Physics 139 Relativity
Relativity Notes 2003

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Notes to be found at:
http://aether.lbl.gov/www/classes/p139/homework/homework.html

1 Uniform Acceleration

This material is to prepare a transition towards General Relativity via the *Equivalence Principle* by first understanding uniform acceleration.

The *Equivalence Principle* stated in a simple form: *Equivalence Principle:* A uniform gravitational field is equivalent to a uniform acceleration.

This is not very precise statement and one lesson we have learned in Special Relativity is the need to be precise in our statements, definitions, and use of coordinates. We will come to a more precise statement of the *Equivalence Principle* in terms like at a space-time point with gravitational acceleration $\vec{g}$ there is a tangent reference frame undergoing uniform acceleration that is equivalent. This is similar to the instantaneous rest frame of Special Relativity in the case of an object undergoing acceleration.

We first need to understand carefully what is a uniform acceleration reference frame, which we will do in steps.

First imagine a reference frame – a rigid framework of rulers and clocks, our standard reference frame – undergoing uniform acceleration. In classical nonrelativistic physics we can imagine a rigid framework to which we can apply a force which will cause it to move with constant acceleration.

However, in Special Relativity no causal impulse can travel faster than the speed of light, thus the frame work cannot be infinitely rigid. When the force causing the acceleration is first applied, the point where the force is first applied begins to accelerate first and as the casual impulse moves out, the other portions join in the acceleration.

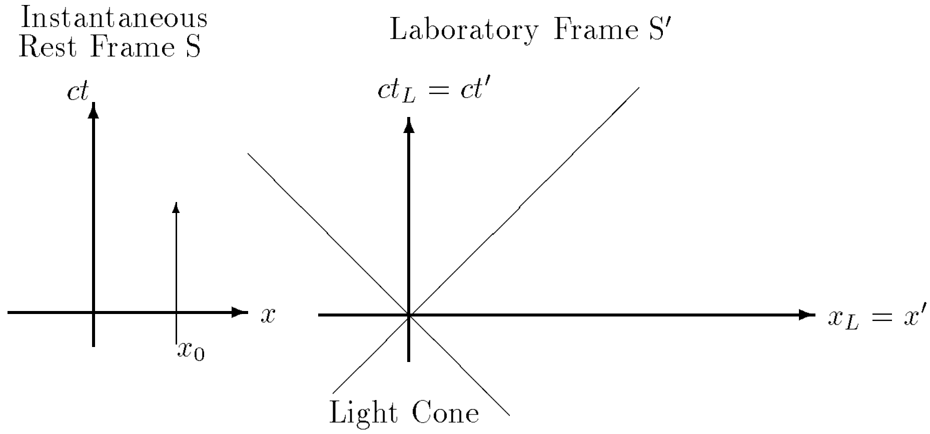
Consider a simple long rod as an example: If one pulls on a long rod, it will lengthen at first as the end being pulled starts moving before the other end even knows it is. Then as it gains speed, Lorentz-FitzGerald contraction will cause it to shorten. If one pushes on the long rod from behind, it will first shorten as the end with the force moves toward the other end which sits there unaware of the some to arrive acceleration. All objects, however rigid, evidently display some degree of elasticity during acceleration. It is clear that in Special Relativity no rod can be infinitely rigid but must be elastic at some level. (Home work problem: prove that since the speed of sound is less than or equal to the speed of light, that the rigidity of any material is less than xxx?)

As a body accelerates, it moves in a continuous fashion from one inertial system to another. If it is to retain its same rest length in its instantaneous rest system, then it length relative to its original inertial system will have to decrease continuously because of Lorentz-FitzGerald length contraction. If, on the other hand, it retained the same length relative to the original inertial system, then the Lorentz-FitzGerald contraction would require its rest length to increase as its gains speed. This is not very satisfactory.

Either way, the metric will depend upon time. If we want a direct comparison to gravity, we need to require an accelerated coordinate system to have a time independent form.

1.1 Accelerating a Point Mass

A uniformly accelerating point mass is one that is subject to the same force in each and every one of its instantaneous rest systems. I.e. a uniformly accelerating point mass is subject to a constant force $\vec{F} = m_o\vec{g}$ along the $+x$-axis in a coordinate system which is the inertial frame where its velocity is zero (instantaneous rest frame).



Acceleration transforms as

$$a_x = \frac{d^2x}{dt^2} = \frac{a'_x}{\gamma^3\left(1 + \frac{vu'_x}{c^2}\right)^3} \tag{1}$$

so that in the instantaneous rest frame $a = \gamma^{-3}a'$. In the instantaneous rest frame $F_x = F'_x$. Now we can solve the equation of motion in either of two ways: from the acceleration or from the force. In Problem Set 2 we solved the problem for a uniformly accelerating rocket using the acceleration transformation. [1]

[1] In rocket frame the acceleration was $a'_x = g$ (Note reversal of S′ and S compared to discussion

Here we use force transformation.

$$\begin{aligned}
F'_x = \frac{dp'_x}{dt'} &= m_o g = F_x \\
\frac{d\gamma m_o \beta c^2}{dct'} &= m_o g \\
d(\gamma\beta) &= \frac{g}{c^2} d(ct') \\
\gamma\beta &= \frac{g}{c^2}(ct') = \frac{gt'}{c} \qquad (2)
\end{aligned}$$

where the constant of integration is set equal to zero because we define the time zero to be when $\beta = 0$. This can be turned into an equation for β alone:

$$\begin{aligned}
\gamma\beta &= \frac{\beta}{\sqrt{1-\beta^2}} = \frac{g}{c^2}(ct') = \frac{gt'}{c} \\
\frac{\beta^2}{1-\beta^2} &= \left(\frac{gt'}{c}\right)^2 \\
\beta &= \frac{(gt'/c)}{\sqrt{1+(gt'/c)^2}} \qquad (3)
\end{aligned}$$

So that, from the laboratory, observing the test particle start from rest we first see its velocity increasing linearly with time as we classically expect for a particle under uniform acceleration. Then as the velocity begins to be a significant fraction of the speed of light, the term in the denominator becomes increasing important and the velocity increases ever more slowly in time and only approaches the speed of light asymptotically. The shape of the trajectory of a particle undergoing uniform acceleration is a hyperbola and not the classical parabola, but for low velocities they are indistinguishable conics.

