SECTION 4

SPACECRAFT EPHEMERIS AND PARTIALS FILE

Contents

4.1 INTRODUCTION

This section gives the equations for the acceleration of the spacecraft relative to the center of integration due to gravity only. These equations include Newtonian and relativistic acceleration terms. The complete formulation for program PV, which generates the spacecraft ephemeris and the corresponding partials file, will eventually be documented by Richard F. Sunseri, the programmer/analyst for program PV. The relativistic equations of motion are given in this section so that this document will contain all of the relativistic equations used in calculating the computed values of observed quantities.

The relativistic equations of motion are given for the Solar-System barycentric frame of reference and also for the local geocentric frame of reference. In deriving these equations, transformations of coordinates between these two relativistic space-time frames of reference are developed. These relativistic transformations are also used in program Regres.

Section 4.2 gives a general description of the spacecraft ephemeris and the corresponding partials file, which are used in program Regres. Section 4.3 develops transformations between the coordinates of the local geocentric frame of reference and the Solar-System barycentric frame of reference. The relativistic equations of motion for the Solar-System barycentric frame of reference, which apply for a spacecraft anywhere in the Solar System, are given in Section 4.4. Section 4.5 gives the corresponding equations for the local geocentric frame of reference. These equations apply for a spacecraft near the Earth, such as an Earth orbiter.

The gravitational equations presented are not complete. The changes in the Earth's harmonic coefficients due to solid-Earth tides and ocean tides are not included. The equations of motion presented in this section use the body-fixed to space-fixed rotation matrices for the various celestial bodies of the Solar System. The rotation matrix used for the Earth is given in Section 5.3. The matrix used for all of the other bodies of the Solar System is given in Section 6.3.

4.2 GENERAL DESCRIPTION OF PROGRAM PV

The spacecraft acceleration relative to the center of integration (COI) is integrated numerically to produce the spacecraft ephemeris. This ephemeris can be represented in the Solar-System barycentric frame of reference for a spacecraft anywhere in the Solar System or in the local geocentric frame of reference for a spacecraft near the Earth. Interpolation of the spacecraft ephemeris for either of these two space-time frames of reference gives the space-fixed position, velocity, and acceleration vectors of the spacecraft relative to the center of integration in km, km/s, and km/s^2. The ephemeris is interpolated at the ET value of the interpolation epoch (coordinate time in the Solar-System barycentric or local geocentric frame of reference). The space-fixed reference frame for the spacecraft ephemeris is the reference frame of the planetary ephemeris used to generate the spacecraft ephemeris.

The COI for the spacecraft ephemeris can be the center of mass of the Sun (S), Mercury (Me), Venus (V), Earth (E), the Moon (M), an asteroid or a comet, the center of mass of the planetary systems Mars (Ma), Jupiter (J), Saturn (Sa), Uranus (U), Neptune (N), or Pluto (Pl), or the planet or a satellite of any of these outer planet systems. The current COI is determined by the spheres of influence centered on each of these points (except the Sun). If the spacecraft is within the sphere of influence of a body or planetary system, the COI is the center of mass of that body or planetary system. Otherwise, the COI is the Sun. The radii of the spheres of influence are parameters on the GIN file, and hence can be varied by the ODP user. Note that the sphere of influence for the Moon is contained within the sphere of influence for the Earth.

The variational equations calculate the partial derivatives of the spacecraft acceleration vector with respect to the parameter vector $\mathbf{q}$ (consisting of solve-for and consider parameters). These partial derivatives are numerically integrated to produce the spacecraft partials file. Interpolation of the spacecraft partials file with an ET epoch produces the partial derivatives of the position, velocity, and acceleration vectors of the spacecraft relative to the COI with respect to $\mathbf{q}$.

4.3 TRANSFORMATIONS BETWEEN COORDINATES OF THE LOCAL GEOCENTRIC FRAME OF REFERENCE AND THE SOLAR-SYSTEM BARYCENTRIC FRAME OF REFERENCE

Section 4.3.1 gives the equation for transforming Earth-centered space-fixed position coordinates of an Earth-fixed tracking station or a near-Earth spacecraft from the local geocentric to the Solar-System barycentric space-time frame of reference. Section 4.3.2 gives the equation relating the differential of coordinate time in the local geocentric frame of reference to the differential of coordinate time in the Solar-System barycentric frame of reference. Section 4.3.3 shows how the expression for coordinate time in the Solar-System barycentric frame of reference minus coordinate time in the local geocentric frame of reference can be obtained from equations in Section 2. Section 4.3.4 gives the equation relating the values of the gravitational constant μ of a celestial body in the local geocentric and Solar-System barycentric frames of reference.

4.3.1 POSITION COORDINATES

4.3.1.1 Derivation of Transformation

The Lorentz transformation given by Eqs. (7a) and (7b) of Hellings (1986) transforms space and time coordinates of the Solar-System barycentric space-time frame of reference to space and time coordinates of the local geocentric frame of reference. The barycentric coordinates are those of a flat space-time which is tangent to the curved space-time of the barycentric frame at the location of the Earth. The geocentric coordinates are those of a flat space-time which is tangent to the curved space-time of the local geocentric frame a large distance from the Earth. Let the space and time coordinates in these two flat space-time frames of reference be denoted by:

$\mathbf{r}'_{\mathrm{BC}}$, $\mathbf{r}'_{\mathrm{GC}}$ = space-fixed geocentric position vectors of tracking station or near-Earth spacecraft expressed in the space coordinates of the flat Solar-System barycentric (BC) and geocentric (GC) frames of reference, respectively.

t'_{BC}, t'_{GC} = coordinate times in the flat Solar-System barycentric and geocentric frames of reference, respectively.

Also, let

$\mathbf{V}_E$ = space-fixed velocity vector of Earth relative to Solar-System barycenter.

V_E = magnitude of $\mathbf{V}_E$

The Lorentz transformation given by Eqs. (7a) and (7b) of Hellings (1986) is:

$$dt'_{GC} = \left(1 + \frac{{V_E}^2}{2c^2}\right)\left(dt'_{BC} - \frac{1}{c^2}\mathbf{V}_E \cdot \mathbf{r}'_{BC}\right) \tag{4–1}$$

$$\mathbf{r}'_{GC} = \mathbf{r}'_{BC} + \frac{1}{2c^2}\left(\mathbf{V}_E \cdot \mathbf{r}'_{BC}\right)\mathbf{V}_E - \left(1 + \frac{{V_E}^2}{2c^2}\right)\mathbf{V}_E \, dt'_{BC} \tag{4–2}$$

which contains terms to order $1/c^2$. Note that if $\mathbf{V}_E$ is along the x axis, these equations reduce to the usual text-book Lorentz transformation to order $1/c^2$.

The metric (the expression for the square of the interval *ds*) in the Solar-System barycentric space-time frame of reference is given by Eqs. (2–16) to (2–18), where the constant L in the scale factor l is defined by Eq. (2–22) evaluated at mean sea level on Earth. The barycentric coordinates t'_{BC} and $\mathbf{r}'_{BC}$, which are flat (Minkowskian) in a local region near Earth, are related to the global coordinates of the barycentric metric (2–16) by what Hellings (1986) refers to as an infinitesimal transformation:

$$dt'_{BC} = \left(1 + L - \frac{U_E}{c^2}\right) dt_{BC} \tag{4–3}$$

$$\mathbf{r}'_{BC} = \left(1 + L + \frac{\gamma U_E}{c^2}\right) \mathbf{r}_{BC} \tag{4–4}$$

where U_{E} is the gravitational potential U given by Eq. (2–17) at the Earth due to all other bodies.

The metric in the local geocentric space-time frame of reference is also given by Eqs. (2–16) to (2–18) and (2–22). However, the gravitational potential U in (2–16) and (2–22) only contains the term of (2–17) due to the Earth. The velocity v in (2–22) changes from the Solar-System barycentric velocity to the geocentric velocity. The constant L_{GC} in the scale factor l_{GC} (where GC refers to the value in the local geocentric frame of reference) is obtained by evaluating (2–22) at mean sea level on Earth. The transformation from the coordinates t'_{GC} and $\mathbf{r}'_{\mathrm{GC}}$ of the flat space-time (which is tangent to the curved space-time of the local geocentric frame a large distance from the Earth) to the coordinates of the local geocentric metric is obtained from the geocentric metric with the gravitational potential U due to the Earth deleted:

$$dt'_{\mathrm{GC}} = \left(1 + L_{\mathrm{GC}}\right) dt_{\mathrm{GC}} \tag{4–5}$$

$$\mathbf{r}'_{\mathrm{GC}} = \left(1 + L_{\mathrm{GC}}\right) \mathbf{r}_{\mathrm{GC}} \tag{4–6}$$

Let the constant L in the barycentric frame minus the constant L_{GC} in the local geocentric frame be denoted as $\tilde{L}$:

$$\tilde{L} = L - L_{\mathrm{GC}} \tag{4–7}$$

Substituting Eqs. (4–3) and (4–4) into the right-hand side of Eqs. (4–1) and (4–2) and substituting (4–5) and (4–6) into the left-hand side and using Eq. (4–7) gives the modified Lorentz transformation which transforms the space and time coordinates of the Solar-System barycentric metric to those of the local geocentric metric:

$$dt_{\mathrm{GC}} = \left(1 + \frac{V_{\mathrm{E}}{}^2}{2c^2}\right)\left[\left(1 + \tilde{L} - \frac{U_{\mathrm{E}}}{c^2}\right)dt_{\mathrm{BC}} - \frac{1}{c^2}\left(1 + \tilde{L} + \frac{\gamma U_{\mathrm{E}}}{c^2}\right)\mathbf{V}_{\mathrm{E}} \cdot \mathbf{r}_{\mathrm{BC}}\right] \tag{4–8}$$

$$\begin{aligned}\mathbf{r}_{\mathrm{GC}} &= \left(1+\tilde{L}+\frac{\gamma U_{\mathrm{E}}}{c^2}\right)\mathbf{r}_{\mathrm{BC}} + \frac{1}{2c^2}\left(1+\tilde{L}+\frac{\gamma U_{\mathrm{E}}}{c^2}\right)\left(\mathbf{V}_{\mathrm{E}}\cdot\mathbf{r}_{\mathrm{BC}}\right)\mathbf{V}_{\mathrm{E}} \\ &\quad -\left(1+\tilde{L}-\frac{U_{\mathrm{E}}}{c^2}\right)\left(1+\frac{{V_{\mathrm{E}}}^2}{2c^2}\right)\mathbf{V}_{\mathrm{E}}\,dt_{\mathrm{BC}}\end{aligned} \tag{4–9}$$

For the next step, we need an expression relating the geocentric space-fixed position vectors of an Earth-fixed tracking station or a near-Earth spacecraft in the local geocentric and Solar-System barycentric space-time frames of reference. Furthermore, the two ends of the position vector in the barycentric frame should be observed simultaneously in coordinate time in that frame. The desired relation is obtained from (4–9) by setting dt_{BC} = 0 and solving for $\mathbf{r}_{\mathrm{BC}}$. Retaining terms to order $1/c^2$ gives:

$$\mathbf{r}_{\mathrm{BC}} = \left(1-\tilde{L}-\frac{\gamma U_{\mathrm{E}}}{c^2}\right)\mathbf{r}_{\mathrm{GC}} - \frac{1}{2c^2}\left(\mathbf{V}_{\mathrm{E}}\cdot\mathbf{r}_{\mathrm{GC}}\right)\mathbf{V}_{\mathrm{E}} \tag{4–10}$$

The inverse transformation, which applies for the condition that $dt_{\mathrm{BC}} = 0$, is:

$$\mathbf{r}_{\mathrm{GC}} = \left(1+\tilde{L}+\frac{\gamma U_{\mathrm{E}}}{c^2}\right)\mathbf{r}_{\mathrm{BC}} + \frac{1}{2c^2}\left(\mathbf{V}_{\mathrm{E}}\cdot\mathbf{r}_{\mathrm{BC}}\right)\mathbf{V}_{\mathrm{E}} \tag{4–11}$$

Section 4.3.1.2 will develop expressions for L, L_{GC}, and their difference $\tilde{L}$ (see Eq. 4–7) and obtain numerical values for these three constants. The gravitational potential U_{E} at the Earth can be calculated from Eq. (2–17) where i = E (Earth). The position vectors of the major bodies of the Solar System relative to the Earth are obtained by interpolating the planetary ephemeris as described in Section 3.1.2.1. The magnitudes of these vectors equal $r_{ij} = r_{\mathrm{E}j}$ in the denominator of Eq. (2–17). The gravitational constants μ_j of the major bodies of the Solar System in the numerator of Eq. (2–17) are obtained from the planetary ephemeris as described in Section 3.1.2.2. When the planetary ephemeris is interpolated, the velocity vector $\mathbf{V}_{\mathrm{E}}$ of the Earth relative to the Solar-System barycenter is also obtained as described in Section 3.1.2.1.

