An Approach for Delaunay Tetrahedralization of Bodies with Curved Boundaries

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Summary. Problem of tetrahedral meshing of three-dimensional domains whose boundaries are curved surfaces is wide open. Traditional approach consists in an approximation of curved boundaries by piecewise linear boundaries before mesh generation. As the result mesh quality may deteriorate. This paper presents a technique for Delaunay-based tetrahedralization in which a set of constrained facets is formed dynamically during face recovery and mechanisms for mutual retriangulation of the curved faces and the tetrahedralization are suggested. The proposed algorithm is constructed in such a way that a facet that was once added in the set of constrained facets is never split into small triangles. It allows retaining the high quality of surface mesh in the tetrahedralization, because during boundary recovery the surface mesh on the curved faces and the tetrahedralization are refined conjointly.

Key words: Delaunay tetrahedralization, curved boundary, boundary recovery, tetrahedral meshes, meshing of parametric models

List of Abbreviations and Symbols

t_{ps}^{Γ} Triangulation in the parametric space of some curved face Γ of input body for which the Delaunay tetrahedralization is being constructed.
t_{3d}^{Γ} An image of t_{ps}^{Γ}in three-dimensional space.
T_{Γ} A set of triangular faces of tetrahedra that are representation of curved face Γ in the tetrahedralization (usually does not coincide with t_{3d}^{Γ}).

1 Introduction

The Delaunay tetrahedralization is widely used for the mesh generation. But the Delaunay tetrahedralization is defined for bodies with piecewise linear boundaries. And when it is needed to tetrahedralize a body with curved boundary usually the input body is approximated with facets and piecewise linear complex (PLC) is created. Then the tetrahedralization is build from the linear model using either the advancing front method or the Delaunay triangulation method [GE91, DU04]. This paper concerns the Delaunay tetrahedralization method in which tetrahedralization is build from the point set and then boundary recovery is performed, while the advancing front method is not considered.

Direct use of the approach when a body with curved boundaries is approximated with piecewise linear elements has two disadvantages. First disadvantage is connected with the fact that the piecewise linear approximation of the input body might have geometrical artifacts (e.g. degenerate and disconnected facets). It might happen, especially if the linear model is taken from a CAD system. However this disadvantage is not essential since 1) there are methods that allow generating surface mesh without geometrical artifacts and 2) CAD data repairing methods are developed.

Second and more serious disadvantage is the fact that quality of resulting tetrahedral mesh is very sensitive to the quality of the initial surface mesh. It occurs because curved face is approximated with linear elements and all linear elements both facets and edges must present in the final tetrahedralization. And if the piecewise linear approximation contains a facet with an acute angle, then this angle remains in tetrahedralization. Thus surface mesh of tetrahedralization cannot be better then the input surface mesh because during tetrahedralization remeshing is possible only within a facet. However, it should be pointed out that there is a common approach when small coplanar facets are unified into one facet. And only this new facet is used during mesh generation instead of the set of the small facets. In some cases it allows significantly improving the quality of the final tetrahedralization. But facets from the approximation of curved face can rarely be unified into one.

Also it should be mentioned that there are methods that allow reconstructing of surface mesh and improving its quality at the same time (e.g. [BE02, FR00]). But all improvements are made before mesh generation and during the tetrahedralization the reconstruction of surface mesh is not executed and each facet is considered individually what can lead to poor quality of the final mesh. Quality deteriorates when Steiner points are inserted into the facets.

Nevertheless, the approach when a body with curved boundaries is approximated with piecewise linear elements is widely used. Usually for general bodies if the initial surface mesh is small enough, then after the tetrahedralization of an initial point set most of facets is already recovered and others can be recovered by adding a few Steiner points. So the quality does not become much worse. Problems can arise 1) if the size of the initial surface mesh is larger than required size of the tetrahedral mesh or 2) in the place where two faces subtend a sharp dihedral angle. In these cases many Steiner are generally added.

This paper considers issues of the Delaunay constrained tetrahedralization for bodies with curved boundaries given analytically, i.e. for each edge a corresponding curve is specified and for each face corresponding parametrization of a surface is specified. Major part of the paper is focused on the problem of boundary recovery.

In comparison with the described above approach when a set of constrained facets F is specified before mesh generation and boundary recovery is performed for the set F, this paper suggests an approach in which the set F is formed dynamically during face recovery and facets (triangles) are added to F when it is known that they have already been recovered. Furthermore, the algorithm is constructed in such a way that a triangle that was once added in F is never split into small triangles. Hence the surface mesh is transferred to the tetrahedralization without changes.

Face recovery is based on comparison of the face triangulation in the parametric space with the tetrahedral mesh. However, it is not required that each triangle of the two-dimensional triangulation in the parametric space presents as a face of some tetrahedron in the tetrahedralization because it is not always possible to get such conformity. An algorithm, which is described in section 3, seeks for the triangles that will be an approximation of the curved face in three-dimensional space. The search is carried out among faces of tetrahedrons of the constructed tetrahedralization. Requirements for such faces are discussed in section 2. A set of the appropriate faces of tetrahedrons may not only consist of images of the triangles of the triangulation in the parametric space. If all appropriate faces are found (it means that a curved face is recovered), then they are added to the set of the constrained facets F, otherwise the curved face is additionally refined. A crucial difference of the suggested approach is the fact that the facets from the approximation of a curved face are added to F if and only if all of them present in the tetrahedralization what means that the curved face is recovered. It allows retaining the high quality of surface mesh in the tetrahedralization, because during boundary recovery the surface mesh on the curved faces and the tetrahedralization are refined conjointly.

