AD/RHIC/AP/TN-86

Accelerator Development Department
BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, NY 11973

AD RHIC Accelerator Physics Technical Note No. 86

Comments on the Effect of Random Errors

A.G. Ruggiero

November 1989

COMMENTS ON THE EFFECT OF RANDOM ERRORS

A.G. Ruggiero

I am somewhat uneasy with the statement that random multiple errors in superconducting magnets cause betatron tune shifts and spreads, and that these tune shifts and spreads are to be taken as criteria on the tolerances of the random errors. If this is so one may end up with quite stringent tolerances indeed.

I like to argue here that only systematic errors cause tune shifts and that random errors do not. Of course all this is due to a definition of systematicity and randomness we all have to agree upon.

To make my point, I will consider the case of octupole errors in the regular arc quadrupoles and dipoles of RHIC. Also I will consider the betatron tune shift (spread) in the horizontal plane for on momentum particles. To first order approximation, which I assume correct as long as the shifts are small, the tune variation due to the quadrupole errors is given by

$$\Delta\nu_H = \frac{3}{8}\ \epsilon_H \ \sum_i \beta_{Hi}^2 b_{3i}$$

where

ϵ_H horizontal emittance

$\sum_i$ is the sum over all the errors assuming they are lumped in the center of quadrupoles

β_{Hi} beta–value of the i–th quadrupole in the horizontal plane

b_{3i} normalized octupole error in the i–th quadrupole

β_{Hi} can take only one of two values: β_{HF} or β_{HD}, depending whether the errors is in a QF or QD magnet respectively. Thus

$$\Delta\nu_H = \frac{3}{8}\ \epsilon_H\ \beta^2_{HF} \underbrace{\sum_i b_{3i}}_{\text{on QF's}}\ +$$

$$+\frac{3}{8}\ \epsilon_H\ \beta^2_{HD} \underbrace{\sum_i b_{3i}}_{\text{QD's}} \tag{1}$$

There are 72 QF–quads and 72 QD–quads. One can also carry a similar expression for the contribution of the b_3 errors in the arc dipoles, which would be

$$\Delta\nu_H = \frac{3}{8}\ \epsilon_H \underbrace{(\sum_i b_{3i})}_{\text{B's}}\ \overline{\beta}^2_H \tag{2}$$

where the sum is over the $N_B = 144$ dipoles and $\overline{\beta}^2_H = \frac{1}{\ell_B} \int \beta^2_H ds$ over the dipole length. This quantity is the same for all dipoles.

The situation is somewhat more complicate for the insertion magnets, because there the lattice function β_H has a less regular behavior. Let us leave for the moment out the contribution of the insertion quads. Let us continue with (1) and (2) where clearly the tune–shift is given directly by the summation of the errors

$$\sum_i b_{3i}$$

over the proper type of magnets.

In practice, and hopefully, the errors b_{3i} will be given from measurements once we have acquired all the magnets. For instance in the case of dipole magnets we shall receive a string of 144 values which will have an average value $\overline{b}_3$ and a more or less random distribution around the average with an rms value $< b_3 >$, so that we can write for any one error

$$b_{3i} = \overline{b}_3 + \delta b_{3i}$$

where δb_{3i} is the deviation from the average. By definition thus

$$\sum_i \delta b_{3i} = 0 \quad . \tag{3}$$

I like to identify here the average value $\bar{b}_3$ of the 144 errors as the systematic error and $< b_3 >$ as the width of the random errors. At this stage we can only speculate on the magnitude of these two errors but their definition is operative and make sense only after we have acquired the magnets and we know their errors.