The derivation of Eqs. (4–10) and (4–11) is a minor variation of a similar derivation in Hellings (1986). The changes to the derivation were suggested by R. W. Hellings. Eq. (4–10) is the same as Eq. (46) of Huang, Ries, Tapley, and Watkins (1990), which will be referred to as HRTW (1990), if the two terms of (46) containing the acceleration of the Earth are ignored. Ignoring these two terms in (4–10) and (4–11) produces an error in the transformed space-fixed position vector of an Earth-fixed tracking station of less than 0.01 mm.

Tracking station coordinates and position coordinates of near-Earth spacecraft ephemerides integrated in the local geocentric frame of reference are expressed in the space coordinates of the local geocentric space-time frame of reference. Eq. (4–10) and will be used to transform the geocentric space-fixed position vector of an Earth-fixed tracking station from the local geocentric space-time frame of reference in which it is computed (Section 5) to the Solar-System barycentric space-time frame of reference. The transformed position vector will be used in the Solar-System barycentric light-time solution (Section 8). Eqs. (4–10) and (4–11) will be used in Section 4.4.5 to calculate the acceleration of a near-Earth spacecraft due to the Earth's harmonic coefficients in the Solar-System barycentric frame of reference.

In transforming the geocentric space-fixed position vector of a fixed tracking station on Earth from the local geocentric frame of reference to the Solar-System barycentric frame of reference using Eq. (4–10), the first term of this equation reduces the geocentric radius of the tracking station by about 16 cm. This term accounts for the different scale factors used in the two frames of reference and the effect of the gravitational potential on measured space coordinates near the Earth in the barycentric frame. The second term of Eq. (4–10) reduces the component of the station position vector along the Earth's velocity vector by up to 3 cm. The diameter of the Earth in the direction of the Earth's velocity is reduced by about 6 cm as viewed in the Solar-System barycentric space-time frame of reference. This effect is due to the different definitions of simultaneity in the two frames of reference, which have a relative velocity of about 30 km/s. The second term of Eq. (4–10) is the Lorentz contraction.

4.3.1.2 Expressions for Scale Factors

The metric (Eq. 2–16) contains the scale factor l given by Eq. (2–18). The constant L in (2–18) is the departure of l from unity. The constant L is defined by Eq. (2–22). The values of L that apply in the Solar-System barycentric frame (L) and the local geocentric frame (L_{GC}) are evaluated from Eq. (2–22) as described in Sections 2.3.1.2 and 2.3.1.3, respectively. From Eq. (4–7), the constant $\tilde{L}$ is L minus L_{GC}. This section will give equations and numerical values for L, L_{GC}, and $\tilde{L}$.

To sufficient accuracy, the numerical value of the constant L, which applies in the Solar-System barycentric space-time frame of reference, can be calculated from:

$$L = \frac{1}{c^2}\left[\frac{1}{AU}\left(\frac{\mu_S + \mu_{Me} + \mu_V}{a_B} + \frac{\mu_{Ma}}{a_{Ma}} + \frac{\mu_J}{a_J} + \frac{\mu_{Sa}}{a_{Sa}} + \frac{\mu_U}{a_U} + \frac{\mu_N}{a_N} + \frac{\mu_{Pl}}{a_{Pl}}\right) + \frac{\mu_M}{a_M} + \frac{\mu_E}{a_e}\left(1 + \frac{J_2}{2}\right) + \frac{\mu_S + \mu_E + \mu_M}{2AU\,a_B} + \frac{1}{2}a_e{}^2\omega_E{}^2\right] \tag{4–12}$$

The number of significant digits given for the parameters in Eq. (4–12) is sufficient to calculate L to seven significant digits. The gravitational constants μ for the Sun (S), Mercury (Me), Venus (V), and the planetary systems Mars (Ma), Jupiter (J), Saturn (Sa), Uranus (U), Neptune (N), and Pluto (Pl), the planetary ephemeris scaling factor AU (which is the number of kilometers per astronomical unit), and the speed of light c were obtained from Standish *et al.* (1995):

$$\begin{aligned}
\mu_S &= 132{,}712{,}440{,}018.\ \text{km}^3/\text{s}^2\\
\mu_{Me} &= 22{,}032.\ \text{km}^3/\text{s}^2\\
\mu_V &= 324{,}859.\ \text{km}^3/\text{s}^2\\
\mu_{Ma} &= 42{,}828.\ \text{km}^3/\text{s}^2\\
\mu_J &= 126{,}712{,}768.\ \text{km}^3/\text{s}^2\\
\mu_{Sa} &= 37{,}940{,}626.\ \text{km}^3/\text{s}^2\\
\mu_U &= 5{,}794{,}549.\ \text{km}^3/\text{s}^2
\end{aligned}$$

$$\mu_{\mathrm{N}} = 6{,}836{,}534.\ \mathrm{km^3/s^2}$$
$$\mu_{\mathrm{Pl}} = 982.\ \mathrm{km^3/s^2}$$
$$AU = 149{,}597{,}870.691\ \mathrm{km/astronomical\ unit}$$
$$c = 299{,}792.458\ \mathrm{km/s}$$

The semi-major axes (a) in astronomical units of the heliocentric orbits of the Earth-Moon barycenter (B) and the planetary systems Mars through Pluto were obtained from Table 5.8.1 on page 316 of the *Explanatory Supplement* (1992). To sufficient accuracy, the values at the epoch J2000.0 can be used:

$$a_{\mathrm{B}} = 1.000{,}000{,}11$$
$$a_{\mathrm{Ma}} = 1.523{,}66$$
$$a_{\mathrm{J}} = 5.203{,}36$$
$$a_{\mathrm{Sa}} = 9.537$$
$$a_{\mathrm{U}} = 19.191$$
$$a_{\mathrm{N}} = 30.069$$
$$a_{\mathrm{Pl}} = 39.482$$

From Standish *et al.* (1995) or Chapter 1 of International Earth Rotation Service (1992), the gravitational constant for the Moon is given by:

$$\mu_{\mathrm{M}} = 4902.8\ \mathrm{km^3/s^2}$$

From Table 15.4 on page 701 of the *Explanatory Supplement* (1992), the semi-major axis of the geocentric orbit of the Moon in kilometers is given to sufficient accuracy by:

$$a_{\mathrm{M}} = 3.844 \times 10^5\ \mathrm{km}$$

From Chapter 1 or Chapter 6 of International Earth Rotation Service (1992), or from Standish *et al.* (1995), values of the gravitational constant of the Earth (μ_{E}), the mean equatorial radius of the Earth (a_{e}), and the second zonal harmonic coefficient of the Earth (J_2), rounded to more than enough significant digits to calculate L to seven significant digits are given by:

$$\mu_E = 398{,}600.44 \text{ km}^3/\text{s}^2$$
$$a_e = 6378.136 \text{ km}$$
$$J_2 = 1.082{,}63 \times 10^{-3}$$

It will be seen in Section 4.3.4 that the gravitational constant of the Earth has slightly different values in the Solar-System barycentric and geocentric frames of reference. However, the difference of about 0.006 km^3/s^2 is not significant here. From Table 15.4 on page 701 of the *Explanatory Supplement* (1992), the inertial rotation rate of the Earth (ω_E) is given by:

$$\omega_E = 0.729{,}2115 \times 10^{-4} \text{ rad/s}$$

Substituting numerical values into Eq. (4–12) and rounding the resulting value of L to seven significant digits gives:

$$L = 1.550{,}520 \times 10^{-8} \tag{4–13}$$

Secular variations in the semi-major axes of the orbits of the planets prevent the calculation of L from Eq. (4–12) to more than seven significant digits.

Of the five terms of Eq. (4–12), only the third and fifth terms apply for L_{GC} in the local geocentric space-time frame of reference:

$$L_{GC} = \frac{1}{c^2}\left[\frac{\mu_E}{a_e}\left(1 + \frac{J_2}{2}\right) + \frac{1}{2} a_e{}^2 \omega_E{}^2\right] \tag{4–14}$$

Substituting numerical values into Eq. (4–14) and rounding the resulting value of L_{GC} to 1×10^{-14} (as in 4–13) gives:

$$L_{GC} = 0.069{,}693 \times 10^{-8} \tag{4–15}$$

From Eq. (4–7), the expression for $\tilde{L}$ is given by Eq. (4–12) minus Eq. (4–14), which is given by terms one, two, and four of (4–12):

$$\tilde{L} = \frac{1}{c^2}\left[\frac{1}{AU}\left(\frac{\mu_S + \mu_{Me} + \mu_V}{a_B} + \frac{\mu_{Ma}}{a_{Ma}} + \frac{\mu_J}{a_J} + \frac{\mu_{Sa}}{a_{Sa}} + \frac{\mu_U}{a_U} + \frac{\mu_N}{a_N} + \frac{\mu_{Pl}}{a_{Pl}}\right) + \frac{\mu_M}{a_M} + \frac{\mu_S + \mu_E + \mu_M}{2AU\, a_B}\right] \tag{4–16}$$

Substituting numerical values into Eq. (4–16) and rounding the resulting value of $\tilde{L}$ to seven significant digits gives:

$$\tilde{L} = 1.480{,}827 \text{ x } 10^{-8} \tag{4–17}$$

The same value is obtained by subtracting Eq. (4–15) from Eq. (4–13), according to Eq. (4–7).

Fukushima (1995) has obtained numerical values for L, L_{GC}, and $\tilde{L}$, which he denotes as L_B, L_G, and L_C, respectively, by numerical integration. His values of these constants (given in his equations (41), (40), and (38)) contain three to four more significant digits than given here and round to the values given in Eqs. (4–13), (4–15), and (4–17).

The numerical values of L and L_{GC} will be used in Sections 11 and 13 to calculate the computed values of one-way doppler (F_1) observables in the Solar-System barycentric and local geocentric frames of reference, respectively. The numerical value of $\tilde{L}$ is used in Eqs. (4–10) and (4–11) and throughout Section 4.3.