As it was noted before, small coplanar facets can be unified into one facet. And only new facet is used during mesh generation instead of the set of the small facets. Actually, the paper proposes an extension of this approach and the facets are unified if they belong to the same curved face and mechanisms for mutual retriangulation of the curved faces and the tetrahedralization are suggested. We compare face triangulation in the parametric space with the tetrahedral mesh as if we tried to reconstruct the triangulation in the parametric space in accordance with the configuration really occurred in the three-dimensional space.

2 Initial mesh generation

The generation of the constrained Delaunay tetrahedralization begins from the Delaunay tetrahedralization of the point set lying on boundary of body. So an initial triangular surface mesh must be constructed prior tetrahedral mesh generation. Since the surface mesh will be compared with the tetrahedralization it is important to choose appropriate surface meshing method, which must satisfy two criteria. Firstly, the generated surface mesh must be as closer as possible to the Delaunay triangulation on a curved surface [CH93]. Secondly, method for surface meshing must allow inserting a new vertex into the existed surface triangulation. The authors use two-dimensional anisotropic mesh generation in the parametric space of surface [CH97, BO96].

As the result of the surface mesh generation a set of 3D vertices is obtained. Next step (see Fig. 1) is the Delaunay tetrahedralization for the set of vertices. Specifically, a three-dimensional algorithm based on flips was used [JO91]. The acquired tetrahedralization will be an initial mesh. As the Delaunay triangulation is the strictly defined triangulation, which does not allow vertices to be connected arbitrary, next step is the boundary recovery in the three-dimensional space.

In the spatial case the boundary recovery problem consists of two tasks: edge recovery and face recovery. In our algorithm the recovery of missing edges must be done prior to the face recovery. Edge recovery is not the complex task for the current state of mesh generation and it has been discussed in many papers. In the particular case a variant of Rupert's algorithm [SC02] was used with modifications for handling curved boundaries [BO02] and with some generalizations to three dimensions (e.g. diametrical spheres were used instead of diametrical circles).

In our algorithm constructing Delaunay tetrahedralization requires the maintenance of tetrahedralization and a separate triangulation in the parametric space of each curved face.

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Generate surface mesh of input body;
Build the Delaunay tetrahedralization of the vertices lying on surface of body;
Perform recovery of missing curved edges;
Perform first stage of curved faces recovery;
For each curved face Γ_k do
        Perform second stage of curved face recovery for face Γ_k; enddo
Insert Steiner points inside of body;
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Fig. 1. An overview of the algorithm for constructing Delaunay tetrahedralization of bodies with curved boundaries.

3 Recovery of missing curved faces

Consider the recovery of missing curved faces, which is the most complex problem of the boundary recovery. First of all, it should be discussed what means that a curved face is recovered. A curved edge (face) will be considered to be recovered in the tetrahedralization if there is a piecewise linear approximation of the curved edge (face) and each linear element of it is present in the tetrahedralization. For a curved edge linear elements are segments, which present as edges of tetrahedra. For a curved face Γ linear elements are triangles and they must present as faces of tetrahedra in the tetrahedralization. A set T_Γ of such triangular faces of tetrahedra must satisfy the requirements:

I. Vertices of the triangular faces lie on the curved face Γ.
II. Each 3D vertex that are an image of some vertex from the triangulation t_{ps}^{Γ} in the parametric space belongs to at least one triangle from the set T_Γ .
III. Interiors of the triangular faces do not intersect each other.
IV. Each edge of triangular face from the set T_Γ that is a segment of the approximation of some external curved edge of Γ belongs to only one triangular face from the set T_Γ . Such edges are called external.
V. Each edge of triangular face from the set T_Γ that is not external belongs to only two triangular faces from the set T_Γ . Such edges are called internal.
VI. Consider triangular faces from T_Γ that share the same vertex v lying on Γ and their edges. These triangular faces form a subset S. For each edge e that has the vertex v sum how many triangular faces from the subset S have the edge e. Internal edge must belong to only two triangular faces from the subset S, whereas external edge must belong to only one triangular face. This requirement is implied by conditions IV and V, nevertheless, it has been formulated here since it will be used in the section 3.2.

These six requirements are substitute for conditions of absence of gaps and self-intersections in comparison with the definition of two-dimensional triangulation.

Thus for the curved face Γ three different objects are defined: triangulation t_{ps}^{Γ} in the parametric space, its image in three-dimensional space t_{3d}^{Γ} and the set T_{Γ} , which is the representation (or approximation) of the curved face Γ in the tetrahedralization. Generally, the sets T_{Γ} and t_{3d}^{Γ} do not coincide.

During face recovery our goal is to find the set T_{Γ} (if it exists) or reconstruct the mesh in such a way that the approximation T_{Γ} is appeared in the tetrahedralization. Faces are recovered in two stages. During the first stage all curved faces are handled together while during the second stage curved faces are handled one by one.

3.1 First stage of curved faces recovery

In some algorithms for constructing Delaunay tetrahedralization equatorial spheres are used for the boundary recovery [SC98]. The equatorial sphere of a triangle is the smallest sphere passes through the three vertices of the triangle. A triangle is encroached if a vertex lies inside or on its equatorial sphere. If a triangle is not encroached it appears in tetrahedralization as a triangular face of tetrahedron. This idea with some modifications can be used for curved faces recovery.

Definition 1. *A vertex v and a triangle t is said to be adjacent if at least one vertex of t and v lie on the same curved face (see Fig. 2).*

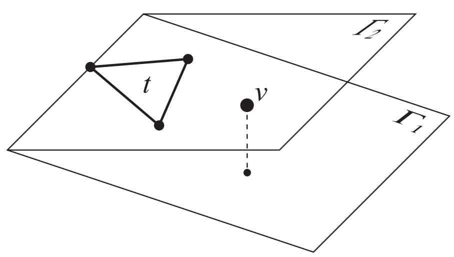

Fig. 2. A triangle t lying on a face Γ_1 and a vertex v lying on a face Γ_2 are adjacent.