But, most important, the average value of the random errors is identically zero. Let us see the consequence of this to the tune–shift. Take for instance eq. (2) which we can write as

$$\Delta\nu_H = \frac{3}{8}\ \epsilon_H\ \overline{\beta}_H^2\ \underbrace{\sum_i (\bar{b}_3 + \delta b_{3i})}_{\text{over B's}}$$

$$= \Delta\nu_{H_{sys}} \ + \ \Delta\nu_{H_{rand}}$$

with

$$\Delta\nu_{H_{sys}} \ = \ \frac{3}{8}\ \epsilon_H\ N_B\ \overline{\beta}_H^2 \bar{b}_3$$

$$\Delta\nu_{H_{rand}} \ = \ \frac{3}{8}\ \epsilon_H\ \overline{\beta}_H^2\ \underbrace{\sum_i \delta b_{3i}}_{\text{over B's}}$$

Because of (3), we see that $\Delta\nu_{H_{rand}} = 0$, that is random errors as defined above do not give a contribution to the betatron tune–shift. This is done only by the systematic error $\bar{b}_3 \neq 0$.

In absence of RHIC and its magnets, we are left with paper exercise and computer simulations. In order to study the effects of systematic errors we introduce a number for each multiple which has the same value for all the magnets of the same type (say dipoles). We vary this value and we plot or register the beam behavior versus the variation and we provide a tolerance criterion. I do not see any problem with this approach and actually we have verified that computer simulations and analytical estimates do indeed agree and that the tune–shifts caused by systematic errors are also good reasons for concern and that they are the criteria for magnet tolerances. Also we have seen that the systematic errors cause significant distortion of lattice functions and a limitation on the dynamical aperture.

I perceive a different story when we do computer simulations to study the effect of the so–called random errors. For instance in the case of the arc dipoles 144 numbers, for a given multiple error, are generated by the computer in a randomly fashion. Nevertheless the computer has a peculiar way of generating random numbers that usually is given for granted and not questioned. The statistical method used by the computer will provide always a significant average value with an expectation value (63% confidence level) to be at least equal to the rms value of the distribution divided by the square root of the number of the errors wanted. Thus in the case of 144 dipoles, when one requires errors distributed over a width of 1 unit, the computer will generate also an average of at least 0.083. This number can be four times larger when the confidence level in the statistics is raised to 98% level, that is one third of the assumed width of the errors. The situation is somewhat worse for each of the two families of quadrupoles QF and QD since they are fewer.

In a computer simulation, at least so far, one has always included an average value of the error distribution which according to the definition given above, is equivalent to a systematic error.

The relative magnitude of this average with respect to the desired width of the distribution may vary from run to run and it has never been checked. It could be that the effects noticed on the beam are due to the statistically introduced average error and not to the random component. I am for sure convinced that the observed tune shifts are due to average values and that the claim they are caused by the width is wrong. I will prove this later (at least I will try).

Let us turn our attention for the moment to Table I. This shows the rms values $< b_n >$ of the random errors b_n in dipoles versus the multiple order n used in tracking. Next we

Table I. Random & Systematic Errors in RHIC Dipoles (in cm^{-n})

n	$< b_n >$	$\bar{b}_n$	$b_n^{(s)}$	$\bar{b}_n^{(t)}$	$\bar{b}_n^{(t)} / < b_n >$	M
1	8.0×10^{-5}	6.7×10^{-6}		2.8×10^{-6}	0.035	800
2	7.4	6.2×10^{-6}	1.5×10^{-5}	2.2	0.030	1000
3	8.1	6.8		1.8	0.022	2000
4	5.6×10^{-6}	4.7×10^{-7}	5.4×10^{-7}	1.4	0.256	15
5	5.4×10^{-7}	4.5×10^{-8}		1.1	2.1	1
6	3.4	2.8×10^{-8}	2.5×10^{-8}	8.2×10^{-7}	2.4	1
7	3.0×10^{-8}	2.5×10^{-9}		6.1	20	1
8	1.8	1.5×10^{-9}	1.3×10^{-9}	4.3	24	1
9	1.6×10^{-9}	1.3×10^{-10}		3.2	197	1
10	9.8×10^{-10}	8.1×10^{-11}	2.0×10^{-10}			