4.3.2 DIFFERENTIAL EQUATION FOR TIME COORDINATES

In order to calculate the acceleration of a near-Earth spacecraft due to the Earth's harmonic coefficients in the Solar-System barycentric frame of reference (in Section 4.4.5), an expression is required for dt_{GC}/dt_{BC} evaluated at the spacecraft. An interval of proper time $d\tau$ recorded on an atomic clock carried by the spacecraft divided by the corresponding interval of coordinate time dt_{BC} in the Solar-System barycentric frame of reference is given by Eq. (2–20):

$$\frac{d\tau}{dt_{BC}} = 1 - \frac{U}{c^2} - \frac{v^2}{2c^2} + L \tag{4–18}$$

where U is the gravitational potential at the spacecraft given by Eq. (2–17), v is the Solar-System barycentric velocity of the spacecraft given by Eq. (2–21), and L is given by Eq. (4–13). The interval $d\tau$ divided by the corresponding interval of coordinate time dt_{GC} in the local geocentric frame of reference is given by:

$$\frac{d\tau}{dt_{GC}} = 1 - \frac{U(E)}{c^2} - \frac{{v_{GC}}^2}{2c^2} + L_{GC} \tag{4–19}$$

where U(E) is the gravitational potential at the spacecraft due to the Earth, v_{GC} is the geocentric velocity of the spacecraft, and L_{GC} is given by Eq. (4–15). If the spacecraft atomic clock were placed at mean sea level on Earth, it would run at the same rate as International Atomic Time TAI. The TAI rate is the same as the rate of coordinate time in the local geocentric frame of reference. The average rate of TAI is the same as the rate of coordinate time in the Solar-System barycentric frame of reference.

If the gradient of the gravitational potential U_E at the Earth due to all other bodies is ignored, the gravitational potential U at a near-Earth spacecraft can be approximated by:

$$U \approx U_E + U(E) \tag{4–20}$$

Also, v^2 in Eq. (4–18) can be expressed as:

$$v^2 = {V_E}^2 + 2\mathbf{V}_E \cdot \dot{\mathbf{r}} + {v_{GC}}^2 \tag{4–21}$$

where $\dot{\mathbf{r}}$ is the geocentric space-fixed velocity vector of the near-Earth spacecraft. Dividing Eq. (4–18) by Eq. (4–19), substituting Eqs. (4–20), (4–21), and (4–7), and retaining terms to order $1/c^2$ gives:

$$\frac{dt_{GC}}{dt_{BC}} = 1 - \frac{U_E}{c^2} - \frac{{V_E}^2}{2c^2} + \tilde{L} - \frac{\mathbf{V}_E \cdot \dot{\mathbf{r}}}{c^2} \tag{4–22}$$

Since terms of order $1/c^4$ are ignored, $\dot{\mathbf{r}}$ can be evaluated in the local geocentric frame of reference or in the Solar-System barycentric frame of reference. The inverse of Eq. (4–22) is Eq. (47) of HRTW (1990), except that I have ignored the term

$$-\frac{\dot{\mathbf{V}}_{\mathrm{E}} \cdot \mathbf{r}}{c^2}$$

in Eq. (4–22) which arises from the gradient of U_{E}.

Eq. (4–22) gives the rate of dt_{GC} relative to dt_{BC} at a point in the local geocentric frame of reference that has a geocentric space-fixed velocity vector of $\dot{\mathbf{r}}$. The first four terms on the right-hand side of Eq. (4–22) represent periodic variations in the rate of geocentric coordinate time with variations in the gravitational potential at the Earth and the Solar-System barycentric velocity of the Earth. The last term on the right-hand side of (4–22) plus the neglected term is the negative of the time derivative of the clock synchronization term in the expression for ET − TAI at an Earth satellite. This is the fourth term on the right-hand side of Eq. (2–23), which is used in Eq. (2–25).

4.3.3 TIME COORDINATES

Eq. (4–22), plus the neglected term listed after it, can be integrated to give an expression for coordinate time t_{BC} in the barycentric frame of reference minus coordinate time t_{GC} in the local geocentric frame of reference. However, this derivation is the same as that for Eq. (2–23) for ET − TAI at a tracking station on Earth. In this equation, ET is coordinate time in the Solar-System barycentric frame of reference and TAI is International Atomic Time obtained from an atomic clock at the tracking station. From Eq. (2–30), TAI plus 32.184 s is coordinate time in the local geocentric frame of reference. Hence, the desired expression for t_{BC} minus t_{GC} is the right-hand side of Eq. (2–23) with the constant 32.184 s deleted. In this expression, $\mathbf{r}_{\mathrm{A}}^{\mathrm{E}}$ is the geocentric space-fixed position vector of the point A where the time difference $t_{\mathrm{BC}} - t_{\mathrm{GC}}$ is desired. The term containing $\mathbf{r}_A^E$ is the time synchronization term, which comes from the Lorentz transformation, and the remaining terms are periodic.

4.3.4 GRAVITATIONAL CONSTANTS

The gravitational constant of a body (defined after Eq. 2–6) has units of km^3/s^2. The "physical" or "measured" or "proper" gravitational constant of a body is measured in the scaled space and time coordinates of the underlying metric. Eq. (2–16) for the metric in the Solar-System barycentric or local geocentric frame of reference shows the space and time coordinates multiplied by the scale factor l given by Eq. (2–18). The equations of motion for bodies and light are independent of the scale factor l. The gravitational constants used in the equations of motion are expressed in the unscaled space and time coordinates of the underlying metric. Since the physical gravitational constant contains three scaled coordinates in the numerator and two scaled coordinates in the denominator, it is equal to the unscaled gravitational constant used in the equations of motion multiplied by the scale factor l. The unscaled gravitational constants μ_{BC} and μ_{GC} of a body used in the equations of motion in the Solar-System barycentric and local geocentric frames of reference, respectively, are given by the following functions of the common physical gravitational constant of the body:

$$\mu_{BC} = \frac{\mu_{physical}}{1+L} \tag{4–23}$$

$$\mu_{GC} = \frac{\mu_{physical}}{1+L_{GC}} \tag{4–24}$$

The gravitational constants μ_{BC} of the celestial bodies of the Solar System are estimated in fitting the planetary ephemeris to the observations of the Solar-System bodies. The corresponding gravitational constants in the local geocentric frame of reference are obtained from Eqs. (4–23) and (4–24) by eliminating $\mu_{physical}$, using Eq. (4–7), and retaining terms to order $1/c^2$:

$$\mu_{\mathrm{GC}} = \left(1 + \tilde{L}\right) \mu_{\mathrm{BC}} \tag{4–25}$$

where $\tilde{L}$ is given by Eq. (4–17). In practice, the only gravitational constant whose value must be transformed from its value in the barycentric frame to its value in the local geocentric frame is the gravitational constant of the Earth.

Eq. (4–23) is the same as Eq. (5) or (15) of Misner (1982) and the same to order $1/c^2$ as Eqs. (21), (23), and (25) of Hellings (1986). Eq. (4–25) is the same to order $1/c^2$ as Eq. (62) of HRTW (1990).

The gravitational constants μ_{BC} of the bodies of the Solar System obtained from the planetary ephemeris and from satellite ephemerides are described in Sections 3.1.2.2 and 3.2.2.1.

4.4 RELATIVISTIC EQUATIONS OF MOTION IN SOLAR-SYSTEM BARYCENTRIC FRAME OF REFERENCE

This section specifies the equations for calculating the acceleration of a spacecraft located anywhere in the Solar System relative to the center of integration (see Section 4.2) due to gravity only. This acceleration is calculated in the Solar-System barycentric space-time frame of reference as the acceleration of the spacecraft relative to the Solar-System barycenter minus the acceleration of the center of integration relative to the Solar-System barycenter. Section 4.4.1 specifies the point-mass Newtonian acceleration plus the relativistic perturbative acceleration due to a body. These equations are used to calculate the acceleration of the spacecraft and the acceleration of the center of integration due to the celestial bodies of the Solar System. The acceleration of a near-Earth spacecraft is affected by geodesic precession, as discussed in Section 4.4.2. The acceleration due to geodesic precession is included in the relativistic point-mass perturbative acceleration specified in Section 4.4.1. The Lense-Thirring relativistic acceleration of a near-Earth spacecraft due to the rotation of the Earth is given in Section 4.4.3. The standard model for calculating the acceleration of a spacecraft due to the harmonic coefficients of a nearby celestial body is discussed in Section 4.4.4. This model uses the formulation in Moyer (1971) and calculates the Newtonian acceleration due to the oblateness of a celestial body in the rest frame of the body. Section 4.4.5 gives a more accurate model for calculating the acceleration of a near-Earth spacecraft due to the harmonic coefficients of the Earth in the Solar-System barycentric frame of reference. Section 4.4.6 gives the formulation for calculating the acceleration of the Earth or Moon (when one of these bodies is the center of integration) due to the oblateness of the Earth and the Moon. This model is also used to calculate the acceleration of the planet or a satellite of one of the outer planet systems due to oblateness when one of these bodies is the center of integration.

The Solar System contains eleven major bodies: the nine planets, the Sun, and the Moon. The input array PERB for program GIN contains an element for each of these bodies, which can be 0, 1, 2, or 3. The value of 3 can only be used for the Earth. The value of the element of the PERB array for a body determines which terms of the acceleration of the spacecraft due to the body and the acceleration of the center of integration due to the body are computed. For a 0, no acceleration terms due to the body are computed. For a 1, the Newtonian acceleration terms due to the body are calculated. For a 2, the Newtonian and relativistic perturbative acceleration terms are calculated. For the Earth, a value of 3 gives these terms plus the acceleration due to geodesic precession, and the Lense-Thirring precession if the spacecraft is within the Earth's sphere of influence. Furthermore, if the spacecraft is within the Earth's oblateness sphere, the acceleration of the spacecraft due to the Earth's harmonic coefficients is calculated in the Solar-System barycentric frame of reference (*i.e.*, from the formulation of Section 4.4.5 instead of Section 4.4.4). The Newtonian acceleration terms due to asteroids and comets on the small-body ephemeris are calculated if the body number is placed into input array XBNUM, the body name is placed into input array XBNAM, and either is placed into input array XBPERB. All three of these inputs are for program GIN.

For a near-Earth spacecraft, all acceleration terms that are of order 10^{-12} or greater relative to the Newtonian acceleration of the spacecraft due to the Earth are retained.

4.4.1 POINT-MASS NEWTONIAN AND RELATIVISTIC PERTURBATIVE ACCELERATIONS

The point-mass Newtonian acceleration plus the point-mass relativistic perturbative acceleration of body i due to each other body j of the Solar System is given by Eq. (54) of Moyer (1971). The ODP contains the PPN (Parameterized Post–Newtonian) parameters β and γ of Will and Nordtvedt (1972). However, Eq. (54) of Moyer (1971) only contains the parameter γ. Eq. (54) can be parameterized with β and γ by comparing the terms of (54) to the corresponding terms of Eq. (6.78) of Will (1981). Will's equation is parameterized with β and γ, which are unity in general relativity, and α_1, α_2, and ξ, which are zero in general relativity. Setting these small parameters to zero in Eq. (6.78) of Will (1981) and comparing the remaining terms to Eq. (54) of Moyer (1971) gives the β and γ parameterized version of Eq. (54) of Moyer (1971):

$$
\begin{aligned}
\ddot{\mathbf{r}}_i = \sum_{j \neq i} \frac{\mu_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{r_{ij}^{\;3}} & \left\{ 1 - \frac{2(\beta+\gamma)}{c^2} \sum_{l \neq i} \frac{\mu_l}{r_{il}} - \frac{2\beta - 1}{c^2} \sum_{k \neq j} \frac{\mu_k}{r_{jk}} \right. \\
& + \gamma \left(\frac{\dot{s}_i}{c}\right)^2 + (1+\gamma)\left(\frac{\dot{s}_j}{c}\right)^2 - \frac{2(1+\gamma)}{c^2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j \\
& \left. - \frac{3}{2c^2} \left[\frac{\left(\mathbf{r}_i - \mathbf{r}_j\right) \cdot \dot{\mathbf{r}}_j}{r_{ij}} \right]^2 + \frac{1}{2c^2} \left(\mathbf{r}_j - \mathbf{r}_i\right) \cdot \ddot{\mathbf{r}}_j \right\} \\
& + \frac{1}{c^2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}^{\;3}} \left\{ \left[\mathbf{r}_i - \mathbf{r}_j\right] \cdot \left[(2+2\gamma)\, \dot{\mathbf{r}}_i - (1+2\gamma)\, \dot{\mathbf{r}}_j \right] \right\} \left(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j\right) \\
& + \frac{3+4\gamma}{2c^2} \sum_{j \neq i} \frac{\mu_j \ddot{\mathbf{r}}_j}{r_{ij}}
\end{aligned}
\qquad (4–26)
$$

where the notation is defined after Eq. (2–6) and by Eqs. (2–7) to (2–9). The space-fixed position, velocity, and acceleration vectors of points i, j, k, and l are referred to the Solar-System barycenter. The rectangular components of these vectors are referred to the space-fixed coordinate system of the planetary ephemeris. The dots denote differentiation with respect to coordinate time of the Solar-System barycentric frame of reference. The gravitational constants of the Sun, Mercury, Venus, the Earth, the Moon, and the planetary systems Mars through Pluto are the values associated with the Solar-System barycentric frame of reference, and they are obtained from the planetary ephemeris. If a satellite ephemeris is used for a planetary system, the gravitational constant for the planetary system obtained from the satellite ephemeris will overstore the value from the planetary ephemeris in the ODP. The gravitational constants of asteroids and comets are obtained from the small-body ephemeris.