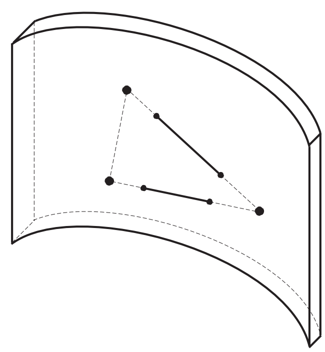

Fig. 3. If two curved faces are close to each other then it may be situation when a triangle of t_{3d}^{Γ} of one face intersects other face. The first stage allows significantly reducing a number of such triangles or eliminating them at all.

During first stage triangles of triangulations t_{3d}^{Γ} and their equatorial spheres are examined. The equatorial sphere of each triangle can contain only allowed vertices. All adjacent vertices of a triangle are allowed to lie inside its equatorial sphere. If the equatorial sphere of a triangle t^{3d} contains a prohibited vertex v^{3d} then a corresponding triangle t^{ps} in the parametric space is split by inserting a new vertex w^{ps}

at the center of circumscribed ellipse of t^{ps} [CH97, BO96]. And a three-dimensional vertex w^{3d} that is an image of w^{ps} is also inserted into the tetrahedralization. Insertion a new vertex both into t_{ps}^{Γ} and into the tetrahedralization implies performing all necessary actions (e.g. flips) to maintain the Delaunay property.

If the new vertex w^{ps} falls within a triangle Δt^{ps}; and if Δt^{3d} and v^{3d} are adjacent, the vertex w^{ps} is not inserted; instead v^{3d} is allowed to lie inside of the equatorial sphere of t^{3d}.

The first stage is executed until all triangles of all curved faces have the equatorial spheres containing only allowed vertices. Since we do not require that the equatorial spheres must strictly be empty the endless cycle of infinite encroachment does not occur and the first stage always terminates. (All modifications have been introduced only to prevent the endless cycle.)

In case of curved boundaries the first stage has very important meaning. If two curved faces are close to each other then a situation is possible when a triangle of t_{3d}^{Γ} of one curved face intersects other curved face (see Fig. 3). The first stage allows significantly reducing a number of such triangles or eliminating them at all (see Fig. 4). Actually, if two curved faces Γ_1 and Γ_2 are close to each other then it is not generally allowed equatorial spheres of triangles of Γ_1 contain vertices of Γ_2. So two triangulations of Γ_1 and Γ_2 "feel" each other and they are conjointly refined during the first stage.

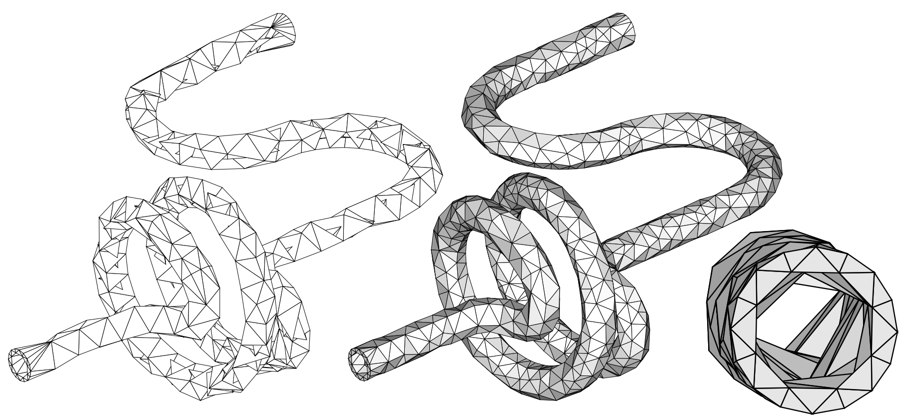

Fig. 4. An example of how the algorithm works with thin-walled bodies. At left, a coarse mesh that consists of triangulations t_{3d}^{Γ} of input body before the first stage of recovery is depicted. The mesh has self-intersections. At right, a tetrahedralization after face recovery (including the second stage) is depicted. During the first stage the surface mesh had been refined in such a way that it became without self-intersections. Although the final tetrahedralization is still coarse but it is valid.

A direct check on self-intersections is not performed because of computational complexity of this task. Thus there is no theoretical guarantee that self-intersections will be removed during the first stage in all cases. Nevertheless, our experiments with thin-walled bodies demonstrated that the algorithm perfectly works with such bodies. Moreover, some self-intersections can be removed during the second stage of face recovery.

It should be mentioned that after the first stage the tetrahedralization is strictly Delaunay.

3.2 Second stage of curved face recovery

Curved faces are recovered one by one and recovery procedure does not switch to the next curved face until previous face is fully recovered. The recovery procedure does not spoil once recovered faces. Each curved face Γ is recovered as follows. First, a list that contains triangular faces of tetrahedra being images of triangles of the triangulation t_{ps}^{Γ} in the parametric space is created. These triangular faces are marked as triangles belonging to the approximation T_{Γ} of the curved face Γ. If each triangle in the parametric space is present in the tetrahedralization as a triangular face and an obtained set of triangular faces satisfies conditions of topological integrity of mesh (see below), then the curved face is already recovered in the tetrahedralization. If the list is empty, i.e. no triangles of t_{ps}^{Γ} are present in the tetrahedralization, then it means that the triangulation in the parametric space t_{ps}^{Γ} differs sharply from the Delaunay triangulation. So it is why so important to choose an appropriate method for surface meshing. In this case the additional refinement of the mesh t_{ps}^{Γ} is performed and the refinement is stopped if at least one triangle of the triangulation t_{ps}^{Γ} is present in the tetrahedralization as a triangular face of some tetrahedron. Usually a few additional points need to be added. It has to be mentioned that face triangulation t_{ps}^{Γ} is refined together with the tetrahedralization.