Note also

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1+\left(\frac{gt'}{c}\right)^2} \qquad (4)$$

If we look at the Lorentz factor γ, we see that it is first very nearly unity and then as the velocity begins to saturate, γ increases linearly with time. This is simply conservation of energy, as the constant acceleration (force in the instantaneous rest frame) is constantly doing work $W = cF$.

Now take a side step and evaluate in terms of proper time.

$$d\tau = \frac{dt'}{\gamma} = \frac{dt'}{\sqrt{1+\left(\frac{gt'}{c}\right)^2}}$$

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in this section.) Thus the acceleration in the Earth frame was $a_x = \left(1 - v^2/c^2\right)^{3/2} g = dv_x/dt$. Regrouping we had $gdt = du_x/\left(1-v^2/c^2\right)^{3/2}$ and integrating gives $gt = v/\sqrt{1-v^2/c^2}$ or $v/c = gt/c/\sqrt{1+(gt/c)^2}$.

$$\tau = \int \frac{dt'}{\sqrt{1+\left(\frac{gt'}{c}\right)^2}} = \frac{c}{g} sinh^{-1}\left(\frac{gt'}{c}\right)$$

or inverting the equation to find t' in terms of proper time τ

$$t' = \frac{g}{c} sinh\left(\frac{g\tau}{c}\right) \tag{5}$$

end aside

Now we can solve for x' using the definition of $\beta = dx'/d(ct')$.

$$\begin{aligned}
dx' &= \beta d(ct') \\
x' &= x_o + \int_{ct'=0}^{ct'=ct'} \beta d(ct') = x_o + \int_{ct'=0}^{ct'=ct'} \frac{gt'/c}{\sqrt{1+(gt'/c))^2}} d(ct') \\
&= \frac{c^2}{g}\sqrt{1+\left(\frac{g}{c^2}(ct')\right)^2}\Big|_{t'=0}^{ct'=ct'} + x'_o \\
&= \frac{c^2}{g}\sqrt{1+(gt'/c)^2} - \frac{c^2}{g} + x'_o
\end{aligned} \tag{6}$$

Define

$$x'_P \equiv x'_o - \frac{c^2}{g} \tag{7}$$

Then our equation becomes

$$\begin{aligned}
x' - x'_P &= \frac{c^2}{g}\sqrt{1+\left(\frac{g}{c^2}ct'\right)^2} \\
\left(\frac{g}{c^2}(x'-x'_P)\right)^2 &= 1 + \left(\frac{g}{c^2}ct'\right)^2 \\
\left(\frac{g}{c^2}(x'-x'_P)\right)^2 - \left(\frac{g}{c^2}ct'\right)^2 &= 1
\end{aligned} \tag{8}$$

This last equation describes a hyperbola. Note that we can find a formula for x in terms of τ given the last equation for x' in terms of t'.

$$(g/c^2)^2(x'-x'_P)^2 - (gt'/c)^2 = 1$$

$$(g/c^2)^2(x'-x'_P)^2 = 1 + (gt'/c)^2 = 1 + sinh^2(g\tau/c) = cosh^2(g\tau/c)$$

or

$$x' = x'_P + \frac{c^2}{g} cosh\left(\frac{g\tau}{c}\right) = x'_0 - \frac{c^2}{g} + \frac{c^2}{g} cosh\left(\frac{g\tau}{c}\right)$$

Note that x'_0 is directly related to the x_0 in the accelerating frame.

Because the world line is a hyperbola in Minkowski space, the world line of the point mass approaches the light line asymptotically. This means all events on the world line will have a space like relationship to all events to the left of the focal point $P \equiv (0, x'_P)$.

$$x'_P = x'_o - \frac{c^2}{g} \tag{9}$$

So that the distance between the rest point and focal point is proportional to the inverse of the acceleration.

insert figure here showing frames with small acceleration and with large accelerations.

$$\beta = \frac{gt'/c}{\sqrt{1+\left(\frac{gt'}{c}\right)^2}} \qquad \frac{g}{c^2}\left(x' - x'_p\right) = \sqrt{1+\left(\frac{gt'}{c}\right)^2} \tag{10}$$

Therefore

$$\beta = \frac{ct'}{x' - x'_P} = tan\theta \tag{11}$$

where θ is the horizontal angle.

insert figure here showing θ etc. The line from point P, $(0, x'_P)$ to point (ct', x') is the x axis in the instantaneous rest frame. Defines simultaneity in instantaneous rest frame is changing constantly since the instantaneous rest frame is continuously changing.

insert figure here showing world lines etc. and that P is a pivot point.

The observer A no matter where along his world lines never knows the future of the observer passing through the pivot point and objects to the left are never in casual contact but if they did would appear to move backward through time. ....

