/2. \tag{13}$$

The inequality in (13) holds for any array $\hat{\mathcal{A}}$ which has one dot per column and at most three dots in any row. Since $2 \le G(N)/N \le 3$, any $N \times G(N)$ radar array $\mathcal{A}$ can be transformed into an array $\hat{\mathcal{A}}$ of this type by performing a sequence of operations, as follows. Each operation moves a single dot. If a row contains d_1 dots and another row contains d_2 dots, with $d_1 \geq d_2 + 2$, then a single-dot operation consists of a dot moving from the row containing d_1 dots to the row containing d_2 dots. This results in one row having $d_1 - 1$ dots and the other row having $d_2 + 1$ dots. The sequence of operations ends when every row has either two dots or three dots, at which point the entire transformation is complete. After this transformation, the resulting array satisfies (13), for it meets the condition that every column has one dot and every row has at most three dots.

Suppose the transformation consists of a total of R operations. The R operations can be reversed to obtain the radar array $\mathcal{A}$ from $\hat{\mathcal{A}}$ again. In each of these reverse operations, consider the effect on the inequality (13). Let a_i denote the net increase in the number of spacings due to the ith reverse operation. (For example, suppose in a reverse operation that a dot is moved from a 3-dot row to another 3-dot row, to yield a 2-dot row and a 4-dot row. A 2-dot row has one spacing, a 3-dot row has three spacings, and a 4-dot row has six spacings, and thus, $a_i = (1+6) - 2 \cdot 3 = 1$.) For the ith reverse operation, T_{M-1} decreases by at most a_i, since at most a_i unique spacings are created in the reverse operation. After the entire sequence of reverse operations, the radar array $\mathcal{A}$ is obtained, and inequality (13) is weakened to

$$T_{M-1} \geq N/4 - r/2 - \sum_{i=1}^{R} a_i. \tag{14}$$

It is easy to show that any array with one dot per column and two or three dots per row contains exactly $2M - 3N$ spacings. By the definition of a_i, the reverse transformation from this type of array to an $N \times M$ radar array $\mathcal{A}$ increases the number of spacings by exactly $\sum_{i=1}^{R} a_i$. Thus,

$$2M - 3N + \sum_{i=1}^{R} a_i \leq |P| = M - 1 - T_{M-1} \leq M - 1 - \left(N/4 - r/2 - \sum_{i=1}^{R} a_i \right). \tag{15}$$

Hence,

$$M \leq 11N/4 + r/2 - 1. \tag{16}$$

Equations (6) and (16) imply

$$\begin{aligned}
\limsup_{N\to\infty} \frac{M}{N} &\leq \limsup_{N\to\infty} \min \left\{ \frac{11}{4} + \frac{r}{2N}, \frac{9 + \sqrt{5 - 4(\frac{r}{N})^2}}{4} \right\} & (17)\\
&\leq \frac{11}{4} + \frac{1}{2}\left(\frac{\sqrt{6} - 2}{4} \right) & (18)\\
&= \frac{20 + \sqrt{6}}{8},
\end{aligned}$$

with equality in (18) when $r = (\sqrt{6} - 2)N/4$. ■

N	G(N) old lower bound	G(N) new lower bound	G(N) new upper bound	G(N) old upper bound
2	4			4
3	7			7
4	10			10
5	12			12
6	15			15
7	18			18
8	21			21
9	23	24		24
10	26		26	27
11	29		29	30
12	32		32	33
13	34	35		35
14	37		37	38
15	40		40	41
16	42	43	43	44
17	45	46		47
18	48			49
19	51			52
20	53			55

Table I: Improved upper and lower bounds on G(N).

$N \times M$	Row Location of dot in each column
9×24	1 2 3 4 5 6 7 8 7 4 3 9 9 5 8 2 6 5 1 4 2 1 3 7
13×35	1 2 3 4 5 6 7 8 9 10 9 11 8 6 12 3 13 13 7 4 12 11 5 2 10 5 8 1 4 7 2 1 3 6 9
16×43	1 2 3 4 5 6 7 8 9 10 11 12 13 7 14 15 12 8 16 4 10 5 16 9 15 13 3 6 14 11 6 9 3 5 2 8 2 1 1 4 7 10 12
17×46	1 2 3 4 5 6 7 8 9 10 11 12 11 13 14 8 15 12 7 6 16 17 17 16 3 10 4 15 9 14 13 2 5 9 1 10 5 7 12 4 2 1 3 6 8 11

Table II: Examples of improved radar arrays, in vector notation. The ith integer of each array indicates the row position of the dot in the ith column.

IV Conclusions

We have shown that $2.733 \approx \frac{276}{101} \leq \limsup_{N\to\infty} G(N)/N \leq (20+\sqrt{6})/8 \approx 2.806$. The remaining gap is approximately 0.0735.

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