APPENDIX C: THEORETICAL ANALYSIS OF GATED GAUSSIAN NOISE

This appendix provides theoretical analyses of gated Gaussian noise expressed mathematically as

$$u(t) = z(t)g(t;T,\tau_{on}) = \left(n(t) * h_{BPF}(t)\right)g(t;T,\tau_{on}) \quad ,$$

where $z(t)$ is a Gaussian noise random process with an ultra-wide transmission bandwidth (B_{UWB}) much greater than the victim receiver bandwidth (B_{DTV}), $n(t)$ is a zero-mean white Gaussian noise random process, $h_{BPF}(t)$ is the impulse response of a band-pass filter that defines B_{UWB} , and * is the convolutional operator. The ideal gating function is described mathematically as

$$g(t;T,\tau_{on}) = \sum_{k=-\infty}^{\infty} R(t-kT;\tau_{on}) \quad , \quad \text{where} \quad R(t;\tau_{on}) = \begin{cases} 1 & \left(S_1 : \quad 0 \le t < \tau_{on}\right) \\ 0 & \left(S_2 : \quad \tau_{on} \le t < T\right) \end{cases}$$

and S_1 and S_2 refer to gated-on and gated-off states, respectively. The complex-baseband (CBB) representation, denoted by ˆ, of $u(t)$ is modeled as

$$\hat{u}(t) = \hat{z}(t)g(t;T,\tau_{on}) \quad ,$$

where $\hat{z}(t)$ is a complex Gaussian-distributed random variable with zero-mean. The variance of $\hat{z}(t)$ is $\sigma_{\hat{z}}{}^2 = 2\sigma^2$, where σ^2 is the variance of the real and imaginary parts. The following amplitude probability distribution (*APD*) and power spectral density (*PSD*) expressions are derived in terms of $\hat{u}(t)$.

C.1. Amplitude Probability Distribution

Computation of the *APD* for gated Gaussian noise begins with the total probability expression

$$\wp(A \le a) = \wp(A \le a \mid S_1)\wp(S_1) + \wp(A \le a \mid S_2)\wp(S_2) \quad , \tag{C-1}$$

where $\wp$ represents probability, $A = \left|\hat{u}(t)\right|$ is the amplitude random variable which takes on values of a ranging from 0 to ∞ , $\wp(S_k)$ represents the probability of being in state S_k, and $\wp(A \le a \mid S_k)$ represents the conditional probability of $A \le a$ given the state S_k.

Amplitudes are Rayleigh distributed in state S_1 giving the probability density function (*PDF*)

$$q(a \mid S_1) = \frac{a}{\sigma^2}\exp\left(\frac{-a^2}{2\sigma^2}\right) \quad .$$

The corresponding cumulative distribution function (*CDF*) is

$$\wp(A \le a \mid S_1) = 1 - \exp\left(\frac{-a^2}{2\sigma^2}\right) \quad .$$

Similarly, the amplitude is always 0 in state S_2 giving the *PDF*

$$q(a \mid S_2) = \delta(a) \quad ,$$

where $\delta(a)$ is the delta function. The corresponding *CDF* is acquired by integrating the *PDF* over all amplitudes which yields the step function $U(a)$. Recall that $a \ge 0$, hence $U(a) = 1$ yielding

$$\wp(A \le a \mid S_2) = 1 \quad .$$

Probabilities of being in each state are expressed in terms of the fractional on-time (ζ)

$$\wp(S_1) = \zeta \quad ; \qquad \wp(S_2) = (1 - \zeta) \quad .$$

Substituting these expressions into the total probability expression (C-1) yields the *CDF*

$$\wp(A \le a) = 1 - \zeta \exp\left(\frac{-a^2}{2\sigma^2}\right) \quad .$$

The corresponding *APD* is obtained by subtracting the *CDF* from 1

$$\wp(A > a) = \zeta \exp\left(\frac{-a^2}{2\sigma^2}\right) \quad . \qquad \text{(C-2)}$$

Figure C-1 provides *APD*s of normalized ($\sigma^2 = 1$) gated Gaussian noise with $\zeta = \{1, 0.5, 0.25, 0.125, 0.0625\}$ in an ultra-wide transmission bandwidth.

Peak-to-average ratio (*P/A*) is computed from the peak and root-mean-square (rms) amplitudes. Peak amplitude (a_{pk}) is defined by the percentile ρ. Replacing $\wp(A > a)$ with ρ and a with a_{pk} in equation (C-2) and solving for a_{pk} yields

$$a_{pk} = \sigma\sqrt{\ln(\zeta/\rho)} \quad .$$

This expression is only valid when $\zeta \ge \rho$.