Now calculate the distance from event P $= (0, x'_P)$ to event (ct', x')

$$\left(\frac{c^2}{g}\right)^2 = (x' - x'_P)^2 - c^2 t'^2 \tag{12}$$

combining that with the equation for β yields

$$\begin{aligned} \beta &= \frac{ct'}{x' - x'_P} \\ \beta^2 \left(x' - x'_P\right)^2 &= c^2 t'^2 \end{aligned} \tag{13}$$

Evaluate this for $t' = 0$ to get the distance, x_{PA}, between event P and where A crosses the x' axis.

$$(x_{PA})^2 = \left(\frac{c^2}{g}\right)^2 = (x' - x'_P)^2 - \beta^2 (x' - x'_P)^2 = \left(1 - \beta^2\right)(x' - x'_P)^2 \tag{14}$$

$$(x' - x'_P) = \gamma x'_{PA} \tag{15}$$

Lorentz contraction $x'_{PA} = x_{PA}/\gamma$.

It is easy to show that the distance from the pivot point to any point on the hyperbolic trajectory is the same. The accelerating system moves in such a way that the distance to the pivot point is increasing in inertial space by precisely its instanteous gamma so that the Lorentz length contraction makes the distance to the pivot point in its rest frame constant. I.e. if the line of simultaneity intersects A's trajectory at point B then from the hyperbola formula above for all B we have

$$(x'_B - x'_P) = \gamma x'_{PA} \tag{16}$$

Thus $x_B - x_P = x_{PB} = \gamma x'_{PA}$ The distance from the pivot point event $(0, x_P)$ to the mass point at B as measured in the accelerated coordinate system is the same as the distance from the pivot point event $(0, x_P)$ to the mass point when it was at rest or any other point on its trajectory. Therefore to an observer in the accelerated system the point mass maintains a fixed distance to the pivot point event $(0, x_P)$ throughout its motion. Thus despite accelerating away continously the eternal moment remains a fixed distance away.

1.2 Uniformly Accelerated Reference Frame

We are now in a position to discuss a uniformly accelerated reference frame.

insert figure of two uniformly accelerating masses with same focal point.

Consider two observers (1) and (2) both with the same focus point x'_p and both cross the x'-axis at the same $t' = 0$. Then there is always the same distance from x'_p and thus each other. As a result they will have to have different accelerations because they have the same focus

$$a_1 = g_1 = c^2/x'_1 \qquad a_2 = g_2 = c^2/x'_2 \tag{17}$$

This is what one sees in the figure with the curves further away from the focal point being flatter. A straight line is a the world line for a non-accelerating particle.

One can make a uniformly accelerated frame, if the acceleration of each point is inversely proportional to its distance from the focus point x'_p. Actually $(ct', x') = (0, x'_p)$.

An observer riding with a meter stick in this accelerated frame would say it maintained a constant length. An observer in an inertial frame (e.g. our Lab frame) claims the rod is shrinking in time as it accelerates away. However, as it approaches the origin, it lengthens and slows down.

A rod on the other side of the origin accelerates to the left rather than the right.

This situation is called a Rindler Space.

Note that the coordinate choices are different from our usual every day conventions. Usually we chose the vertical axis to be the z-axis and have the effective acceleration "downward" toward negative z. What we would observe conventionally from our inertial frame would be an elevator rushing doward towards us at high speed and decellerating at a rate g coming to a stop at a distance and then accelerating upwards away retracing its path.

Note that the acceleration depends on distance away from pivot point. That is the only system in which the clocks in the instanteous rest frame do not have to continuously be reset relative to each other.

1.2.1 Coordinate System for a Uniformly Accelerating System

There are two natural frames in which to describe a uniformly accelerating system. They are the accelerating coordinates τ the proper (rest frame) time and related coordinates and the coordinates of one inertial frame that matches the accelerating frame at one instant.

The transformation between the coordinate systems is given by

$$\begin{aligned} x' &= -\frac{c^2}{g} + \left(x + \frac{c^2}{g}\right) cosh\left(\frac{g\tau}{c}\right) \\ ct' &= \left(x + \frac{c^2}{g}\right) sinh\left(\frac{g\tau}{c}\right) \end{aligned} \tag{18}$$

where g is standing in for the uniform 3-acceleration magnitude a. Note that this is what one must expect for the form of the time transformation. At the pivot point $x = x_P = -c^2/g$ time must be frozen (thus the name pivot point). If we put that value in, then we find $ct' = 0$ independent of τ As one moves away from the pivot point, the conversion to t' must increase linearly with the distance from the pivot point. Thus it must be proportional to $x + c^2/g$.

1.2.2 The Metric for a Uniformly Accelerating System

We want to find $g_{\mu\nu}$

$$(cd\tau)^2 = ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

For an inertial frame we have the Minkowski metric $\eta_{\mu\nu}$ which in Cartesian coordinates is diagonal and constant

$$\eta_{\mu\nu} = [1, -1, -1, -1]$$

How about expressing this in terms of the proper time τ and the instanteous rest frame coordinates x, y, and z?

$$dx^0 = cdt = sinh(g\tau/c)dx^1 + (1 + gx/c^2)cosh(g\tau/c)d\tau$$

$$dx^1 = dx' = cosh(g\tau)dx^1 dx^1 + (1 + gx/c^2)sinh(g\tau/c)d\tau$$

$$(cd\tau)^2 = ds^2 = (1 + gx/c^2)^2(cd\tau)^2 - (d\vec{x})^2$$

$$g_{\mu\nu} = [(1 + gx/c^2)^2, -1, -1, -1] \qquad g^{\mu\nu} = [(1 + gx/c^2)^{-2}, -1, -1, -1]$$

Note the difference between covariant and contravariant.

1.3 Alternate Discussion to be integrated

We revisited the Lorentz transformation in the case of circular motion, that is, motion with a uniform speed but continously changing direction, in the case that results in Thomas precession. Now we consider the velocity transformation.

