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A SEMI-IMPLICIT SCHEME FOR GRID POINT ATMOSPHERIC MODELS OF THE PRIMITIVE EQUATIONS

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ABSTRACT

A semi-implicit time integration scheme tested earlier with a spectral model is now adapted to a grid point model of the primitive equations. Predictions prepared by the implicit method compare quite favorably with the forecasts produced by an explicit technique. The implicit model runs about four times faster; and after 5 days of integration, the forecasts differ by less than 20 m.

1. INTRODUCTION

Motion in compressible fluids can generate time oscillations extending over a very broad spectrum of frequencies. Numerical models concerned with a simulation of the essential features of the fluid motion need to carry only that part of the frequency spectrum that directly affects the large-scale circulations.

The filtering approximations used in the early atmospheric models eliminated both sound and gravity waves, but they produced equations that became difficult to integrate. Current models retain the gravity modes, provide greater accuracy, and do not require any transformation of the basic equations for the numerical calculations.

Unfortunately, the primitive equations models will not give stable solutions with long time steps. The short 10-min steps used in these models reduce the truncation errors to a level two orders of magnitude lower than the error level associated with the spatial resolution.

Scientists in the Soviet Union devoted a considerable amount of time to studies of stable integration algorithms. The work of Marchuk (1965) in this field deserves special attention. In the United States, Holton (1967) and Kurihara (1965) also tested simple implicit models with success. All of these schemes allow long time steps and retain the main advantages of the primitive equations formulation. The particular method discussed in this report differs only slightly from one of the techniques proposed by Kurihara (1965).

2. THE METHOD OF INTEGRATION

Oscillatory systems generally behave in accordance with relatively simple physical laws. Trigonometric functions or complex exponential functions represent this type of motion adequately, and a first-order differential equation will describe the time variations accurately. This equation will be given the following form

$$\frac{\partial F}{\partial t}=i\omega F \tag{1}$$

where the constant ω represents the frequency of the oscillation. Using the method of Kurihara (1965), we will arbitrarily break down this frequency into two components α and β:

$$\alpha+\beta=\omega, \tag{2}$$

and we will use the following finite-difference algorithm to approximate eq (1):

$$\frac{F(t+\Delta t)-F(t-\Delta t)}{2\Delta t}=i\alpha F(t)+i\beta\left[\frac{F(t+\Delta t)+F(t-\Delta t)}{2}\right]. \tag{3}$$

We must note here that, in the absence of the β term in the above equation, we have a centered time step. When α vanishes, we have a fully implicit step; and we may consider that β represents that part of the frequency ω that we will treat implicitly.

For simple wave motion, we will use the prediction operator X that transforms $F(t)$ into $F(t+\Delta t)$. In terms of this operator, (3) becomes an algebraic equation

$$(1-i\beta\Delta t)X^2-2i\alpha\Delta tX-(1+i\beta\Delta t)=0, \tag{4}$$

and this equation gives the two solutions

$$X=\frac{i\alpha\Delta t\pm\sqrt{1+\beta^2\Delta t^2-\alpha^2\Delta t^2}}{1-i\beta\Delta t}. \tag{5}$$

The properties of the two solutions depend on the condition

$$\alpha^2\Delta t^2\leq 1+\beta^2\Delta t^2. \tag{6}$$

If this condition is satisfied, the magnitude of the two roots will be unity, and the numerical integration will remain stable.

Consider now a high-frequency perturbation where the restoring influence of gravity is accounted for by β while the advective effect of a basic current U is included in α:

$$k^2U^2\Delta t^2\leq 1+k^2gh_0\Delta t^2. \tag{7}$$

Here, h_0 represents the mean depth of the fluid, and k

is the wave number of the perturbation. The condition given above will be satisfied if

$$|U| \leq \sqrt{gh_0}, \tag{8}$$

and clearly this condition presents no problem in the atmosphere. If we make an allowance for the rotation of the earth in the model or let the temperature vary with latitude, these changes will alter the phase speed of gravity waves. If we incorporate these effects in α, then the left-hand side of eq (6) will increase, but not sufficiently to violate this condition. We conclude that the proposed treatment of the gravity waves will remain stable under nearly all conditions.

We may now consider the meteorological perturbations. Since gravity has little effect on the phase speed of these waves, we may delete the corresponding term from eq (7). With centered finite differences of second order in space, this equation becomes

$$U^2 \frac{\Delta t^2}{d^2} \sin^2 (kd) \leq 1 \tag{9}$$

where d is the mesh length of the grid. This condition is the criterion normally used with filtered models.

The above analysis suggests that all terms in the meteorological equations should be evaluated explicitly with only a few exceptions. Gravity waves operate through the pressure gradient term and the divergence term. These two quantities should be averaged in time as proposed in the β term of eq (3).