give the expectation values $\bar{b}_n$ of average values from 144 numbers randomly generated at the computer. We also give the <u>systematic</u> values $b_n^{(s)}$ of the errors used in tracking. Observe that the expectation values of averages from random numbers are comparable to the used <u>systematic</u> values. All these values appear in Tables IV.3–3 and 4 of the RHIC CDR. Continuing the inspection of Table I we give in one column the tolerances $\bar{b}_n^{(t)}$ of the <u>systematic</u> values for b_n according to our recent estimate based on the calculation of the tune–shifts (Table II of AD/RHIC–58), followed by the systematic/random ratio $\bar{b}_n^{(t)} / < b_n >$. Interesting to note that this ratio increases very fast with the multiple order. The square of the inverse of the ratio gives an estimate on the number of magnets M to be produced and measured to arrive at a statistically significant estimate of averages and rms widths of the errors. For instance, in order to get a good confidence of the performance of the dipoles with respect to the octupole error b_3, essentially all the magnets will have to be produced and measured. For the decapole error b_4, 15 magnets are to be produced, but this number would correspond only to a 63% confidence level. Twice that number would be preferable.

Going back to our issue, it is seen that the tolerance given on the systematic octupole $\bar{b}_3^{(t)}$ is smaller by a factor of four than the value of the average $\bar{b}_3$ one can statistically expect from the generation by computer of 144 random numbers. And this is exactly the point of my concern.!

Computer usually generate random numbers u (variates) with uniform distribution in the range between 0 and 1. To generate random numbers x with gaussian distribution of "zero average" and "rms width 1" (normal distribution), the following procedure is often used. Two variates u_1 and u_2 are generated by the computer and two random numbers x_1 and x_2 satisfying the normal distribution are given by

$$x_1 \doteq (-2\log u_1)^{1/2} \quad \cos 2\pi u_2$$
$$x_2 = (-2\log u_1)^{1/2} \quad \sin 2\pi u_2$$

I always follow this procedure. When numbers are generated in this fashion their absolute magnitude is checked, and if any one is larger than 2.5, the pair itself is rejected and replaced by a new one satisfying the criterion.

We have generated in this fashion several sequences of random numbers with the computer, each sequence is being made of 144 numbers. We, of course, demanded zero–average and one unit for rms width. The actual results are given in Table II.

In this table

$$\text{average} = \frac{1}{144}\sum_{i=1}^{144} b_i = \bar{b}$$

$$\text{rms width} = \left\{ \frac{\sum_{i=1}^{144}(b_i - \bar{b})^2}{144} \right\} = < b >$$

The statistically expected average is

$$1/\sqrt{144} = 0.083$$

Table II. Computer Generated Random Number Statistics

Sequence No.	Average	rms Width	$\Delta\nu_H$
1	-.1326	0.963	-0.00947
2	-.1123	1.062	-0.00802
3	-.0107	.997	-0.00076
4	0.0285	1.055	0.00204
5	-0.0592	0.0982	-0.00423
6	-0.01316	0.998	-0.00940
7	0.0355	0.868	0.00254
8	0.0779	0.986	0.00557
9	-0.0798	.907	0.00570

The fluctuation on the observed rms width I believe is of no consequence to the beam dynamics. More to the point is the observed presence of the statistically significant average which causes indeed a betatron tune–shift as reported on the last column of Table II. These shift values are of magnitude comparable to those found by George Parzen. My suspicion is that Parzen's results have been biased by the presence of average values in the generated random numbers.

At this point I question seriously the validity of the work done in the past also for what concern tracking and the search for the dynamical aperture. All the results we have seen before are really due to the average (systematic) effects embedded in the generation of the random number.

To resolve the issue, one should repeat the analysis but in the following way. To study the effect of random errors for a particular multiple order and for a particular type of magnet, one generates as usual random numbers at the computer, but before attempting anything, each of the computer generated number is modified by subtracting the common average value. These modified numbers are those to be used (i) for estimating tune–shifts, and (ii) for tracking to determine dynamic aperture.