Then a main loop of recovery procedure is started. While the list is non-empty the following actions take place. A triangular face t is removed from the beginning of the list. For each internal edge e of t other triangular face that will be a neighbor for the triangular face t through the edge e has to be found. First, the search is performed among already marked triangles. And if such triangular face exists (it can be only one), then corresponding neighbor is found and next internal edge of t is considered. Certainly, many triangular faces of tetrahedralization can share the same edge e but only two marked triangular faces can share the edge e (see the requirement V) including the triangular face t. So for a marked triangular face through an internal edge only one neighbor is possible. The requirement V is always fulfilled for marked triangles if surface of face does not have singularities and seam edges (see details in section 3.3).

If the neighbor has not been found among marked triangles, then triangular faces, satisfying conditions of topological integrity of mesh, are searched among all triangular faces that have the edge e. These triangular faces will be candidates to be the neighbor for the triangular face t through the edge e.

Elucidate aforesaid in figure 6. At left a fragment of triangulation t_{ps}^{Γ} of curved face in the parametric space is shown, while at right a corresponding configuration in the three-dimensional space. For a triangle AMN from the parametric space a corresponding triangular face $A'M'N'$, which is a face of some tetrahedron of tetrahedralization, exists. So the triangular face $A'M'N'$ is marked as belonging to the approximation T_{Γ} of the curved face and pushed into the list. Let the triangular face $A'M'N'$ be now extracted from the list. Adjoining triangular faces that also will belong to the approximation T_{Γ} must be found for the triangular face $A'M'N'$. For this purpose internal edges of $A'M'N'$ are examined. Consider one such edge

Find triangular faces of tetrahedra being images of triangles of t_{ps}^{Γ};
Mark them as belonging to T_{Γ};
Add them to the end of the list L;
While the list L is nonempty do //It is a main loop
    Take first element (triangular face t) of the list L and remove it from the list L;
    For each internal edge e_i of t, $i \in \{0, 1, 2\}$
        If (other triangular face that shares the same edge e_i exists)
            {go to the next edge e_{i+1};}
        Else Switch (how many possible triangular faces to be the neighbor for t through e_i have been found)
            1: {marked this triangular face as belonging to T_{Γ} and add it to the end of the list L;}
            0 or > 1: {neighbor is not found}
        end Switch
        end If
    end For
    If (at least one neighbor is not found)
        {Add t to the end of the list L; } end If
    If (a pass over the list does not give any new marked triangular faces)
        {curved face is not recovered, break;} end If
end do
If (face is not recovered) {Remesh curved face and tetrahedralization; repeat the second stage again;} end If
Curved face has been successfully recovered

Fig. 5. Second stage of curved face recovery.

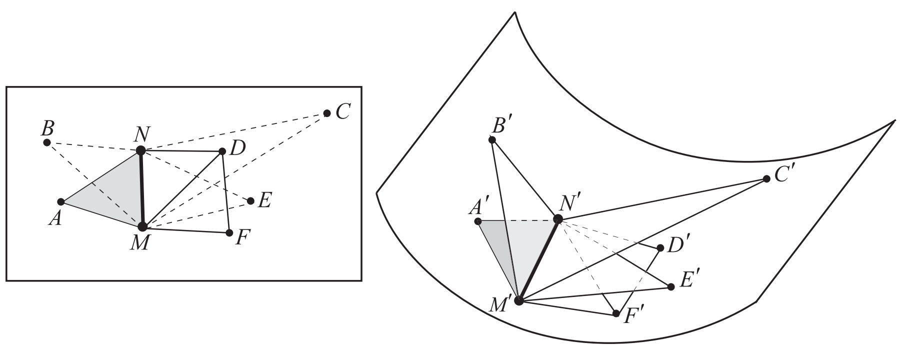

Fig. 6. Left: Fragment of two-dimensional triangulation of curved face in the parametric space. Right: the same configuration in the three-dimensional space.

$M'N'$. First, it is checked if there is other marked triangular face with the edge $M'N'$. Such triangle face can be only one by construction. If the marked triangle face has been found, then it means that the neighbor for the triangular face $A'M'N'$ through the edge $M'N'$ is found.

Suppose the marked triangle face has not been found. Return to the figure 6. The edge MN also belongs to the triangle MND but there is no triangular face $M'N'D'$ in the three-dimensional space. Instead of it the triangular face $M'N'F'$ presents in the tetrahedralization but a corresponding triangle MNF is absent from the parametric space. Besides the triangular face $A'M'N'$ and $M'N'F'$, the edge $M'N'$ also belongs to following triangular faces of tetrahedra: $M'N'B'$, $M'N'C'$ and $M'N'E'$. All of them ($M'N'F'$, $M'N'B'$, $M'N'C'$ and $M'N'E'$) are candidates to be the neighbor for the triangular face $A'M'N'$ through the edge $M'N'$. But only one of them can really become such neighbor. Among all possible candidates find those that meet conditions of topological integrity of mesh. These conditions are similar to requirements described at the beginning of the section 3.

Condition 1. Vertices of the triangular face lie on curved face that is being recovered.

Condition 2. A vertex (C' see Figure 7) of a triangular face-candidate ($M'N'C'$) that is opposite of considering edge ($M'N'$) is not fully recovered. A vertex is said to be fully recovered if it satisfies with the requirement VI provided that the set T_Γ is constituted only from marked triangular faces adjoining the vertex. Thus the vertex C' depicted in figure 7 is fully recovered and the triangular face-candidate $M'N'C'$ will be rejected as not meeting the condition 2.