1.3.1 Instanteous Velocity Transformation

The Lorentz transformations of space-time coordinates

$$\begin{aligned} t' &= \gamma\,(t - \beta x/c) \\ x' &= \gamma\,(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned} \tag{19}$$

and their converse (primes exchanged with unprimes and $\beta = v/c$ with $-\beta$ are differentiated with respect to t' and used to find the velocity

$$\begin{aligned} \vec{u} &= (u_1, u_2, u_3) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \\ \vec{u}' &= (u_1', u_2', u_3') = (\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}) \end{aligned} \tag{20}$$

$$u_1' = \frac{u_1 - v}{1 - u_1 v/c^2}, \qquad u_2' = \frac{u_2}{\gamma(1 - u_1 v/c^2)}, \qquad u_3' = \frac{u_3}{\gamma(1 - u_1 v/c^2)} \tag{21}$$

$$u_1 = \frac{u_1' - v}{1 - u_1 v/c^2}, \qquad u_2 = \frac{u_2'}{\gamma(1 - u_1' v/c^2)}, \qquad u_3 = \frac{u_3'}{\gamma(1 - u_1' v/c^2)} \tag{22}$$

No assumption as the uniformity of $\vec{u}$ (or $\vec{u}'$) has been made. These equations apply equally to the instantaneous velocity in non-uniform (or circular) motion.

Now consider the magnitudes u and u' defined as

$$u^2 = u_1^2 + u_2^2 + u_3^2, \qquad u'^2 = u_1'^2 + u_2'^2 + u_3'^2 \tag{23}$$

Now we can readily calculate the $\gamma(u)$ transformation laws by factoring out the $(dt)^2$ and $(dt')^2$ from $(cd\tau)^2 = (cdt)^2 - (d\vec{r})^2 = (cdt')^2 - (d\vec{r}')^2$ and substituting in for u_i'

$$dt^2(c^2 - u^2) = (dt')^2(c^2 - u'^2) = dt^2\gamma^2(v)(1 - u_1 v/c^2)^2(c^2 - u'^2). \tag{24}$$

$$c^2 - u'^2 = \frac{c^2(c^2 - u^2)(c^2 - v^2)}{(c^2 - u_1 v)^2} \tag{25}$$

$$\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left(1 - \frac{u_1 v}{c^2}\right) \qquad\qquad \frac{\gamma(u)}{\gamma(u')} = \gamma(v)\left(1 + \frac{u_1' v}{c^2}\right) \tag{26}$$

Now note how simple this is in the instanteous rest frame:

$$\gamma(u') = \gamma(v)\gamma(u)$$

This should remind you of the rapidity formulation given in the homework. where the rapidity is defined as the rotation angle

$$\phi(u) = tanh^{-1}\left(\frac{u}{c}\right), \qquad tanh(\phi(u) = \frac{u}{c} \tag{27}$$

$$\phi(u) = \phi(u') + \phi(v) \tag{28}$$

Differentiating this with respect to time gives us a simple way to work out the acceleration transformation.

1.3.2 Acceleration Transformation

$$\frac{d}{dt}\phi(u) = \frac{d}{dt'}\phi(u')\frac{dt'}{dt} \tag{29}$$

Since the derivative of the hyperbolic tangent is the hyperbolic secant

$$\frac{d}{dt}\phi(u) = \frac{1}{c}\gamma^2(u)\frac{du}{dt} \tag{30}$$

Since

$$\frac{dt'}{dt} = \frac{\gamma(u')}{\gamma(u)} \tag{31}$$

Substituting we obtain the acceleration transformation formula

$$\gamma^3(u')\frac{du'}{dt'} = \gamma^3(u)\frac{du}{dt} \tag{32}$$

Under the Galilean transformation, the acceleration is invariant; but, acceleration is not in Special Relativity.

We need to define the proper acceleration

$$|\tilde{a}| \equiv \alpha \equiv = \gamma^3(u)\frac{du}{dt} = \frac{d}{dt}\left[\gamma(u)u\right] \tag{33}$$

where α is measured in the instantaneous rest frame.

Now constant instanteous acceleration (constant proper acceleration) is a particularly simple case. Integrating and chosing $u = 0$ at $t = 0$ (or vice versa) one finds

$$\alpha t = \gamma(u)u \tag{34}$$

Thus at low velocity u increases linearly with t and as $u \to c$ $\gamma(u)$ grows linearly with time. Squaring, solving for u, and integrating again, chosing zero as the constant of integration yields

$$x^2 - (ct)^2 = c^4/\alpha \equiv X^2 \tag{35}$$

Thus, for obvious reasons, rectilinear motion with constant proper acceleration is called hyperbolic motion.

1.4 Rindler Space, Symmetry and GR

The equivlance principle implies a new symmetry and thus associated invariance. With a realization and the uniqueness of solutions give a formulation to the theory of gravity.

The strong and weak Equivalence Principle: The weak equivalence principle is that gravitational and inertial masses are precisely equal (also includes Lorentz invariance). The strong equivalence principle applies to all laws of nature that no experiment can distinguish between an accelerating frame of reference and a uniform gravitational field.

We can also use this symmetry approach to find the Rindler space. Consider an "generalized elevator" as a kind of rocket ship in outer space far from the strong influence of Earth or any other body. Now give the "elevator" a constant acceleration g upwards. All inhabitants of the "elevator" will feel the pressure from the floor, just as if they were living in the gravitational field at the surface of the Earth (or equivalent). This is a method of constructing "artificial" gravitational field. We now consider this artificial gravitational field more carefully.