The first term of Eq. (4–26) is the point-mass Newtonian acceleration of body *i*:

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i} \frac{\mu_j \left(\mathbf{r}_j - \mathbf{r}_i \right)}{r_{ij}{}^3} \tag{4–27}$$

The remaining terms of Eq. (4–26) are the point-mass relativistic perturbative acceleration of body *i*. The acceleration of the spacecraft (point *i*) relative to the Solar-System barycenter due to the Sun, Mercury, Venus, the Earth, the Moon, the barycenters of the planetary systems Mars through Pluto, and asteroids and comets is calculated from Eq. (4–26). However, the terms included in the calculation are controlled by the arrays PERB and XBPERB. If the element of the PERB array for a perturbing body *j* in (4–26) is 1, the acceleration of the spacecraft due to that body is calculated from Eq. (4–27). If PERB is 2 or 3, the acceleration of the spacecraft due to body *j* is calculated from Eq. (4–26). If PERB is 0, the acceleration of the spacecraft due to body *j* is not calculated. The acceleration of the spacecraft due to each asteroid and comet included in the XBPERB array is calculated from Eq. (4–27). The acceleration of the center of integration (if it is the Sun, Mercury, Venus, the Earth, the Moon, the barycenter of one of the planetary systems Mars through Pluto, an asteroid, or a comet) relative to the Solar-System barycenter is also calculated from Eq. (4–26) using the PERB and XBPERB arrays as described above. The perturbing bodies for the center of integration are the same as those for the spacecraft except that the center of integration is excluded. The acceleration of the spacecraft relative to the center of integration is the acceleration of the spacecraft minus the acceleration of the center of integration.

Evaluation of the relativistic perturbative acceleration terms of Eq. (4–26) requires the acceleration ($\ddot{\mathbf{r}}_j$) of body j in two places. Since terms of order $1/c^4$ are ignored, the Newtonian acceleration given by Eq. (4–27) or Eq. (2–12) can be used. Calculation of the relativistic perturbative acceleration of body i due to perturbing body j using Eq. (4–26) requires the just-mentioned Newtonian acceleration of body j, the gravitational potential at body j, and the gravitational potential at body i. The contribution to these gravitational potentials and accelerations due to the mass of a Solar-System body will not be computed if the element of the PERB array for that body is zero or the body (if it is an asteroid or a comet) is not included in the XBPERB array. Note that the mass of body i contributes to the Newtonian acceleration of each perturbing body j and the gravitational potential at each perturbing body j.

If the spacecraft is outside the sphere of influence of a planetary system (Mars, Jupiter, Saturn, Uranus, Neptune, or Pluto), the acceleration of the spacecraft due to that planetary system is calculated from the gravitational constant of the planetary system located at the barycenter of the planetary system (obtained from the planetary ephemeris). However, if the spacecraft is inside the sphere of influence of a planetary system and a satellite ephemeris for that planetary system is used, then the acceleration of the spacecraft due to each satellite and the planet of the planetary system is calculated. The gravitational constants of each of these bodies and their positions relative to the barycenter of the planetary system are obtained from the satellite ephemeris as described in Section 3.2.2.1. If the element of the PERB array for the planetary system is 1, the acceleration of the spacecraft due to each body of the planetary system is Newtonian (*i.e.*, calculated from Eq. 4–27). If the element of the PERB array is 2, these acceleration terms are relativistic (*i.e.*, calculated from Eq. 4–26).

If the center of integration (COI) for the spacecraft ephemeris is the planet or a satellite of one of the outer planet systems, the acceleration of the COI due to the distant bodies of the Solar System is calculated from Eq. (4–26) as described above, except that the position of the planet or satellite which is the

COI is used instead of the position of the barycenter of the planetary system. The acceleration of the COI due to each of the other bodies of the planetary system is calculated from Eq. (4–26) if PERB for the planetary system is 2 and from Eq. (4–27) if PERB for the planetary system is 1.

The remainder of this section will show how the n-body point-mass relativistic equations of motion (Eq. 4–26) can be derived from the n-body point-mass metric tensor and related equations (Eqs. 2–1 to 2–15). The trajectory of a massless particle or a celestial body in the gravitational field of n other celestial bodies is a geodesic curve which extremizes the integral of the interval ds between two points:

$$\delta \int ds = 0 \tag{4–28}$$

Special conditions for treating the mass of body i whose motion is desired will be given below. In order to obtain the equations of motion with coordinate time t of the Solar-System barycentric frame of reference as the independent variable, Eq. (4–28) is written as

$$\delta \int L \, dt = 0 \tag{4–29}$$

where the Lagrangian L is given by:

$$L = \frac{ds}{dt} \tag{4–30}$$

An expression for L^2 is obtained from Eq. (2–15) for ds^2 by replacing differentials of the space coordinates of body i by derivatives of the space coordinates with respect to coordinate time t multiplied by dt, and then dividing the resulting equation by dt^2. The Lagrangian L could be obtained by expanding the square root of L^2 in powers of $1/c^2$. Given L, the equations of motion that extremize the integral (4–29) are the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad x \to y, z \tag{4–31}$$

where

$$\dot{x}_i = \frac{dx_i}{dt} \qquad x \to y, z \tag{4–32}$$

A simpler procedure for obtaining the equations of motion directly from derivatives of L^2 is developed as follows. The Euler-Lagrange equations are unchanged by multiplying both terms by L:

$$L\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - L\frac{\partial L}{\partial x_i} = 0 \qquad x \to y, z \tag{4–33}$$

Differentiating $L\left(\partial L/\partial \dot{x}_i\right)$ with respect to t gives:

$$\frac{d}{dt}\left(L\frac{\partial L}{\partial \dot{x}_i}\right) = \left(\frac{\dot{L}}{L}\right)\left(L\frac{\partial L}{\partial \dot{x}_i}\right) + L\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \qquad x \to y, z \tag{4–34}$$

where

$$\dot{L} = \frac{dL}{dt} \tag{4–35}$$

The equations of motion are obtained by substituting the last term of Eq. (4–34) into (4–33):

$$\frac{d}{dt}\left(L\frac{\partial L}{\partial \dot{x}_i}\right) - \left(\frac{\dot{L}}{L}\right)\left(L\frac{\partial L}{\partial \dot{x}_i}\right) - \left(L\frac{\partial L}{\partial x_i}\right) = 0 \qquad x \to y, z \tag{4–36}$$

The derivatives $L\left(\partial L/\partial x_i\right)$ and $L\left(\partial L/\partial \dot{x}_i\right)$ are obtained by differentiation of the expression for L^2. Because the equations of motion contain terms to order $1/c^2$ only,

$$\frac{\dot{L}}{L} = \frac{L\dot{L}}{L^2} \approx \frac{L\dot{L}}{c^2} \tag{4–37}$$

where L^2 has been replaced by its leading term c^2 and $L\dot{L}$ is obtained by differentiating a simplified expression for L^2 containing terms to order $1/c^0$ only. The required expression for L^2 is obtained from Eq. (2–15) as described above:

$$\begin{aligned} L^2 = c^2 g_{44} &+ g_{11}\left(\dot{x}_i{}^2 + \dot{y}_i{}^2 + \dot{z}_i{}^2\right) \\ &+2c\, g_{14}\dot{x}_i + 2c\, g_{24}\dot{y}_i + 2c\, g_{34}\dot{z}_i \end{aligned} \tag{4–38}$$

where the components of the n-body metric tensor are obtained from Eqs. (2–1) to (2–6) and Eq. (2–11). The n-body point-mass relativistic equations of motion (Eq. 4–26) can be derived by evaluating Eq. (4–36) using Eqs. (4–37) and (4–38). However, in evaluating the partial derivatives of Eq. (4–38) with respect to the position components of body i, the gravitational potential at each perturbing body j and the Newtonian acceleration of each perturbing body j must be considered to be functions of coordinate time t only. These functions must not be differentiated with respect to the position components of body i. These special conditions were pointed out to me by Dr. Frank B. Estabrook and Dr. Hugo Wahlquist of the Jet Propulsion Laboratory. This particular derivation of Eq. (4–26), for the case where $\beta = 1$, is given in Section II of Appendix A of Moyer (1971).

4.4.2 GEODESIC PRECESSION

Geodesic precession is due to the motion of the Earth through the Sun's gravitational field. It causes the pole of the orbit of an Earth satellite to precess about the normal to the ecliptic at the rate of 19.2″ x 10^{-3}/year. This causes the ascending node of the orbit of an Earth satellite on the ecliptic to increase in celestial longitude by 19.2 mas/year. This same effect decreases the general precession in longitude by the same amount (see *Explanatory Supplement* (1961), p. 170).

In the Solar-System barycentric space-time frame of reference, the geocentric acceleration of a near-Earth spacecraft due to geodesic precession is included in the point-mass relativistic perturbative acceleration of the spacecraft calculated from Eq. (4–26) minus the point-mass relativistic perturbative acceleration of the Earth calculated from the same equation (see Dickey, Newhall, and Williams (1989) and HRTW (1990)). When the geocentric acceleration of a near-Earth spacecraft is calculated in the local geocentric space-time frame of reference, a separate equation is required for calculating the acceleration due to geodesic precession (Section 4.5.3).