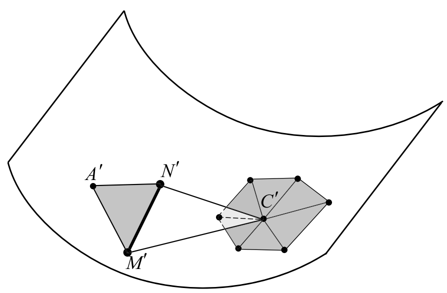

Fig. 7. A vertex C' is fully recovered. Therefore, a triangular face $M'N'C'$ will be rejected as not meeting the condition 2.

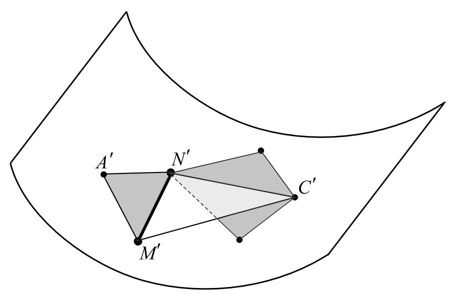

Fig. 8. An edge of tetrahedron $N'C'$ is fully recovered. Therefore, a triangular face $M'N'C'$ will be rejected as not meeting the condition 3.

Condition 3. Two edges of a triangular face-candidate ($M'C'$, $N'C'$ see Figure 8) connecting vertices (M' and N') to opposite vertex (C') of the considering edge ($M'N'$) are not fully recovered. An internal edge is said to be fully recovered if it belongs to two marked triangular faces. An external edge is said to be fully recovered if it belongs to one marked triangular face. This condition is equivalent to the requirements III and IV. Illustrated in figure 8 the edge $N'C'$ is fully recovered and the triangular face-candidate $M'N'C'$ will be rejected as not meeting the condition 3.

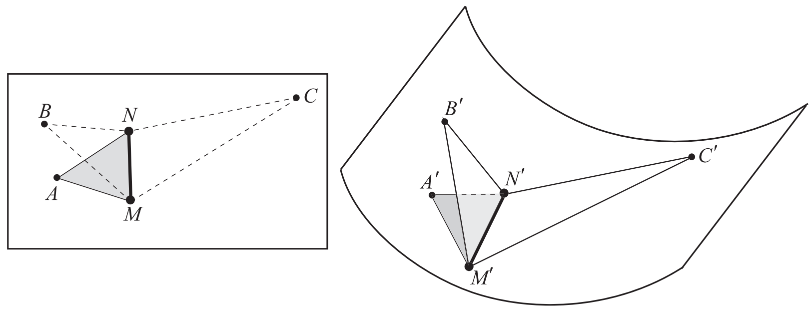

Fig. 9. Marked faces of tetrahedra must not produce mesh in the parametric space with self-intersections.

Condition 4. A vertex (A' see Figure 9) of extracted from the list triangular face ($A'M'N'$) and a vertex of a triangular face-candidate (B' or C'), which are opposite of the considering edge ($M'N'$), lie in the parametric space in different half-planes separated by line (MN) that connects the vertices of the considering edge in the parametric space. Thus the triangular face-candidate $B'M'N'$ does not meet condition 4, whereas the triangular face-candidate $M'N'C'$ meets this condition.

It is easy to notice that conditions 2 and 3 do not allow an internal edge to belong to more than two marked triangular faces and an external edge to belong to more than one marked triangular face.

The condition 4 differs from other conditions because it is not a part of the requirements described at the beginning of the section 3. The first three conditions are needed to guarantee that obtained approximation T_Γ is correct, whereas the condition 4 allows finding the approximation T_Γ that is different from t_{3d}^Γ. Thus if the condition 4 were not required, face recovery would not be possible in many cases, because it rarely occurs when each triangle of the triangulation t_{ps}^Γ is present in the tetrahedralization as a triangular face.

3.3 Handling seams and singularities

There are some difficulties with checking of the condition 4. These difficulties are connected with the fact that mapping from the parametric space to the three-dimensional space may not be biunique. Usually non-uniqueness arises at the boundaries of parametrization domain and can be two types. First type is seam edges. For example (see Figure 10), when a cylinder with a hole is unfolded, edges A_1B_1, A_2B_2, C_1D_1 and C_2D_2 are arisen in the parametric space what leads to creation of artificial seam edges $A'B'$ and $C'D'$ in the three-dimensional space. It takes place because parametrization of cylinder is periodic in one parameter. Seam edges in the parametric space must be handled together in such a way that it will be possible to join surface mesh in the three-dimensional space.

Second type of non-uniqueness is degeneracy (or singularity) on a surface. Singularity is a place where the parametrization collapses so that a range of parameter values corresponds to a single three-dimensional point. For example, sphere has two singularities corresponding with its poles. Cone has one singularity (its vertex).

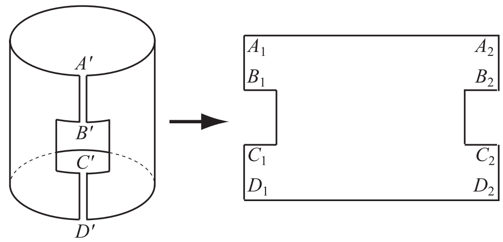

Fig. 10. When a cylinder with a hole is unfolded, edges A_1B_1, A_2B_2, C_1D_1 and C_2D_2 are arisen in the parametric space what leads to creation of artificial seam edges $A'B'$ and $C'D'$ in the three-dimensional space.