Suppose we want this artificial gravitational field to be constant in space and time. We will find that we can make the artificial gravitational field uniform in time and two spatial directions but it must decrease in the direction of the field itself. The inhabitants will feel a constant acceleration.

Consider a coordinate grid for an elevator free to accelerate uniformly or be in a uniform gravitational field, which we take to be ξ^μ inside the elevator, such that points on the elevator wall and floor are given by ξ^i and are constant. The zeroth component $\xi^0 = c\tau$, where τ is the proper time (elapsed instanteous rest time in the elevator). An observer in outer space uses a standard Cartesian grid x^μ in an inertial frame there. The motion of the elevator is described by the function $x^\mu(\tilde{\xi})$.

That is the elevator is free to move only along one axis (the "vertical" axis). We designate the "vertical" direction to be the z-axis. The origin of the $\tilde{\xi}$ coordinates is a point in the middle of the floor of the elevator, which for convenience coincides with the origin of the $\tilde{x}$ coordinates at $t = \tau = \xi^0(\tau) = 0$. Thus the coordinates of the origin (center point of elevator floor) will be

$$\tilde{\xi} = (c\tau, 0, 0, 0) \qquad \tilde{x}_c = (ct(\tau), 0, 0, z(\tau)) \tag{36}$$

Time (τ) run at a constant rate for the observer inside the elevator.

$$\left(\frac{\partial x^\mu}{\partial \tau}\right)^2 = \left(\frac{\partial ct}{\partial \tau}\right)^2 - \left(\frac{\partial z}{\partial \tau}\right)^2 = c^2. \tag{37}$$

The acceleration is set to be $\vec{g}$, which is the spatial portion of the four-acceleration:

$$\tilde{a} = \frac{\partial^2 x^\mu}{\partial \tau^2} = g^\mu. \tag{38}$$

At $\tau = 0$ we can specify that the velocity of the elevator is zero:

$$\frac{\partial x^\mu}{\partial \tau} = (c, \vec{0}) \quad (\text{at } \tau = 0). \tag{39}$$

We can make use of the differential proper time along any world line $d\tau = dt/\gamma$. Using the relation

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1 + \left(\frac{gt}{c}\right)^2} \tag{40}$$

we find

$$\tau = \int \frac{dt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}} = \frac{g}{c} sinh^{-1}\left(\frac{gt}{c}\right) \tag{41}$$

Inverting this equation we find a relationship for t in terms of τ

$$\frac{gt}{c} = sinh\left(\frac{g\tau}{c}\right) \tag{42}$$

This equation works for the origin. The acceleration depends upon location so that the more general formula becomes

$$ct = \left(\xi^3 + \frac{c^2}{g}\right) sinh\left(\frac{g\tau}{c}\right) \tag{43}$$

$$z = x^3 = \left(\xi^3 + \frac{c^2}{g}\right) cosh\left(\frac{g\tau}{c}\right) - \frac{c^2}{g} \tag{44}$$

At that moment t and τ coincide, and if the acceleration $\vec{g}$ is to be be constant, then at $\tau = 0$, $\partial\vec{g}/\partial\tau = 0$, so that

$$\frac{\partial}{\partial\tau} g^\mu = (F, \vec{0}) = \frac{F}{c}\frac{\partial}{\partial\tau} x^\mu \qquad \tau = 0, \tag{45}$$

where F is an unknown constant.

Now this equation is Lorentz covariant. So not only at $\tau = 0$, but also at all times we should have

$$\frac{\partial}{\partial\tau} g^\mu = \frac{F}{c}\frac{\partial}{\partial\tau} x^\mu \tag{46}$$

Combining equations x and y gives

$$\begin{aligned} g^\mu &= \frac{F}{c}(x^\mu + A^\mu) = \frac{g^2}{c^2}(x^\mu + A^\mu) = \frac{g^2}{c^2}(x^\mu + \delta^\mu_3 \frac{c^2}{g}), \\ x^\mu(\tau) &= B^\mu cosh(g\tau/c) + C^\mu sinh(g\tau/c) - A^\mu, \end{aligned} \tag{47}$$

F^μ, A^μ, B^μ, and C^μ are constants. $F = g^2/c$ can be found from the derivative of four acceleration evaluated at $\beta = 0$. Then from equations 16, 17, and the boundary conditions:

$$(g^\mu)^2 = cF = g^2, \quad B^\mu = \frac{c^2}{g}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad C^\mu = \frac{c^2}{g}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad A^\mu = B^\mu, \tag{48}$$

and since at $\tau = 0$, the acceleration is purely spacelike. We find that the parameter g is the absolute value of the acceleration.

We notice that the position of the elevator floor at "inhabitant time" τ is obtained from the position at $\tau = 0$ by a Lorentz boost around the point $x^\mu = -A^\mu$. This must imply that the entire elevator is Lorentz-boosted. The boost is given by the rotation matrix with angle $\chi = g\tau/c$. This observation immediately gives the coordinates of all other points in the elevator. Suppose at $\tau = 0$,

$$x^\mu(0, \vec{\xi}) = (0, \vec{\xi}) \tag{49}$$

Then at other τ values

$$x^\mu(c\tau, \vec{\xi}) = \begin{pmatrix} sinh(g\tau/c)\left(\xi^3 + \frac{c^2}{g}\right) \\ \xi^1 \\ \xi^2 \\ cosh(g\tau/c)\left(\xi^3 + \frac{c^2}{g}\right) - \frac{c^2}{g} \end{pmatrix} \tag{50}$$

The 0 and 3 (height) components of the ξ coordinates, imbedded in the x coordinates, are pictured in the next figure. The light cone defines the boundary of the space at $\tau = 0$ the coordinates lie on the positive x^3 axis in a very ordinary way. Each x^3 coordinate follows a hyperbola in x^3 and $c\tau$ that keep it in the right quadrant (in x^3- $c\tau$ plane. The description of the quadrant of space time in terms of the ξ coordiates is called "Rindler space".