4.4.3 LENSE-THIRRING PRECESSION

The unit vector **S** in the direction of the north pole of the orbit of an Earth satellite undergoes the general relativistic Lense-Thirring precession due to the rotation of the Earth. The unit vector **S** precesses at the rate:

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\Omega} \times \mathbf{S} \qquad (4\text{–}39)$$

where **Ω** is the Lense-Thirring angular velocity vector. From Will (1981), Eq. (9.5), term 2,

$$\boldsymbol{\Omega} = \frac{(1+\gamma)\, G}{2c^2 r^3}\left[-\mathbf{J} + \frac{3(\mathbf{J}\cdot\mathbf{r})\,\mathbf{r}}{r^2}\right] \qquad \frac{\text{rad}}{\text{s}} \qquad (4\text{–}40)$$

where

G = constant of gravitation
 = 6.67259×10^{-20} km^3/s^{2}kg

r = space-fixed geocentric position vector of near-Earth spacecraft, km

r = magnitude of **r**

J = angular momentum vector of the Earth

Eq. (4–40) with γ equal to its general relativistic value of unity is also given in Misner, Thorne, and Wheeler (1973), Eq. (40.37) (with $\Delta_1 = \Delta_2 = 1$, their general relativistic values). The angular momentum vector of the Earth is given by:

$$\mathbf{J} = (0.33068)\, m_E\, a_e^{\,2} \omega_E\, \mathbf{P} \qquad \frac{\text{kg km}^2}{\text{s}} \tag{4–41}$$

where a_e and ω_E are defined after Eq. (4–12) and:

m_E = mass of the Earth, kg

$\mathbf{P}$ = unit vector aligned with the Earth's spin axis and directed toward the north pole

The constant 0.33068 in Eq. (4–41) is the polar moment of inertia C of the Earth divided by $m_E a_e^{2}$. It was computed by J. G. Williams of the Jet Propulsion Laboratory as J_2 (definition and numerical value given after Eq. 4–12) which is equal to $(C - A)/m_E a_e^{2}$ (see Kaula (1968), p. 68, Eq. 2.1.32), where A is the equatorial moment of inertia of the Earth, divided by $(C - A)/C = 0.0032739935$ (see Seidelmann (1982), p. 96, parameter H).

The angular velocity vector $\mathbf{\Omega}$ given by Eq. (4–40) is the local rotation rate of the inertial geocentric frame of reference relative to a non-rotating geocentric frame. The equations of motion in a non-rotating geocentric frame are those of a coordinate system rotating with the angular velocity $-\,\Omega$. So, to the equations of motion in a non-rotating geocentric frame of reference , we must add the Coriolis acceleration $-\,2\boldsymbol{\omega}\times\dot{\mathbf{r}}$, where ω is the angular velocity $-\,\Omega$ and $\dot{\mathbf{r}}$ is the geocentric space-fixed velocity vector of the near-Earth spacecraft. Thus, in the non-rotating local geocentric space-time frame of reference or the Solar-System barycentric space-time frame of reference, the acceleration of a near-Earth spacecraft due to the Lense-Thirring precession is given by:

$$\ddot{\mathbf{r}} = 2\mathbf{\Omega}\times\dot{\mathbf{r}} \tag{4–42}$$

The ratio of this acceleration to the Newtonian acceleration of an Earth satellite is a maximum for a very near Earth satellite. The angular rate $|\mathbf{\Omega}|$ is about 2×10^{-14}

rad/s for the TOPEX satellite (semi-major axis = 7712 km). The corresponding acceleration computed from Eq. (4–42) is about 3 x 10^{-13} km/s^2. Since the Newtonian acceleration of the TOPEX satellite is about 0.7 x 10^{-2} km/s^2, the Lense-Thirring acceleration is approximately 4 x 10^{-11} times the Newtonian acceleration. In the non-rotating geocentric frame, we should also add the centrifugal acceleration $-\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}$. However, this acceleration is a maximum of about 10^{-21} times the Newtonian acceleration, which can safely be ignored.

Substituting Eq. (4–41) into Eq. (4–40) and substituting the result into Eq. (4–42) and using

$$\begin{aligned} \mu_{\mathrm{E}} &= Gm_{\mathrm{E}} \\ &= \text{gravitational constant of the Earth, km}^3/\text{s}^2 \end{aligned}$$

as defined after Eq. (2–6) gives:

$$\ddot{\mathbf{r}} = \frac{(0.33068)(1+\gamma)\mu_{\mathrm{E}}\, a_{\mathrm{e}}{}^2 \omega_{\mathrm{E}}}{c^2 r^3}\left[\frac{3}{r^2}(\mathbf{r}\times\dot{\mathbf{r}})(\mathbf{r}\cdot\mathbf{P}) + (\dot{\mathbf{r}}\times\mathbf{P})\right] \qquad \frac{\text{km}}{\text{s}^2} \tag{4–43}$$

In order to calculate the Lense-Thirring acceleration from Eq. (4–43), an expression is required for the pole vector **P** in the space-fixed coordinate system of the planetary ephemeris (see Section 3.1.1). It could be calculated from polynomials for the right ascension and declination of the Earth's mean north pole of date. However, the following simpler algorithm was suggested by J.G. Williams. In the Earth-fixed coordinate system aligned with the true pole, prime meridian, and equator of date,

$$\mathbf{P} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4–44}$$

In the space-fixed coordinate system of the planetary ephemeris, **P** is given by:

$$\mathbf{P} = T_{\mathrm{E}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4–45}$$

where T_{E} is the 3 x 3 rotation matrix from Earth-fixed coordinates referred to the true pole, prime meridian, and equator of date to the space-fixed coordinate system of the planetary ephemeris. The algorithm for calculating the transformation matrix T_{E} for the Earth is given in Section 5.3. Rather than formally calculating the space-fixed pole vector $\mathbf{P}$ from Eq. (4–45), it is given simply by the third column of T_{E}.

Eq. (4–43) with $\gamma = 1$ (general relativity) and expressed in terms of $\mathbf{J}$ given by Eq. (4–41) or $\mathbf{J}/m_{\mathrm{E}}$ instead of $\mathbf{P}$ is given by Eq. (9.5.19) on p. 232 of Weinberg (1972) and Eq. (41) of HRTW (1990), respectively.

4.4.4 NEWTONIAN ACCELERATION OF SPACECRAFT DUE TO THE HARMONIC COEFFICIENTS OF A CELESTIAL BODY

This section presents the model for the Newtonian acceleration of the spacecraft due to the oblateness of a nearby celestial body. This acceleration is only calculated if the spacecraft is within the oblateness sphere of the body. Section 4.4.5 gives the model for the relativistic acceleration of a near-Earth spacecraft due to the oblateness of the Earth. This more-accurate model will be used if the element of the PERB array for the Earth is set to 3 instead of 1 or 2. The relativistic model of Section 4.4.5 may eventually be applied to other Solar-System bodies in addition to the Earth. The relativistic acceleration of a near-Earth spacecraft due to the Earth's oblateness includes the calculation of the Newtonian oblateness acceleration from the equations of this section.

The acceleration of the center of integration due to oblateness is calculated when the center of integration is the Earth or the Moon. This model is given in Section 4.4.6 and includes the effects of the oblateness of the Earth and the Moon. The acceleration of the center of integration due to the oblateness of the Sun is not calculated because the Sun cannot currently be modelled as an oblate body in the ODP. If the center of integration is the planet or a satellite of one of the outer planet systems, the acceleration of the center of integration due to the oblateness of the bodies of the planetary system is calculated from the above model as described in Section 4.4.6.

The Newtonian acceleration of the spacecraft due to the oblateness of a nearby celestial body can be calculated for any body in the Solar System except the Sun. These bodies consist of the nine planets, the Moon, the satellites of the outer planets Mars through Pluto, asteroids, and comets. Calculation of the acceleration due to a satellite or the planet of one of the outer planet systems requires the use of a satellite ephemeris for that system.

Calculation of the Newtonian acceleration of the spacecraft due to the oblateness of a nearby body B requires the 3 x 3 body-fixed to space-fixed rotation matrix T_{B} for body B. If body B is the Earth (E), the body-fixed to space-fixed transformation matrix T_{E} for the Earth rotates from one of two possible Earth-fixed coordinate systems selected by the user to the space-fixed coordinate system of the planetary ephemeris (see Section 3.1.1). One of these Earth-fixed coordinate systems is aligned with the mean pole, prime meridian, and equator of 1903.0. The other Earth-fixed coordinate system is aligned with the true pole, prime meridian, and equator of date. For the former case, the matrix T_{E} includes rotations through the X and Y angular coordinates of the true pole of date relative to the mean pole of 1903.0. The formulation for calculating either version of the transformation matrix T_{E} for the Earth is given in Section 5.3. For every other body B in the Solar System except the Earth, the transformation matrix T_{B} rotates from the body-fixed coordinate system aligned with the true pole, prime meridian, and equator of date to the space-fixed coordinate system of the planetary ephemeris. The formulation for calculating T_{B} is given in Section 6.3. If nutation terms are not included in calculating T_{B}, the body-fixed coordinate system is aligned with the mean pole, prime meridian, and equator of date.

Note that if the body-fixed coordinate system for the Earth is aligned with the mean pole of 1903.0 instead of the true pole of date, the tesseral harmonic coefficients C_{21} and S_{21} for the Earth must be non-zero to account for the offset of the mean pole of date (assumed to be the mean figure axis) from the mean pole of 1903.0. This is discussed further in Section 5.2.8.

The Newtonian acceleration of the spacecraft due to the oblateness of a nearby body B is obtained by rotating the space-fixed position vector of the spacecraft to the body-fixed coordinate system, calculating the oblateness acceleration in the body-fixed coordinate system, and then rotating the acceleration of the spacecraft due to oblateness from the body-fixed coordinate system to the space-fixed coordinate system. However, the oblateness acceleration is not calculated in one of the body-fixed equatorial coordinate systems described above, but in the body-fixed up-east-north coordinate system. So, one additional rotation matrix is needed in addition to the matrix T_{B}. The following paragraph gives the equations for rotating between the space-fixed coordinate system of the planetary ephemeris and the body-fixed up-east-north coordinate system. The equations for calculating the Newtonian oblateness acceleration in the body-fixed up-east-north coordinate system are given in Moyer (1971).

Let

$\mathbf{r}$ = space-fixed position vector of the spacecraft relative to the center of integration (COI) of the spacecraft ephemeris. This vector is represented in the space-fixed coordinate system of the planetary ephemeris.

$\mathbf{r}_{\mathrm{B}}^{\mathrm{COI}}$ = space-fixed position vector of the oblate body B relative to the center of integration. This vector is obtained from the planetary ephemeris as described in Section 3.1.2.1 and, if necessary, a satellite ephemeris as described in Section 3.2.2.1.

Then, the space-fixed position vector of the spacecraft (S/C) relative to the oblate body B is given by:

$$\mathbf{r}_{\mathrm{S/C}}^{\mathrm{B}} = \mathbf{r} - \mathbf{r}_{\mathrm{B}}^{\mathrm{COI}} \tag{4–46}$$

It is related to the corresponding body-fixed position vector $\mathbf{r}_{\mathrm{b}}$ of the spacecraft by:

$$\mathbf{r}^{\mathrm{B}}_{\mathrm{S/C}} = T_{\mathrm{B}}\, \mathbf{r}_{\mathrm{b}} \tag{4–47}$$

where the body B can be the Earth (E). The specific equatorial body-fixed coordinate system that $\mathbf{r}_{\mathrm{b}}$ is referred to is the body-fixed coordinate system of the body-fixed to space-fixed transformation matrix T_{B}, as discussed above. The inverse transformation of Eq. (4–47) is:

$$\mathbf{r}_{\mathrm{b}} = T_{\mathrm{B}}{}^{\mathrm{T}}\, \mathbf{r}^{\mathrm{B}}_{\mathrm{S/C}} \tag{4–48}$$

where the superscript T indicates the transpose of the matrix.

Let $\mathbf{r}'$ denote the position vector of the spacecraft relative to the oblate body B in the body-fixed up-east-north rectangular coordinate system (x′y′z′). The x′ axis is directed outward along the radius to the spacecraft, the y′ axis is directed east, and the z′ axis is directed north. The transformation from equatorial body-fixed coordinates to up-east-north body-fixed coordinates is given by:

$$\mathbf{r}' = R\, \mathbf{r}_{\mathrm{b}} \tag{4–49}$$

where the 3 x 3 rotation matrix R is given by Eq. (161) of Moyer (1971). The matrix R is a function of sines and cosines of the latitude ϕ and longitude λ of the spacecraft measured in the body-fixed equatorial coordinate system. Given the rectangular components of $\mathbf{r}_{\mathrm{b}}$ from Eq. (4–48), the sines and cosines of ϕ and λ are given by Eqs. (165) to (168) of Moyer (1971).