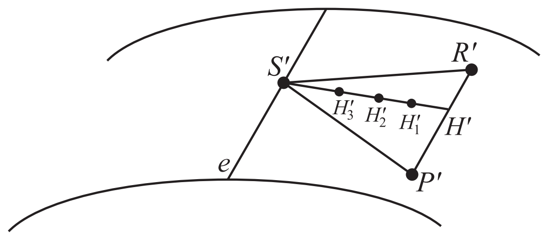

Fig. 11. Determination of parametric coordinates (u, v) for a point S' lying on a seam edge e.

In case of seam edge usually it is possible to perform backward transformation and find parametric coordinates (u, v) for a point lying on a seam edge e. Let u be the periodic coordinate. Let S' be a vertex of triangular face $P'R'S'$ and S' lies on seam edge (fig. 11). Let H' be any point lying in the interior of segment $P'R'$. Consider three points H_1', H_2' and H_3' lying in the interior of $S'H'$ so that $d(S', H_3') < d(S', H_2') < d(S', H_1')$, where d — distance between points, and for these points find the closest points $(H_1(u_1, v_1), H_2(u_2, v_2), H_3(u_3, v_3))$ lying on the curved face. If $u_1 < u_2 < u_3$, then $S(u_S, v_S)$ lies in the parametric space on a seam edge with the largest value of periodic coordinate, if $u_1 > u_2 > u_3$, then $S(u_S, v_S)$ lies on a seam edge with the lowest value of periodic coordinate; otherwise the periodic coordinate cannot be determined. The value of parametric coordinate depends on triangular face in which a point lies. For example, if we consider other triangular face to which the vertex S' belongs, then the value of the periodic coordinate might be other.

In case of singularities it is not possible to find parametric coordinates for a point lying on degeneracy. If the condition 4 cannot be checked, assume that it is fulfilled. A main problem of curved face recovery is that usually more than one neighbor is possible. Our task is to find only one appropriate neighbor using some criteria (four conditions). If condition 4 is assumed to be fulfilled, it makes the task of face recovery be more difficult. If condition 4 is not checked, it does not lead to incorrect mesh, since conditions 1-3 guarantee that that obtained approximation T_Γ is correct.

Presence of seam edges and singularities is the cause why it is impossible to introduce a definition determining that a curved face Γ is recovered if every non-degenerate triangle of the triangulation t_{ps}^{Γ} in the parametric space is present as a triangular face of tetrahedron in the tetrahedralization. A non-degenerate triangle is a triangle with three different 3D vertices (some vertices of t_{ps}^{Γ} may coincide in the three-dimensional space). Examine a triangulation t_{ps}^{Γ} in the parametric space depicted in figure 12 and suppose that every triangle has corresponding triangular face in the three-dimensional space. Let e_1 and e_2 be seam edges, which coincide in

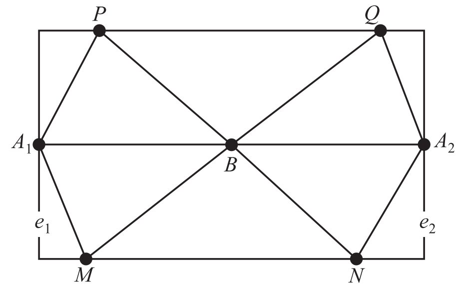

Fig. 12. Two-dimensional triangulation of a curved face in the parametric space t_{ps}^{Γ}. Edges e_1 and e_2 are seam edges. If each of the triangles A_1PB, A_1MB, BQA_2 and BNA_2 is present as a triangular face of some tetrahedron in the tetrahedralization, then four triangular faces of the set T_{Γ} share the same edge $A'B'$.

the three-dimensional space. In this case an image of the vertices A_1 and A_2 is the same point A' in the three-dimensional space. Respectively, edges A_1B and BA_2 are the same edge $A'B'$ of some tetrahedron. As the result four triangular faces that are a part of the approximation T_{Γ} share the same edge $A'B'$ but it is inadmissible. Thus even if every triangle has corresponding triangular face in three-dimensional space it does not exempt from necessity to check conditions of topological integrity of mesh. So before starting the main loop of recovery, already marked triangles must be checked if they meet requirements IV and V. If not, additional refinement of the face mesh is performed.

3.4 Remeshing curved face and tetrahedralization

Return to the face recovery method. If only one triangular face is remained after all four conditions have been checked then this triangular face is marked as belonging to the approximation T_{Γ} and added to the end of the list, then the next internal edge of t is considered. If more then one or no triangular faces fulfill all four conditions, the extracted triangular face t, for which neighbors have been searched, is added to the end of the list again. The triangular face t is added to the end of the list (not to the beginning) since others triangular faces of the list should be examined before t is considered again. They can produce new marked triangular faces and when t is extracted again, neighbors for t might be found in spite of the fact that previous attempt to do it ended in failure.

The main loop continues until one more pass over the list gives any new marked triangular faces (the recursion is stopped to prevent endless cycle) or the list becomes empty. In the former case the curved face is not recovered and a new vertex should be inserted into the triangulation t_{ps}^{Γ} and the tetrahedralization to locally change triangulation in a non-recovered area.

The list contains triangular faces for which neighbors have not been found yet, so when the list is empty, it means that separate marked triangles are joined to each other and a curved face is recovered. Triangular faces that represent a curved face and are in the tetrahedralization (they are called "marked" triangles and have abbreviation T_{Γ}) should be stored somewhere. These triangular faces are added to the set of constrained facets F. Also they are locked to prevent their deletion during maintaining the Delaunay property. Since the triangular faces are locked, the recovery of a next face never makes a previously recovered face disappear. Usually this approach leads to the constrained Delaunay tetrahedralization. But it has been shown [JO89] that in some cases "locking" faces does not always give the constrained Delaunay tetrahedralization. For example, during a 3-2 flip, when two triangular faces are eliminated and a new is created, if the first face is locked whereas the second is not, then the flip cannot be performed and the second triangular face might not be locally Delaunay. Nevertheless, this fact is not essential. First, if an input body does not have an edge that belongs to only one face (e.g. no internal boundaries) then the second triangular face will also be locked after a corresponding curved face is recovered. Second, usually the Delaunay tetrahedralization itself is rarely required since it is not optimal in three-dimensional space. After tetrahedral mesh improvement (smoothing and local transformations) it looses all properties of Delaunay triangulation anyway.