It should be clear that an observer inside the elevator feels no effects that depend explicity on his time coordinate τ, since a transition for τ to t' is nothing but a Lorentz transformation.

We also notice some important effects:

(i) Equal τ lines (lines of simultaneity) converge at the left (at the $x^3 - c\tau$ origin). It follows that the local clock speed, which is given by $\eta = \sqrt{(\partial x^\mu/\partial c\tau)^2}$. varies with height $\equiv x^3$:

$$\eta = 1 + g\xi^3/c^2, \tag{51}$$

(ii) The acceleration or gravitational field strength felt locally is $\eta^{-2}\vec{g}(\xi)$, which is proportional to the distance to the point $x^\mu = -A^\mu$. So even though the field is constant in the transverse direction and with time, it decreases with height (x^3).

(iii) The region of space-time described by the observer in the elevator is only part of all of space-time, where $x^3 + c^2/g > |x^0|$. The boundary lines are called (past and future) horizons.

All of these are trypically relativistic effects. In the non-relativistic limit $(g \to 0)$ the coordinates simplfy to

$$x^3 = \xi^3 + \frac{1}{2} g \tau^2; \qquad x^0 = c\tau. \tag{52}$$

According to the equivalence principle the relativistic effects discovered here should also be features of gravitational fields generated by matter (or energy). Let us inspect them individually.

Observation (i) suggest that clocks will run slower, if they are deep down in a gravitational field. Indeed as one suspects equation x will generalize to

$$\eta = 1 + \Phi(x)/c^2 \tag{53}$$

where $\Phi(x)$ is the gravitational potential. This will be true, provided that the gravitational field is stationary (not time varying). This effect is called the gravitational redshift.

Relativistic effect (ii) could have been predicted by the following argument. The energy density of a gravitational potential is negative. Since the energy of two masses M_1 and M_2 at a distance r apart is $E = -G_n M_1 M_2 / r$, we can calculate the energy density of a field $\vec{g}$ as $T_{00} = -(1/8\pi G_n)|\vec{g}|^2$. If we have normalized $c = 1$, this is also its mass density. But then this mass density in turn should generate a gravitational field! This would imply

$$\vec{\partial} \cdot \vec{g} = 4\pi G_n T_{00} = -\frac{1}{2} |\vec{g}|^2$$

so that the field strength should decrease with height. However, this reasoning is too simplistic, since the field obeys a differential equation but without the coefficient 1/2.

The possible emergence of horizons (iii) turns out to be a new feature of relativistic gravitational fields. Under normal circumstances the fields are so weak that no horizon will be seen, but gravitational collapse may produce horizons. If this happens, there will be regions of space-time from which no signals can be observed.

The most important conclusion to be drawn is that in order to describe a gravitational field, one may have to perform a transformation from the coordinates ξ^μ that were used inside the elevator where one feels the gravitational field, toward coordinates x^μ that describe empty space-time, in which freely falling objects move along straight lines. Now we know that in an empty space without gravitational fields the clock speeds and the lengths of the rulers are described by a distance fuction $c\tau$ or ℓ as

$$(cd\tau)^2 = -(d\ell)^2 = g_{\mu\nu} dx^\mu dx^\nu; \text{ where } g_{\mu\nu} = \eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1) \tag{54}$$

In tems of the coordinates ξ^μ appropriate for the elevator, we have for infinitesimal displacement $d\xi^\mu$,

$$\begin{aligned} dx^0 &= sinh(g\tau/c)d\xi^3 + (1+g\xi^3/c^2)cosh(g\tau/c)d\tau, \\ dx^3 &= cosh(g\tau/c)d\xi^3 + (1+g\xi^3/c^2)sinh(g\tau/c)d\tau. \end{aligned} \quad (55)$$

This implies

$$(cd\tau)^2 = -(d\ell)^2 = (1+g\xi^3/c^2)^2(dc\tau)^2 - (d\vec{\xi})^2. \quad (56)$$

If we write this in the form

$$(cd\tau)^2 = -(d\ell)^2 = g_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu = (1+g\xi^3/c^2)^2(dc\tau)^2 - (d\vec{\xi})^2. \quad (57)$$

then we see that all effects that the gravitational field have on rulers and clocks can be described in terms of space and time dependent field $g_{\mu\nu}(\xi)$. Only in the gravitational field of a Rindler space can one find coordinates x^μ inerms of these the function $g_{\mu\nu}$ takes the simple form shown. We will see that $g_{\mu\nu}(\xi)$ is all that is ned to describe the gravitational field completely.

Spaces in which the infinitesimal distance $cd\tau$ or $d\ell$ is described by a space time dependent fuction $g_{\mu\nu}(\xi)$ are called curved or Riemann spaces. Space-time is apparently a Riemann space.

We can write the metric more explicitly as

$$g_{\mu\nu} = \begin{pmatrix} \eta(\xi)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1/\eta(\xi)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (58)$$

where $\eta(\xi) = 1 + g\xi^3/c^2$.

$$dx^\mu = (cd\tau, d\vec{r}) \qquad sx_\mu = \eta^2 cd\tau, d\vec{r}) \quad (59)$$

Note since the metric has the form

$$(cd\tau)^2 = -(d\ell)^2 = (1+g\xi^3/c^2)^2(dct^2 - (d\vec{r})^2. \quad (60)$$

then an object stationary at fixed $\vec{r}$ has proper time $d\tau = \eta dt$. A particle moving with velocity $\vec{v} = d\vec{r}/dt$ will have proper time

$$d\tau = \sqrt{(\eta dt)^2 - (d\vec{r}/c)^2} = dt\sqrt{\eta^2 - \beta^2} = dt/\gamma^* \quad (61)$$

where $\gamma^* = 1/\sqrt{\eta^2 - \beta^2}$.