Rms amplitude is found by substituting the probability densities given previously into the derivative of (C-1) with respect to a to obtain

$$q(a) = q(a \mid S_1)\wp(S_1) + q(a \mid S_2)\wp(S_2) = \zeta \frac{a}{\sigma^2} \exp\left(\frac{-a^2}{2\sigma^2}\right) + (1-\zeta)\delta(a) \quad ,$$

and using this *PDF* to calculate rms amplitude

$$a_{rms} = \sqrt{\int_0^\infty a^2 q(a) da} = \sigma\sqrt{\zeta} \quad .$$

Finally, the expression for *P/A* is

$$P / A = \left(\frac{a_{pk}}{a_{rms}}\right)^2 = \frac{\ln(\zeta/\rho)}{\zeta} \quad .$$

Figure C-2 provides *P/A* as a function of ζ for gated noise in the ultra-wide transmission bandwidth; peak definitions considered in Figure C-2 are ρ = {0.001, 0.01, 0.1, 1.0} %.

C.2. Power Spectral Density

Derivation of the *PSD* for gated Gaussian noise begins with the autocorrelation function

$$R_{\hat{u}\hat{u}}(t, t+\tau) = E\{\hat{u}(t)\hat{u}^*(t+\tau)\} \quad ,$$

where $E\{\ \}$ denotes expectation and * denotes conjugation. Pulling the deterministic gating function out of the expectation gives

$$R_{\hat{u}\hat{u}}(t, t+\tau) = g(t) g^*(t+\tau) R_{\hat{z}\hat{z}}(\tau) \quad .$$

Expressing the gating function as a Fourier series

$$g(t) = \sum_{k=-\infty}^{\infty} c_k \exp(j2\pi kt/T) \quad , \qquad c_k = \frac{1}{T}\int_0^T g(t) \exp(-j2\pi kt/T) dt$$

yields

$$R_{\hat{u}\hat{u}}(t, t+\tau) = \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_l c_k^* \exp(j2\pi(l-k)t/T) \exp(-j2\pi k\tau/T) R_{\hat{z}\hat{z}}(\tau) \quad .$$

The first complex-exponential term causes the autocorrelation to be time-varying. This variation is periodic, in that

$$R_{\hat{u}\hat{u}}(t,t+\tau) = R_{\hat{u}\hat{u}}(t+T,t+\tau+T) \quad ,$$

and is removed by averaging over the gating period. The resulting time-averaged autocorrelation function is

$$\langle R_{\hat{u}\hat{u}}(\tau)\rangle = \frac{1}{T}\int_0^T R_{\hat{u}\hat{u}}(t,t+\tau)dt = \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_l c_k^* \exp\left(-j2\pi k\tau/T\right) R_{\hat{z}\hat{z}}(\tau)\frac{1}{T}\int_0^T \exp\left(j2\pi(l-k)t/T\right)dt \quad ,$$

which evaluates to

$$\langle R_{\hat{u}\hat{u}}(\tau)\rangle = \begin{cases} \sum_{k=-\infty}^{\infty}\left|c_k\right|^2 \exp\left(-j2\pi k\tau/T\right) R_{\hat{z}\hat{z}}(\tau) & (l=k) \\ 0 & (l \neq k) \end{cases} \quad .$$

The *PSD* is derived from the Fourier transform of the time-averaged autocorrelation function

$$\langle S_{\hat{u}\hat{u}}(f)\rangle = \int_{-\infty}^{\infty}\langle R_{\hat{u}\hat{u}}(\tau)\rangle \exp\left(-j2\pi f\tau\right)d\tau \quad .$$

Substitution yields the following expression for the *PSD* of gated Gaussian noise

$$\langle S_{\hat{u}\hat{u}}(f)\rangle = \sum_{k=-\infty}^{\infty}\left|c_k\right|^2 \delta(f+\frac{k}{T}) * S_{\hat{z}\hat{z}}(f) \quad .$$

Note that $S_{nn}(f)$ is constant, yielding

$$S_{\hat{z}\hat{z}}(f) = n_0\left|H_{LPF}(f)\right|^2 \quad ,$$

where n_0 is the power density of the underlying white Gaussian process and $H_{LPF}(f)$ is the CBB representation of $H_{BPF}(f)$. This simplifies the *PSD* expression

$$\langle S_{\hat{u}\hat{u}}(f)\rangle = n_0\sum_{k=-\infty}^{\infty}\left|c_k\right|^2 \delta(f+\frac{k}{T}) * \left|H_{LPF}(f)\right|^2 \quad .$$

In this experiment we assume the transmission filter, $H_{LPF}(f)$, is constant over the bandwidth of the victim receiver. Hence, the *PSD* of gated noise is constant for all frequencies considered.