Additional refinement of curved face and tetrahedralization is performed in the following cases.

1. If none of triangles of the triangulation t_{ps}^{Γ} is present in the tetrahedralization. A few points must be inserted into the triangulation t_{ps}^{Γ} and their images into the tetrahedralization. Then it is necessary to check if this requirement is satisfied.

2. To resolve the configuration depicted in figure 12. In that case a bad triangle (e.g. $\triangle A_1PB$) is eliminated by inserting a new vertex at the center of circumscribed ellipse of the bad triangle [CH97, BO96]. A three-dimensional image of the inserted vertex must also be inserted into the tetrahedralization. Insertion a new vertex both into t_{ps}^{Γ} and into the tetrahedralization implies performing all necessary actions (e.g. flips) to maintain the Delaunay property.

3. When an attempt to find the neighbor for the triangle t through the edge e ended in failure. There are two subcases here. 3a) The triangular face t has a corresponding triangle t_{Γ} in the triangulation t_{ps}^{Γ}. In this subcase the edge e has a corresponding edge e_{Γ}. The edge e_{Γ} belongs to two triangles: t_{Γ} and Δt_{Γ}. A new vertex is inserted at the center of circumscribed ellipse of the triangle Δt_{Γ}. 3b) This subcase occurs when the subcase 3a is not applicable. Take the edge e and find its midpoint. For the midpoint find the closest point $p(u, v)$ lying on the face. For the point $p(u, v)$ find a triangle Δt_{Γ} of the triangulation t_{ps}^{Γ} in which the point lies. A new vertex is inserted at the center of circumscribed ellipse of the triangle Δt_{Γ}. A three-dimensional image of the inserted vertex must also be inserted into the tetrahedralization. After the new vertex is inserted the second stage must be repeated.

4. Aforesaid method for the face recovery does not guarantee that each three-dimensional vertex, being an image of some vertex of the triangulation t_{ps}^{Γ} in the parametric space, belongs to at least one marked triangular face (requirement II). A roundabout path not including some vertices lying on the curved face may be found. To resolve this situation for vertices that are missed in T_{Γ} additional refinement of face mesh around them must be done. Then the second stage must be repeated.

When mesh is being refined, curved edges must remain to be recovered.

So the face recovery procedure is the heuristic iterative process. During one iteration the triangular faces that belong to face three-dimensional triangulation T_{Γ} are looked for. The triangulation in the parametric space t_{ps}^{Γ} is compared with the tetrahedralization and corresponding triangular faces are marked. It is similar as if we tried to reconstruct triangulation t_{ps}^{Γ} in accordance with that actually takes place in the three-dimensional space.

3.5 Termination

In this section some possible termination issues are discussed.

In order to start the main loop of the recovery procedure it is required that at least one triangle of the triangulation t_{ps}^{Γ} is present in the tetrahedralization as a triangular face of some tetrahedron. While this usually works out in practice; there is no guarantee, in theory, no matter how finely face mesh is refined.

The main loop of recovery is terminated when all neighbors for all marked triangles are found. It might get stuck in an attempt to find a neighbor for some triangular face.

To investigate these issues, this algorithm has been tested for real-world bodies with curved geometry. Bodies contained up to 2000 curved faces with different geometries. Surfaces of curved faces varied form the simplest (planes, cylinders, cones and spheres) to fairly complicated (tori, B-surfaces, swept and spun surfaces). Our attempts to find a body for which the algorithm doesn't work out ended in failure. It always terminated.

Nevertheless, in order to guarantee termination in all cases this algorithm is deliberately stopped to prevent endless loop in the following cases.

1) When quantity of point of the triangulation t_{ps}^{Γ} has been doubled but no triangles of the triangulation t_{ps}^{Γ} are present in the tetrahedralization.

2) When quantity of the marked triangular faces for which neighbors are not found is slightly changed from iteration to iteration. After algorithm is stopped curved face is replaced with piecewise linear approximation and each facet is recovered separately. It should be emphasize that there was no need to do it as these two cases never occurred and the algorithm terminated for all bodies on which it was tested.

So described in the paper algorithm can be regarded as a preprocessor step before boundary recovery using piecewise linear complex. Most likely, boundary recovery will be done during the preprocessor step; if not, piecewise linear complex can be considered.

Carried out tests demonstrated good performance. For example, for bodies depicted in an appendix mesh generation time does not exceed some minutes on a computer with Pentium IV processor. And this time includes all stages of mesh generation: surface mesh generation, initial mesh generation, boundary recovery and refinement of tetrahedral mesh.