1.4.1 Uniformly Accelerating Clocks - Gravitational Freqency Shift

We could have derived these results by simply considering two clocks in a spaceship ("rocket elevator") with constant acceleration or the Doppler effect on a photon emitted at one end of the spaceship and received at the other and comparing this to our thought experiment about a photon in a gravitational field. This approach is much more physical but does not show all the features of the Rindler space.

Consider that the inertial (and by inference gravitational) "mass" of a particle is given by the sum of its rest energy plus all other energies divided by c^2. Thus the inertial and gravitational mass of photon is $E/c^2 = h\nu/c^2$. If a photon changes its gravitational potential through simple propagation then it must change its energy by and amount $\Delta E = Egh/c^2$ where g is the acceleration of gravity and h is the height change. Thus

$$\frac{\Delta E}{E} = \frac{\Delta \nu}{\nu} = \frac{gh}{c^2} \tag{62}$$

The fractional frequency change is the change in gravitational potential divided by c^2.

For comparison consider a lab accelerating at rate g with the two clocks separated by a instantaneous distance h along the acceleration direction. If the first clock sends a photon of frequency ν_{source}, then the second clock recieves a photon observed at frequency ν_{observed} we know that they are related by the Doppler formula by

$$\nu_{\mathrm{observed}} = \nu_{\mathrm{source}}\gamma\left(1 + \beta cos\theta\right) \tag{63}$$

Since the angle is either 0 or 180°,

$$\frac{\nu_{\mathrm{observed}}}{\nu_{\mathrm{source}}} = \gamma\left(1 \pm \beta\right) \tag{64}$$

The time it takes for the photon to get from the first to the second clock is approximately $\Delta t = h/c$ and the velocity change is $\Delta v = g\Delta t = gh/c$ or $\beta = gh/c^2$ Differentially one has $\nu_2/\nu_1 = 1 + gx/c^2$

The gravitational redshift was fist measured directly in the laboratory in 1960 by Pound and Rebka where they let a 14.4 keV γ-ray, emitted in the radioactive decay of ^{57}Fe, to fall 22.6 meters down an evacuated shaft where $gh/c^2 = 2.47 \times 10^{-15}$, and they measured a fractional change in frequency of $(2.57 \pm 0.26) \times 10^{-15}$, thus verifying to that level the equivalence principle.

One could anticipate that for a spherical mass (M) in an otherwise flat space-time that the rate of clocks would vary as

$$\frac{dt(r)}{dt(\infty)} = 1 - \frac{GM}{c^2 r} \tag{65}$$

1.5 Local Coordinates

It is sometimes better to use local standard clocks for the determination of velocity and acceleration at each point, rather than referring to a single coordinate clock located at the origin. The former run faster than the latter by the factor η, so that the local velocity β_L of an object moving over coordinate intervals dx and dt is given by

$$(\beta_L)^i = \frac{d}{\eta d\tau}(x^i) \quad \text{or} \quad (\beta_L)^i = \frac{1}{\eta}\beta^i \tag{66}$$

A second application of this time derivative operator to $(\beta_L)^i$ gives the connection between coordinate and local acceleration:

$$\left(\dot{\beta}_L\right)^i = \frac{d}{\eta d\tau}(\beta_L)^i = \frac{d}{\eta d\tau}\left(\frac{dx^i}{\eta d\tau}\right) = \frac{1}{\eta^2}\left[\dot{\beta}^i - \frac{\partial_x \eta}{\eta}\beta^i \beta_x\right] \tag{67}$$

since $\partial_\tau \eta = (\partial_x \eta)\beta_x$; hence,

$$(a_L)^i = \frac{1}{\eta^2}\left[a^i - \frac{g}{\eta}\beta^i \beta_x\right] \tag{68}$$

Thus the local acceleration of a free-falling body is

$$(a_L)_x = -\frac{g}{\eta}\left[1 - \frac{\beta_x^2}{\eta^2}\right], \quad (a_L)_y = \frac{g}{\eta}\frac{\beta_x \beta_y}{\eta^2}, \quad (a_L)_z = \frac{g}{\eta}\frac{\beta_x \beta_z}{\eta^2} \tag{69}$$

Thus the acceleration depends upon the local velocity and the local value of g at any point is found to be $g_L = g/\eta$ with the local velocity $(\beta_L)^i = \beta^i/\eta$ so that one can write

$$(a_L)_x = -g_L\left[1 - (\beta_L)_x^2\right], \quad (a_L)_y = -g_L(\beta_L)_x(\beta_L)_y, \quad (a_L)_z = -g_L(\beta_L)_x(\beta_L)_z \tag{70}$$

Freee-falling local acceleration appear here exclusively in terms of local velocities and the local acceleration constant g_L.

When an object falls vertically, its acceleration $(a_L)_x$ ranges between $-g_L$ and 0 depending n $(\beta_L)_x$, rather than between $-g$ and $+g$ as it does at the origin.