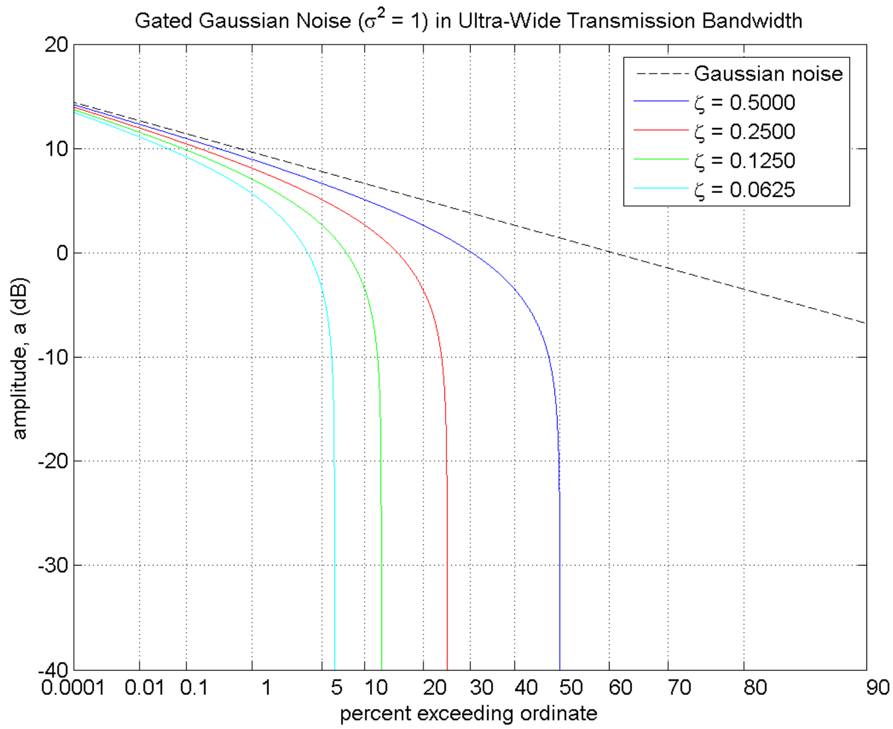

Figure C-1. Theoretical *APD*s of gated noise in B_{UWB}.

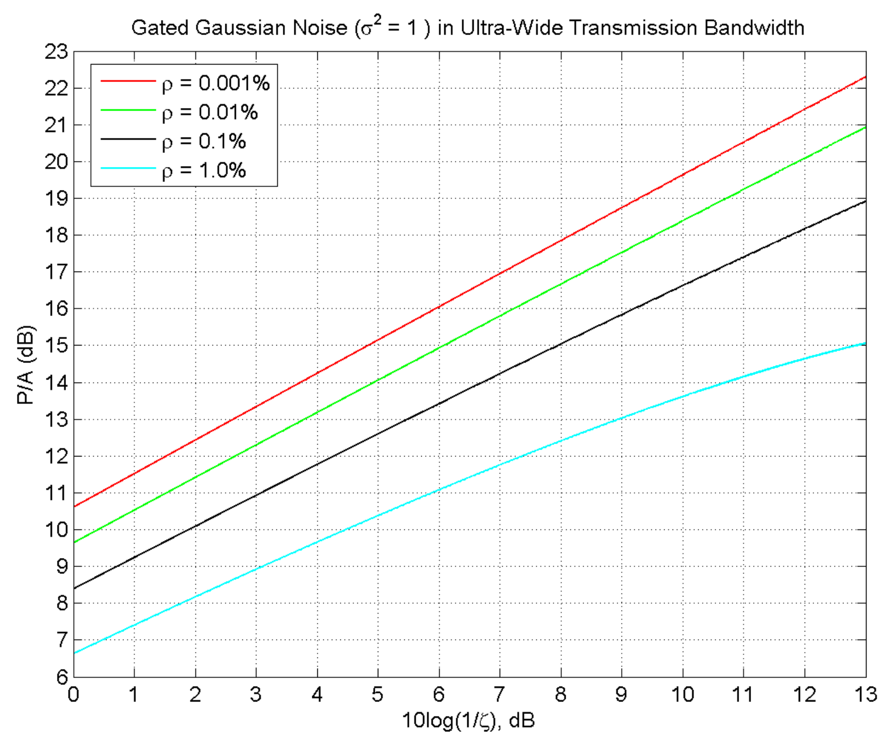

Figure C-2. Theoretical *P/A* of gated noise in B_{UWB}.