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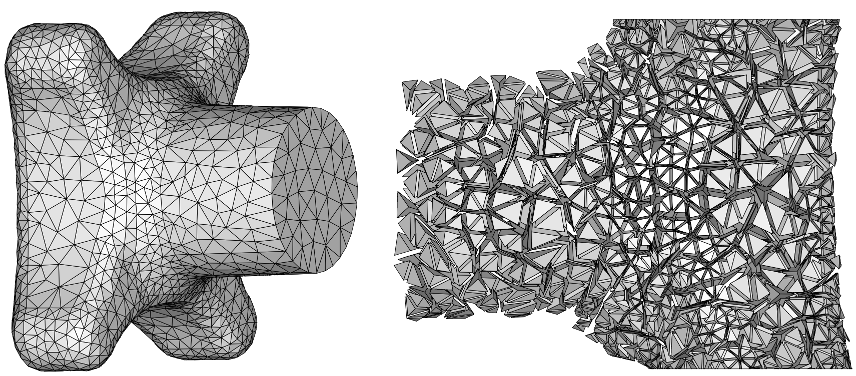

Fig. 13. Example 1. Boundary of the body consists of 51 curved faces separated by 120 curved edges. Mesh contains 2920 vertices and 10760 tetrahedra.

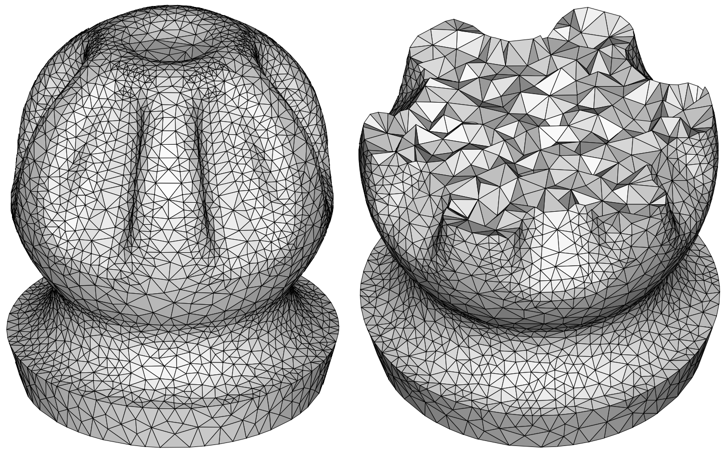

Fig. 14. Example 2. Boundary of the body consists of 87 curved faces separated by 204 curved edges. Mesh contains 7489 vertices and 34082 tetrahedra.

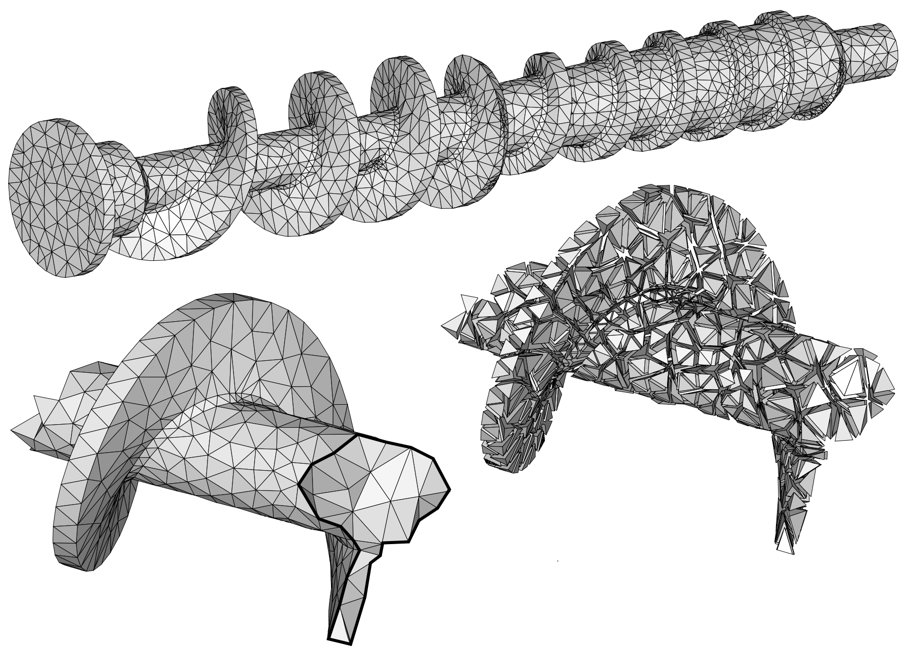

Fig. 15. Example 3. Boundary of the body consists of 75 curved faces separated by 186 curved edges. Mesh contains 4989 vertices and 18856 tetrahedra.

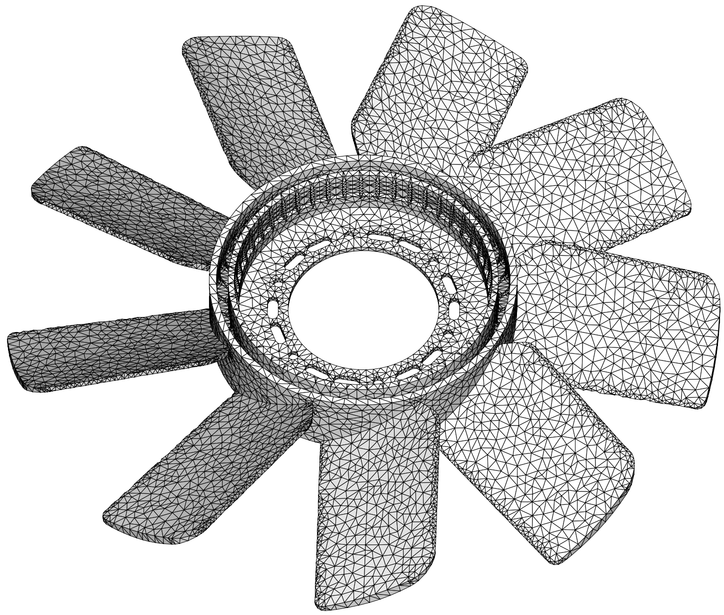

Fig. 16. Example 4. Boundary of the body consists of 753 curved faces separated by 1982 curved edges. Mesh contains 18571 vertices and 53444 tetrahedra.