Substituting Eq. (4–48) into (4–49) gives the transformation from the space-fixed position vector of the spacecraft relative to the oblate body B to the corresponding body-fixed position vector in the up-east-north coordinate system:

$$\mathbf{r}' = R\, T_{\mathrm{B}}{}^{\mathrm{T}} \mathbf{r}^{\mathrm{B}}_{\mathrm{S/C}} \equiv G\, \mathbf{r}^{\mathrm{B}}_{\mathrm{S/C}} \tag{4–50}$$

The inverse transformation is:

$$\mathbf{r}_{S/C}^{B} = T_B\, R^T \mathbf{r}' \equiv G^T \mathbf{r}' \tag{4–51}$$

The following paragraph will describe the calculation of the acceleration of the spacecraft due to the oblateness of body B in the body-fixed up-east-north coordinate system. Given this acceleration, $\ddot{\mathbf{r}}'$, the oblateness acceleration in the space-fixed coordinate system of the planetary ephemeris is given by the second derivative of Eq. (4–51), obtained holding the transformation matrix G fixed:

$$\ddot{\mathbf{r}} = G^T \ddot{\mathbf{r}}' \tag{4–52}$$

The rotation matrix G is not differentiated because the oblateness acceleration $\ddot{\mathbf{r}}'$ is the inertial acceleration of the spacecraft with rectangular components along the instantanteous positions of the axes of the body-fixed up-east-north coordinate system.

Given the space-fixed position vector of the spacecraft relative to the oblate body B given by Eq. (4–46), calculate the rectangular components of $\mathbf{r}_b$ from Eq. (4–48). Using these rectangular components, calculate the radius r from the oblate body B to the spacecraft and the sines and cosines of the latitude ϕ and longitude λ of the spacecraft measured in the body-fixed equatorial coordinate system from Eqs. (165) to (168) of Moyer (1971). Given r, ϕ, and λ, calculate the acceleration of the spacecraft due to the oblateness of body B in the body-fixed up-east-north coordinate system from the sum of Eqs. (173) and (174) of Moyer (1971). Eq. (173) gives the acceleration due to the zonal harmonic coefficients J_n, and Eq. (174) gives the acceleration due to the tesseral harmonic coefficients C_{nm} and S_{nm}. These equations are a function of the Legendre polynomial P_n of degree n in sin ϕ, the associated Legendre function P_n^m defined by Eq. (155) of Moyer (1971), and the derivatives of both of these functions with respect to sin ϕ. These four functions are functions of sin ϕ and cos ϕ and are computed recursively from Eqs. (175) to (183) of Moyer (1971). Given the oblateness acceleration $\ddot{\mathbf{r}}'$ in the body-fixed up-east-north coordinate system, rotate it into the space-fixed coordinate system of the planetary ephemeris using Eq. (4–52).

Eqs. (173) and (174) of Moyer (1971) can be derived from the expressions for the gravitational potential, which are given by Eqs. (154) to (159) of Moyer (1971).

4.4.5 RELATIVISTIC ACCELERATION OF SPACECRAFT DUE TO THE HARMONIC COEFFICIENTS OF THE EARTH

In the Solar-System barycentric space-time frame of reference, the Earth is foreshortened in the direction of motion, which distorts the harmonic expansion of its gravitational potential. In the geocentric frame, however, the shape of the Earth and its gravitational potential are unaffected. The acceleration of a near-Earth spacecraft in the Solar-System barycentric frame of reference due to the oblateness of the Earth is calculated from the algorithm obtained from HRTW (1990), which is detailed in the following paragraphs. This algorithm calculates the oblateness acceleration in the local geocentric space-time frame of reference, where the gravitational potential of the Earth is known, and utilizes the relativistic coordinate transformations between the Solar-System barycentric and local geocentric frames of reference developed in Section 4.3.

The trajectory of a near-Earth spacecraft in the Solar-System barycentric space-time frame of reference is obtained by numerical integration with coordinate time t_{BC} of the barycentric frame as the independent variable. At each integration step, the current Earth-centered space-fixed position vector of the spacecraft $\mathbf{r}_{BC}$ in the Solar-System barycentric space-time frame of reference, calculated from Eq. (4–46), where B is the Earth E, is transformed to $\mathbf{r}_{GC}$ in the geocentric space-time frame of reference using Eq. (4–11), which is evaluated as described after it. Then, using $\mathbf{r}_{GC}$ as the input, the Newtonian acceleration of the near-Earth spacecraft $\ddot{\mathbf{r}}_{GC}$ due to the harmonic coefficients of the Earth in the geocentric frame of reference is calculated from the algorithm of Section 4.4.4. In evaluating this algorithm, the gravitational constant of the Earth should be the value in the local geocentric frame of reference calculated from the value in the barycentric frame (obtained from the planetary ephemeris) using Eq. (4–25). The acceleration $\ddot{\mathbf{r}}_{GC}$ is then transformed to the acceleration $\ddot{\mathbf{r}}_{BC}$ in the barycentric frame using the second derivative of Eq. (4–10), which is derived in the next paragraph.

In Eq. (4–10), $\mathbf{r}_{BC}$ is a function of coordinate time t_{BC} in the barycentric frame, and $\mathbf{r}_{GC}$ is a function of coordinate time t_{GC} in the geocentric frame. First, Eq. (4–10) will be differentiated with respect to t_{BC}. In carrying out this differentiation, $\mathbf{V}_E$ and U_E are considered to be constant, and $\tilde{L}$ is constant. It is shown in HRTW (1990) that differentiation of $\mathbf{V}_E$ and U_E yields (after differentiating Eq. 4–10 twice) acceleration terms which are of order 10^{-14} or smaller relative to the Newtonian acceleration of a near-Earth spacecraft. In differentiating the right-hand side of Eq. (4–10),

$$\frac{d\mathbf{r}_{GC}}{dt_{BC}} = \frac{d\mathbf{r}_{GC}}{dt_{GC}} \frac{dt_{GC}}{dt_{BC}} \tag{4–53}$$

where dt_{GC}/dt_{BC} is given by Eq. (4–22). Differentiating Eq. (4–10) with respect to t_{BC} using Eqs. (4–53) and (4–22), and retaining terms to order $1/c^2$ gives:

$$\dot{\mathbf{r}}_{BC} = \left[1 - \frac{(1+\gamma)U_E}{c^2} - \frac{V_E{}^2}{2c^2} - \frac{\mathbf{V}_E \cdot \dot{\mathbf{r}}}{c^2}\right]\dot{\mathbf{r}}_{GC} - \frac{1}{2c^2}\left(\mathbf{V}_E \cdot \dot{\mathbf{r}}_{GC}\right)\mathbf{V}_E \tag{4–54}$$

where $\dot{\mathbf{r}}_{BC} = d\mathbf{r}_{BC}/dt_{BC}$, $\dot{\mathbf{r}}_{GC} = d\mathbf{r}_{GC}/dt_{GC}$, and $\dot{\mathbf{r}} = \dot{\mathbf{r}}_{BC}$ or $\dot{\mathbf{r}}_{GC}$ since terms of order $1/c^4$ are ignored. Differentiating Eq. (4–54) with respect to t_{BC} using Eqs. (4–53) and (4–22), holding $\mathbf{V}_E$ and U_E fixed, and retaining terms to order $1/c^2$ gives:

$$\ddot{\mathbf{r}}_{BC} = \left[1 - \frac{(2+\gamma)U_E}{c^2} - \frac{V_E{}^2}{c^2} + \tilde{L} - \frac{2\mathbf{V}_E \cdot \dot{\mathbf{r}}}{c^2}\right]\ddot{\mathbf{r}}_{GC} - \frac{1}{2c^2}\left(\mathbf{V}_E \cdot \ddot{\mathbf{r}}_{GC}\right)\left(\mathbf{V}_E + 2\dot{\mathbf{r}}\right) \tag{4–55}$$

where $\ddot{\mathbf{r}}_{BC} = d^2\mathbf{r}_{BC}/dt_{BC}{}^2$ and $\ddot{\mathbf{r}}_{GC} = d^2\mathbf{r}_{GC}/dt_{GC}{}^2$. Since it is not necessary to transform $\dot{\mathbf{r}}_{BC}$ to $\dot{\mathbf{r}}_{GC}$ from the inverse of Eq. (4–54) in order to calculate $\ddot{\mathbf{r}}_{GC}$ as described in the preceding paragraph, the geocentric space-fixed velocity vector $\dot{\mathbf{r}}$ of the near-Earth spacecraft can most conveniently be evaluated with $\dot{\mathbf{r}} = \dot{\mathbf{r}}_{BC}$, which is given by the derivative of Eq. (4–46), where B is the Earth E.

Given the acceleration of a near-Earth spacecraft due to the oblateness of the Earth calculated in the local geocentric space-time frame of reference as described above, Eq. (4–55) transforms this acceleration to the corresponding acceleration in the Solar-System barycentric space-time frame of reference. The acceleration $\ddot{\mathbf{r}}_{BC} - \ddot{\mathbf{r}}_{GC}$ obtained from Eq. (4–55), with γ set equal to its general relativistic value of unity, is Eq. (59) of HRTW (1990). The relativistic acceleration of a near-Earth spacecraft due to the oblateness of the Earth minus the corresponding Newtonian acceleration is of order 10^{-8} relative to the Newtonian oblateness acceleration, which is of order 10^{-3} relative to the Newtonian acceleration of the spacecraft due to the point-mass Earth. Hence, the relativistic oblateness acceleration minus the Newtonian oblateness acceleration of a near-Earth spacecraft is of order 10^{-11} relative to the Newtonian acceleration of the spacecraft due to the Earth.

4.4.6 ACCELERATION OF THE CENTER OF INTEGRATION DUE TO OBLATENESS

The acceleration of the center of integration due to oblateness is calculated when the center of integration is the Earth or the Moon and accounts for the oblateness of both of these bodies. The model for this acceleration is derived below. If the center of integration is the planet or a satellite of one of the outer planet systems, this model is used to calculate the acceleration of the center of integration due to the oblateness of the bodies of the planetary system as described at the end of this section.

The force of attraction between the Earth and the Moon consists of:

1. The force of attraction between the point-mass Earth and the point-mass Moon.

2. The force of attraction between the oblate part of the Earth (*i.e.*, the Earth's harmonic coefficients) and the point-mass Moon.

3. The force of attraction between the oblate part of the Moon (*i.e.*, the Moon's harmonic coefficients) and the point-mass Earth.

4. The force of attraction between the oblate part of the Earth and the oblate part of the Moon.

The force 1 is accounted for in Section 4.4.1. The formulation of this section accounts for the forces 2 and 3, but ignores the force 4, which is negligible.

Let

$\ddot{\mathbf{r}}_M(E)$	=	acceleration of point-mass Moon due to the oblateness of the Earth
$\ddot{\mathbf{r}}_E(M)$	=	acceleration of point-mass Earth due to the oblateness of the Moon

These accelerations, with rectangular components referred to the space-fixed coordinate system of the planetary ephemeris, are computed from the Newtonian formulation of Section 4.4.4. In calculating $\ddot{\mathbf{r}}_M(E)$, the Moon is treated as the spacecraft and Eq. (4–46) for the space-fixed position vector of the spacecraft relative to the oblate body is replaced by the space-fixed geocentric position vector of the Moon $\mathbf{r}_M^E$ interpolated from the planetary ephemeris (see Section 3.1.2.1). Similarly, in calculating $\ddot{\mathbf{r}}_E(M)$, the Earth is treated as the spacecraft, and Eq. (4–46) is replaced by $-\mathbf{r}_M^E$.