1.6 Dynamics

The 4-D momentum is defined in an accelerated system just as it is defined in an inertial frame.

$$\tilde{p} = m_0\tilde{u} \qquad p^\mu = m_0 u^\mu = (p^0, \vec{p}) = m_0\gamma^*(1, \beta_x, \beta_y, \beta_z) \tag{71}$$

where $\vec{p} = \gamma^* m_0 c^2 \beta$ and $p^0 = \gamma^* m_0 c^2$ where $E = m_0 c^2$ is the proper energy. For $m_0 = 0$ in these equations one replaces $\gamma^* m_0 c^2$ by E_0. Because this is a 4-D vector multiplied by an invariant, it can be found either by using he known values of β^μ in the accelerated system or by transforming the inertial 4-D momentum to the accelerated system.

In its covariant form, it is

$$p_\mu = g_{\mu\nu} p^\nu = (\eta^2 p^0, -\vec{p}) = (p_0, -\vec{p}) \tag{72}$$

so that

$$p_0 = \gamma^* \eta^2 m_0 c^2$$

There are evidently two entirely different energies of an object in the accelerated system; the covariant and he contravariant energies.

If momentum and energy are conserved in a local interaction in an inertial system, then the momentum and the covariant and contravariant energies are conserved in the accelerated system. ($\Delta p = 0$ is an invariant.)

However, if there a quantity is conserved over the path of a freely-falling particle? The answer as we will show later is that the covariant energy is.

More generally, the covariant energy of an object is constant for any time-independent metric.

1.7 Gravitational Redshift

The Equivalence principle leads directly to two interesting predictions about the behavior of light in the presence of gravity. The first effect is that as light climbs up a gravitational gradient, its frequency decreases. The second is that light is deflected by a gravitational field.

These effects are obvious, if one knows that light consists of photons where $E = h\nu$ is the relation between the photon's kinetic energy E and the photon's frequency ν. Einstein's formula relating inertial mass m_I to energy $E = m_I c^2$. The weak Equivalence Principle states $m_I = m_G$. For the work done by a gravitational field with potential Φ on a particle of gravitational mass m_G as it traverses a potential difference $d\Phi$ is $-m_G d\Phi$. This must equal DE, the gain in the particle's kinetic energy. For a photon, $dE = h d\nu$, and so

$$h d\nu = -m_G d\Phi = -m_I d\Phi = -\frac{E}{c^2} d\Phi = -\frac{h\nu}{c^2} d\Phi, \tag{73}$$

and thus

$$\frac{d\nu}{\nu} = -\frac{d\Phi}{c^2} \tag{74}$$

Integrating this equation over a finite path from A to B, one fines

$$\frac{\nu_A}{\nu_B} = e^{-(\Phi_B - \Phi_A)/c^2} = \frac{e^{-\Phi_B/c^2}}{e^{-\Phi_A/c^2}} \tag{75}$$

As for light bending in a gravitational field, imagine a ray of light as a stream of photons; since these photons have inertial and gravitational mass, we expect them to obey Galileo's principle and follow a curved path just like a Newtonian bullet traveling at velocity c. That wold make, for example, the downward curvature of a horizontal beam in the earth's gravitational field with x horizontal and z vertical equal to

$$\frac{d^2z}{dx^2} = \frac{d^2z}{c^2dt^2} = -\frac{g}{c^2}. \tag{76}$$

In units of years and light-yeas $c = 1$, and it so happens that $g \approx 1$.

1.8 Static and Stationary SpaceTimes

A stationary field is one that does not change in time and a static one is one where the sources do not move. The most important property of stationary spacetimes is that they admit a preferred time. The metric of every static field can be brougt to the canonical form

$$(cd\tau)^2 = ds^2 = e^{2\Phi/c^2}c^2dt^2 - d\vec{r}^2 = \eta^2(cdt)^2 - d\vec{r}^2 \tag{77}$$

where the last part uses our previous notation. We can calculate the elapsed proper time

$$d\tau^2 = dt^2\left(e^{2\Phi/c^2} - \beta^2\right) = dt^2\left(\eta^2 - \beta^2\right) \tag{78}$$

$$d\tau = dt\sqrt{e^{2\Phi/c^2} - \beta^2} = dt\sqrt{\eta^2 - \beta^2} = dt/\gamma^* \tag{79}$$

In the weak field limit$\Phi/c^2 << 1$, $e^{2\Phi/c^2} \simeq 1 + 2\Phi/c^2$.

$$ds^2 \simeq (1 + 2\Phi/c^2)c^2dt^2 - dl^2$$

For a particle-worldline between two events P_1 and P_2, we have

$$\int_{P_1}^{P_2} ds = \int_{t_1}^{t_2} \frac{ds}{dt}dt = c\int_{t_1}^{t_2}\left(1 + \frac{2\Phi}{c^2} - \frac{v^2}{c^2}\right)^{1/2} dt, \tag{80}$$

where $v = dl/dt$ is the coordinate velocity of the particle. The binomial approximation gives

$$\int_{P_1}^{P_2} ds = c\int_{t_1}^{t_2}\left(1 + \frac{2\Phi}{c^2} - \frac{v^2}{c^2}\right)^{1/2} dt = c\int_{t_1}^{t_2}\left(1 + \frac{\Phi}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right) dt = C(T_1 - T_2) - \frac{1}{c}\int_{t_1}^{t_2}\left(\frac{1}{2}v^2 - \Phi\right) dt. \tag{81}$$

The condition that $\int ds$ be maximal is therefore equivalent to the last integral being minmal. That is exacatly Hamilton's Principle.

One consequence which we can read off immediately is what is called the Shapiro time delay. A light-ray satisfies $ds^2 = 0$ and thus $e^{\Phi}cdt = \pm dl$, the two signs corresponding to the two possible directions of travel. Consequently a radar, or other light signal, reflected from a distant object will return to its emission point after a coordinate time

$$\Delta t = 2\int e^{-\Phi}dl \tag{82}$$

has elapsed there, where the integration is performed over the path of the signal.