Consider the force of attraction between the Earth and the Moon due to the oblateness of the Earth, assuming the Moon to be a point mass. This force produces $\ddot{\mathbf{r}}_M(E)$ and:

$\ddot{\mathbf{r}}_E(E)$	=	acceleration of the Earth due to the force of attraction between the oblate part of the Earth and the point-mass Moon

Since these two accelerations are derived from equal and opposite forces,

$$\ddot{\mathbf{r}}_E(E) = -\frac{\mu_M}{\mu_E}\ddot{\mathbf{r}}_M(E) \tag{4–56}$$

where μ_E and μ_M are the gravitational constants of the Earth and Moon, obtained from the planetary ephemeris. Similarly, consider the force of attraction between the Earth and the Moon due to the oblateness of the Moon, assuming the Earth to be a point mass. This force produces $\ddot{\mathbf{r}}_E(\mathrm{M})$ and:

$\ddot{\mathbf{r}}_M(\mathrm{M})$ = acceleration of the Moon due to the force of attraction between the oblate part of the Moon and the point-mass Earth

Since these two accelerations are derived from equal and opposite forces,

$$\ddot{\mathbf{r}}_M(\mathrm{M}) = -\frac{\mu_E}{\mu_M}\ddot{\mathbf{r}}_E(\mathrm{M}) \tag{4–57}$$

The acceleration of the Earth due to the oblateness of the Earth attracting the point-mass Moon and the oblateness of the Moon attracting the point-mass Earth is given by:

$$\begin{aligned} \ddot{\mathbf{r}}_E &= \ddot{\mathbf{r}}_E(\mathrm{M}) + \ddot{\mathbf{r}}_E(\mathrm{E}) \\ &= \ddot{\mathbf{r}}_E(\mathrm{M}) - \frac{\mu_M}{\mu_E}\ddot{\mathbf{r}}_M(\mathrm{E}) \end{aligned} \tag{4–58}$$

Similarly, the acceleration of the Moon due to the oblateness of the Earth attracting the point-mass Moon and the oblateness of the Moon attracting the point-mass Earth is given by:

$$\begin{aligned} \ddot{\mathbf{r}}_M &= \ddot{\mathbf{r}}_M(\mathrm{E}) + \ddot{\mathbf{r}}_M(\mathrm{M}) \\ &= \ddot{\mathbf{r}}_M(\mathrm{E}) - \frac{\mu_E}{\mu_M}\ddot{\mathbf{r}}_E(\mathrm{M}) \end{aligned} \tag{4–59}$$

The accelerations $\ddot{\mathbf{r}}_E$ and $\ddot{\mathbf{r}}_M$ are functions of the harmonic coefficients of the Earth and the Moon. Also, $\ddot{\mathbf{r}}_E$ is proportional to μ_M and $\ddot{\mathbf{r}}_M$ is proportional to μ_E. The ODP evaluates Eqs. (4–58) and (4–59) using the harmonic coefficients J_2, C_{22}, and S_{22} only for the Earth and the Moon. The negative of Eqs. (4–58) and (4–59) are contributions to the acceleration of the spacecraft relative to the Earth and the Moon, respectively.

The acceleration of the Earth due to its own oblateness in the presence of the point-mass Moon is approximately 5 x 10^{-11} times the Newtonian acceleration of a GPS (Global Positioning System) satellite (semi-major axis $a \approx 26{,}560$ km). This ratio is smaller for the TOPEX satellite. For a GPS satellite, the acceleration of the satellite due to the oblateness of the Moon minus the acceleration of the Earth due to the oblateness of the Moon is of order 10^{-13} relative to the Newtonian acceleration of the satellite.

The Earth (or the Moon) is also accelerated due to the oblateness of the Earth (or the Moon) attracting the point-mass Sun. The acceleration of the Earth due to its own oblateness in the presence of the point-mass Sun is about 6 x 10^{-14} times the Newtonian acceleration of a GPS satellite. In the ODP, Eqs. (4–58) and (4–59) do not include the acceleration of the Earth and the Moon, respectively, due to the interaction of the oblateness of these bodies with the point-mass Sun.

If the center of integration is the planet or a satellite of one of the outer planet systems, the above model is used to calculate the acceleration of the center of integration due to the oblateness of the bodies of the planetary system.

If the center of integration is satellite i of one of the outer planet systems, the acceleration of satellite i due to the oblateness of the planet and due to the oblateness of satellite i acting on the point mass of the planet is calculated from Eq. (4–59), where M refers to satellite i and E refers to the planet. If the spacecraft is within the harmonic sphere of satellite j, the acceleration of satellite i due to the oblateness of satellite j and due to the oblateness of satellite i acting on the point mass of satellite j is calculated from Eq. (4–59), where M refers to satellite i and E refers to satellite j. Note that the masses of the satellites and the planet are obtained as described in Section 3.2.2.1.

If the center of integration is the planet of one of the outer planet systems, the acceleration of the planet due to the oblateness of satellite i and due to the oblateness of the planet acting on the point mass of satellite i is calculated from

Eq. (4–58), where E refers to the planet and M refers to satellite i. This calculation is performed for each satellite of the planetary system and the resulting accelerations of the planet are summed.

If the center of integration is the barycenter of one of the outer planet systems, the acceleration of the barycenter due to the oblateness of the bodies of the outer planet system is zero.

4.5 RELATIVISTIC EQUATIONS OF MOTION IN LOCAL GEOCENTRIC FRAME OF REFERENCE

This section specifies the equations for calculating the acceleration of a near-Earth spacecraft (typically, an Earth satellite) relative to the center of mass of the Earth due to gravity only. This acceleration is calculated in the local geocentric space-time frame of reference. Section 4.5.1 specifies the Newtonian point-mass acceleration of a near-Earth spacecraft due to the Sun, the Moon, the planets, asteroids, and comets minus the corresponding acceleration of the Earth. In the local geocentric space-time frame of reference, the n-body point-mass relativistic perturbative acceleration reduces to the acceleration obtained from the 1-body Schwarzschild isotropic metric for the Earth (specified in Section 4.5.2) plus the acceleration due to geodesic precession (specified in Section 4.5.3). The Lense-Thirring relativistic acceleration of a near-Earth spacecraft due to the rotation of the Earth is given in Section 4.5.4. Section 4.5.5 specifies the calculation of the acceleration of a near-Earth spacecraft due to the oblateness of the Earth and the Moon from the Newtonian model of Section 4.4.4. Section 4.5.6 specifies the calculation of the acceleration of the Earth (which is subtracted from the acceleration of the spacecraft) due to the oblateness of the Earth and the Moon using the model of Section 4.4.6.

The time argument used to evaluate all acceleration models and interpolate the spacecraft ephemeris is coordinate time t_{GC} of the local geocentric space-time frame of reference. It is also used to interpolate the planetary ephemeris instead of the actual argument, which is coordinate time t_{BC} of the Solar-System barycentric frame of reference. The gravitational constant of the Earth used in all models is the value calculated from the corresponding value in the Solar-System barycentric frame (obtained from the planetary ephemeris) using Eq. (4–25).

All acceleration terms which are of order 10^{-12} or greater relative to the Newtonian acceleration of the spacecraft due to the Earth are retained.

4.5.1 POINT-MASS NEWTONIAN ACCELERATION

The point-mass Newtonian acceleration of a near-Earth spacecraft relative to the center of mass of the Earth in the local geocentric space-time frame of reference is calculated the same as in the Solar-System barycentric space-time frame of reference as described in Section 4.4.1 (when the center of integration in the barycentric frame is the Earth). The point-mass Newtonian acceleration is the acceleration of the near-Earth spacecraft calculated from Eq. (4–27) minus the acceleration of the Earth calculated from the same equation. Terms are obtained for the Sun, Mercury, Venus, the Earth, the Moon, the planetary systems Mars through Pluto, asteroids, and comets. The Earth accelerates the spacecraft. The remaining bodies accelerate the spacecraft relative to the Earth. If the element of the PERB array for any of these bodies is 0, or an asteroid or a comet is not included in the XBPERB array, the acceleration due to that body is not calculated. The only difference from the calculations in the barycentric frame is that the value of the gravitational constant of the Earth in the local geocentric frame is calculated from the value in the barycentric frame (obtained from the planetary ephemeris) using Eq. (4–25).

The independent variable for the equations of motion in the geocentric frame of reference is coordinate time t_{GC} of the geocentric frame. However, the time argument for interpolating the planetary ephemeris for the position vectors of the perturbing bodies is coordinate time t_{BC} of the barycentric frame. It could be obtained by adding $t_{BC} - t_{GC}$ to t_{GC}. From Section 4.3.3, the time difference $t_{BC} - t_{GC}$ is given by the right-hand side of Eq. (2–23) with the constant 32.184 s deleted. For this application, the clock synchronization term, which is the third dot product term, is evaluated with the geocentric space-fixed position vector of the near-Earth spacecraft. The remaining terms of Eq. (2–23) are periodic terms. The time difference $t_{BC} - t_{GC}$ affects the position vectors of the perturbing bodies and hence the acceleration of a near-Earth spacecraft relative to the Earth. For a GPS satellite, this effect is of order 10^{-18} relative to the Newtonian acceleration of the satellite due to the Earth, which is negligible. Therefore, in the local geocentric space-time frame of reference, the planetary ephemeris can be interpolated with coordinate time t_{GC} of the geocentric frame in order to obtain the position vectors of the perturbing bodies.

Lengths and times in the Solar-System barycentric space-time frame of reference are smaller than those of the local geocentric space-time frame of reference by the factor $1 + \tilde{L}$ (*i.e.*, the barycentric frame values are the geocentric frame values divided by this factor), where $\tilde{L}$ is given by Eq. (4–17). From Eq. (4–25), gravitational constants in the barycentric frame are also smaller than those of the local geocentric frame by the same factor $1 + \tilde{L}$. The point-mass Newtonian acceleration of a near-Earth spacecraft relative to the Earth due to all perturbing bodies except the Earth is computed from gravitational constants and distances in the barycentric frame (both obtained from the planetary ephemeris). This differential inverse radius-squared perturbative acceleration is high by the factor $1 + \tilde{L}$, and can be converted to the correct value in the local geocentric frame of reference by multiplying it by $1 - \tilde{L}$. For a GPS satellite, the resulting correction is of order 10^{-13} relative to the Newtonian acceleration of the satellite due to the Earth. It is doubtful if such a small effect could be seen in the data and hence, the point-mass Newtonian acceleration of a near-Earth spacecraft relative to the Earth due to all perturbing bodies except the Earth is not multiplied by the correction factor $1 - \tilde{L}$. The point-mass Newtonian acceleration of a near-Earth spacecraft due to the Earth is computed from the gravitational constant of the Earth in the geocentric frame calculated from Eq. (4–25) and the geocentric radius to the spacecraft represented in the geocentric frame. Hence, this calculation is correct in the geocentric frame.

4.5.2 POINT-MASS RELATIVISTIC PERTURBATIVE ACCELERATION DUE TO THE EARTH

HRTW (1990) show that the n-body point-mass relativistic perturbative acceleration in the Solar-System barycentric space-time frame of reference reduces to the relativistic perturbative acceleration obtained from the one-body Schwarzschild isotropic metric for the Earth (specified in this section) plus the acceleration due to geodesic precession (specified in the next section) in the local geocentric space-time frame of reference.

The n-body point-mass metric tensor is given by Eqs. (2–1) to (2–12). Simplifying these equations to the case of one massive body (the Earth) and a massless particle (a near-Earth spacecraft) and substituting them into Eqs. (2–13) to (2–15) for the interval ds gives:

$$ds^2 = \left(1 - \frac{2\mu_{\mathrm{E}}}{c^2 r} + \frac{2\beta \mu_{\mathrm{E}}^2}{c^4 r^2}\right) c^2 dt^2 - \left(1 + \frac{2\gamma \mu_{\mathrm{E}}}{c^2 r}\right)\left(dx^2 + dy^2 + dz^2\right) \qquad (4\text{–}60)$$

where the subscript i has been removed from the coordinates of the spacecraft. The gravitational constant of the Earth μ_{E} in the local geocentric frame of reference is calculated from the corresponding value in the Solar-System barycentric frame of reference using Eq. (4–25). When β and γ are equal to their general relativistic values of unity, this is the Schwarzschild isotropic one-body point-mass metric, which has been expanded, retaining all terms to order $1/c^2$. See Moyer (1971), Eq. (8). Dividing Eq. (4–60) by dt^2 according to Eq. (4–30) and denoting dx/dt as $\dot{x}$, etc., gives the expression for the square of the Lagrangian L. Differentiation of L^2 gives expressions for $L\partial L/\partial x$, $L\partial L/\partial \dot{x}$, and $L\dot{L}$. Also, the second of these three expressions must be differentiated with respect to coordinate time t of the local geocentric frame. Substituting all four of these expressions into Eqs. (4–36) and (4–37) gives the point-mass equations of motion due to the Earth in the local geocentric frame of reference. Subtracting the point-mass Newtonian acceleration of a near-Earth spacecraft due to the Earth gives the following expression for the point-mass relativistic perturbative acceleration of a near-Earth spacecraft in the local geocentric space-time frame of reference:

$$\ddot{\mathbf{r}} = \frac{\mu_{\mathrm{E}}}{c^2 r^3}\left\{\left[2(\beta + \gamma)\frac{\mu_{\mathrm{E}}}{r} - \gamma \dot{s}^2\right]\mathbf{r} + 2(1+\gamma)(\mathbf{r} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}}\right\} \qquad (4\text{–}61)$$

This same equation can be obtained by simplifying Eq. (4–26) to the case of one perturbing body (the Earth) and removing the Newtonian term. In Eq. (4–61),

$\mathbf{r}$, $\dot{\mathbf{r}}$	=	geocentric space-fixed position and velocity vectors of near-Earth spacecraft
r	=	magnitude of $\mathbf{r}$
$\dot{s}$	=	magnitude of $\dot{\mathbf{r}}$

For an Earth satellite, the relativistic perturbative acceleration given by Eq. (4–61) will always be less than 10^{-8} times the Newtonian acceleration of the satellite. It will usually be of order 10^{-9} or smaller.

4.5.3 GEODESIC PRECESSION

Geodesic precession was introduced in Section 4.4.2. In the Solar-System barycentric space-time frame of reference, the acceleration due to geodesic precession is included in the point-mass relativistic perturbative acceleration calculated from Eq. (4–26). However, in the local geocentric space-time frame of reference, it must be calculated separately.

The precession rate of the north pole $\mathbf{S}$ of the orbit of an Earth satellite about the normal to the ecliptic is given by Eq. (4–39), where $\mathbf{\Omega}$ is the angular velocity vector due to geodesic precession. From Will (1981), p. 209, Eq. (9.5), the first term,

$$\mathbf{\Omega} = \frac{1}{c^2}\left(\gamma + \frac{1}{2}\right)\sum_j \dot{\mathbf{r}}_{\mathrm{E}}^j \times \nabla\left(\frac{\mu_j}{r_{\mathrm{E}j}}\right) \tag{4–62}$$

where $\mathbf{r}_{\mathrm{E}}^j$ and $\dot{\mathbf{r}}_{\mathrm{E}}^j$ are space-fixed position and velocity vectors of the Earth relative to body j, $r_{\mathrm{E}j}$ is the magnitude of $\mathbf{r}_{\mathrm{E}}^j$, and μ_j is the gravitational constant of body j. The second vector in the cross product is the gradient of the gravitational potential $U > 0$ at the Earth due to body j. In Eq. (4–62), the only body j which can produce an acceleration of a near-Earth spacecraft greater than order 10^{-14} relative to the Newtonian acceleration of the spacecraft is the Sun. Setting Ω equal to the term due to the Sun (j = S) and evaluating that term gives:

$$\boldsymbol{\Omega} = \frac{\mu_{\mathrm{S}}\left(\gamma + \frac{1}{2}\right)}{c^2 r_{\mathrm{ES}}{}^3}\left(\mathbf{r}_{\mathrm{E}}^{\mathrm{S}} \times \dot{\mathbf{r}}_{\mathrm{E}}^{\mathrm{S}}\right) \tag{4–63}$$

When $\gamma = 1$, this equation is equal to Eq. (43) of HRTW (1990). Eq. (4–63) in the form of one term of Eq. (4–62) is given in Misner, Thorne, and Wheeler (1973), Eq. (40.33b), term 3, and Eq. (40.34), line 3. When reading the references given in this section, consider the geocentric orbit of the Earth satellite to be a gyroscope in orbit about the Sun.

The inertial geocentric frame of reference is rotating with the angular velocity Ω given by Eq. (4–63) relative to the Solar-System barycentric frame of reference. However, the ODP uses a non-inertial geocentric frame of reference, which is non-rotating relative to the barycentric frame of reference. When formulating the equations of motion in the non-inertial geocentric frame of reference, it must be considered to be rotating with the angular velocity – Ω. Hence, in addition to the usual equations of motion in the non-rotating geocentric frame of reference, we must add the centrifugal acceleration $-\omega \times \omega \times \mathbf{r}$ and the Coriolis acceleration $-2\omega \times \dot{\mathbf{r}}$, where the angular velocity ω of the coordinate system relative to the inertial frame is – Ω. The ratio of the centrifugal acceleration to the Newtonian acceleration increases with distance from the Earth. For a GPS satellite, it is of order 10^{-21}, which is negligible. The Coriolis acceleration of a near-Earth spacecraft due to geodesic precession is:

$$\ddot{\mathbf{r}} = 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} \tag{4–64}$$

where Ω is given by Eq. (4–63) and $\dot{\mathbf{r}}$ is the space-fixed geocentric velocity vector of the near-Earth spacecraft. The ratio of this acceleration to the Newtonian acceleration increases with distance from the Earth. For a GPS satellite, it is about 4×10^{-11}. Eq. (4–64) is also given by the second term of Eq. (40) of HRTW (1990).

The position and velocity vectors in Eq. (4–63) are interpolated from the planetary ephemeris, and the additional velocity vector in Eq. (4–64) is interpolated from the geocentric spacecraft ephemeris. The argument for each of these interpolations is coordinate time t_{GC} of the local geocentric space-time frame of reference.

4.5.4 LENSE-THIRRING PRECESSION

The acceleration of a near-Earth spacecraft due to the Lense-Thirring precession is calculated from the formulation of Section 4.4.3, specifically Eqs. (4–43) and (4–45). In Eq. (4–43), the geocentric space-fixed position and velocity vectors of the near-Earth spacecraft are interpolated from the geocentric spacecraft ephemeris using coordinate time t_{GC} in the local geocentric frame of reference as the argument. In Eq. (4–45), T_E is the rotation matrix from Earth-fixed coordinates referred to the true pole, prime meridian, and equator of date to the space-fixed coordinate system of the planetary ephemeris. It is calculated from the formulation given in Section 5.3. The time argument for calculating T_E is coordinate time ET (coordinate time t_{BC} of the Solar-System barycentric frame or coordinate time t_{GC} of the local geocentric frame). It will be seen in Section 5.3 that the internal time transformation from the argument ET to universal time UT1 used in calculating T_E in the local geocentric frame of reference is different from the time transformation used in calculating T_E in the Solar-System barycentric frame of reference. Furthermore, the time transformation used in program PV in the barycentric frame is simpler than the one used in program Regres in that frame. Because the acceleration due to the Lense-Thirring precession is so small (see Section 4.4.3), the gravitational constant of the Earth in Eq. (4–43) can be the value in the Solar-System barycentric frame or the corresponding value in the local geocentric frame computed from Eq. (4–25).

4.5.5 NEWTONIAN ACCELERATION OF NEAR-EARTH SPACECRAFT DUE TO THE HARMONIC COEFFICIENTS OF THE EARTH AND THE MOON

In the local geocentric space-time frame of reference, the acceleration of a near-Earth spacecraft due to the oblateness of the Earth and the Moon is calculated from the Newtonian model of Section 4.4.4. In Eq. (4–46), the space-fixed position vector **r** of the spacecraft relative to the center of integration (the Earth in the local geocentric frame of reference) is interpolated from the geocentric spacecraft ephemeris as a function of coordinate time t_{GC} of the local geocentric frame of reference. The second term of Eq. (4–46) is interpolated from the planetary ephemeris as a function of t_{GC}. When the oblate body B is the Earth E, the second term of Eq. (4–46) is zero. When the oblate body B is the Moon M, the second term of Eq. (4–46) is the geocentric position vector of the Moon.

In calculating the acceleration of a near-Earth spacecraft due to the oblateness of the Earth, the gravitational constant of the Earth must be the value in the local geocentric frame of reference, calculated from the corresponding value in the Solar-System barycentric frame using Eq. (4–25). In calculating the acceleration due to the oblateness of the Moon, the gravitational constant of the Moon can be the value in the Solar-System barycentric frame of reference, obtained from the planetary ephemeris. The same value must be used in the next section in calculating the acceleration of the Earth due to the oblateness of the Earth and the Moon.

In Eqs. (4–48), (4–51), and (4–52), the Earth-fixed to space-fixed transformation matrix T_{E} and the Moon-fixed to space-fixed transformation matrix T_{M} are evaluated from the formulations given in Sections 5.3 and 6.3, respectively, as a function of coordinate time t_{GC} of the local geocentric frame of reference. The correct argument for evaluating T_{E} and T_{M} is coordinate time t_{BC} of the Solar-System barycentric frame of reference. Approximating it with t_{GC} produces errors in the calculated oblateness accelerations which are of order 10^{-16} relative to the Newtonian acceleration of the spacecraft due to the Earth.

4.5.6 ACCELERATION OF THE CENTER OF INTEGRATION DUE TO OBLATENESS

In the local geocentric frame of reference, the center of integration is the Earth. The acceleration of the Earth due to oblateness accounts for the oblateness of the Earth and the Moon. This acceleration is calculated from the formulation of Section 4.4.6, specifically Eq. (4–58). In this equation, the acceleration of the point-mass Earth due to the oblateness of the Moon and the acceleration of the point-mass Moon due to the oblateness of the Earth are calculated from the Newtonian model of Section 4.4.4. Both of these calculations require the geocentric space-fixed position vector of the Moon. To sufficient accuracy, it can be interpolated from the planetary ephemeris using coordinate time t_{GC} of the local geocentric frame of reference as the argument. Also, to sufficient accuracy, t_{GC} can be used as the argument for calculating the body-fixed to space-fixed transformation matrix T_E for the Earth and T_M for the Moon.

In Eq. (4–58), the acceleration of the Earth due to the oblateness of the Earth and the Moon is proportional to the gravitational constant of the Moon. It can be the value in the Solar-System barycentric frame of reference, which is the same value used in the preceding section to calculate the acceleration of a near-Earth spacecraft due to the oblateness of the Moon.

The negative of the acceleration of the Earth due to the oblateness of the Earth and the Moon is a contribution to the acceleration of a near-Earth spacecraft relative to the Earth in the local geocentric frame of reference.