3. THE MODELING EQUATIONS

We will use the primitive meteorological equations and assume that the atmosphere is incompressible and inviscid. Shuman and Stackpole (1969) proposed that the resulting equations be written in the following form

$$K = \tfrac{1}{2} (u^2 + v^2), \tag{10}$$

$$Q = f + m^2 \left[\frac{\partial}{\partial x} \left(\frac{v}{m} \right) - \frac{\partial}{\partial y} \left(\frac{u}{m} \right) \right], \tag{11}$$

$$\frac{\partial u}{\partial t} = -m \frac{\partial \phi}{\partial x} + Qv - m \frac{\partial K}{\partial x}, \tag{12}$$

$$\frac{\partial v}{\partial t} = -m \frac{\partial \phi}{\partial y} - Qu - m \frac{\partial K}{\partial y}, \tag{13}$$

and

$$\frac{\partial \phi}{\partial t} = -m^2 \left[\frac{\partial}{\partial x} \left(\frac{\phi u}{m} \right) + \frac{\partial}{\partial y} \left(\frac{\phi v}{m} \right) \right] \tag{14}$$

that they call the invariant form of the meteorological equations. This set of equations is valid in any conformal projection, and the variables are defined as: x and y are the coordinates of the projection, u and v are the components of the wind vector along the axes of the coordinate system, ϕ is the geopotential, m the map scale, K the kinetic energy, and Q is the absolute vorticity.

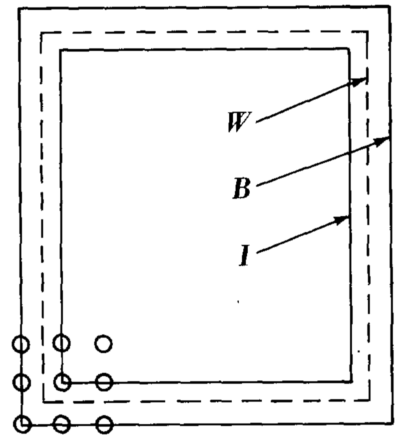

FIGURE 1.—Rectangular 47×51 grid point network. The dashed line W represents the solid walls in the fluid. The full line B represents the boundary of the grid, and I represents the first set of interior points.

One advantage of the invariant form is that the gradient of the map scale is incorporated into the evaluation of vorticity and divergence.

4. THE INITIALIZATION

The numerical integration will be performed on the grid shown in figure 1. It is assumed that the fluid is bounded by solid walls located along the dashed line W. The stream function ψ is the only field used as initial data. Since this stream function normally gives flow through the wall, we replace it by a modified version ψ'. We first define the average value of the stream function along a closed curve

$$\tilde{\psi} = \frac{\oint \psi ds}{\oint ds}. \tag{15}$$

The average value along the wall is then obtained from the averages around the rectangles B and I:

$$\tilde{\psi}_W = \tfrac{1}{2} (\tilde{\psi}_B + \tilde{\psi}_I). \tag{16}$$

By using a distance R of five grid lengths, the stream function is modified as

$$\psi' = \psi \text{ if } r > R, \tag{17}$$

$$\psi' = \tilde{\psi}_W + \frac{r}{R} (\psi - \tilde{\psi}_W) \quad \text{if } 0 < r < R, \tag{18}$$

and

$$\psi_B' = 2\tilde{\psi}_W - \psi_I' \tag{19}$$

where r is the shortest distance to the wall. An influence region of five grid lengths keeps the boundary region smooth and does not produce unreasonable values for vorticity near the boundaries.

All finite-difference calculations will use the three algorithms

$$\overline{F}^{\alpha}=\tfrac{1}{2}\,[F(\alpha+\tfrac{1}{2}\Delta\alpha)+F(\alpha-\tfrac{1}{2}\Delta\alpha)], \tag{20}$$

$$F_{\alpha}=\frac{1}{\Delta\alpha}\,[F(\alpha+\tfrac{1}{2}\Delta\alpha)-F(\alpha-\tfrac{1}{2}\Delta\alpha)], \tag{21}$$

and

$$\overline{F}^{2\alpha}=\tfrac{1}{2}[F(\alpha+\Delta\alpha)+F(\alpha-\Delta\alpha)] \tag{22}$$

where α stands for any one of the independent variables.

The winds are then calculated as

$$u=-m\overline{\psi'_y}^{x}, \tag{23}$$

$$v=m\overline{\psi'_x}^{y}, \tag{24}$$

$$U^*=\frac{u}{m}, \tag{25}$$

$$V^*=\frac{v}{m}, \tag{26}$$

$$U=\overline{U^*}^{xy}, \tag{27}$$

and

$$V=\overline{V^*}^{xy}. \tag{28}$$

The intermediate variables U^* and V^* are evaluated in the middle of the grid squares while U and V are grid point values. The model uses the variables U and V; and to obtain values at the boundary points, we use the following conditions along the walls parallel to the y axis:

$$\overline{U}^{x}=0 \tag{29}$$

and

$$V_x=0. \tag{30}$$

A similar procedure is applied to the other two walls.

To obtain the geopotential ϕ, we use the balance equation:

$$K=\frac{m^2}{2}\,(U^2+V^2), \tag{31}$$

$$Q=\overline{f}^{xy}+\overline{m^2}^{xy}(\overline{V}^{y}_{x}-\overline{U}^{x}_{y}), \tag{32}$$

and

$$\overline{\phi}^{yy}_{xx}+\overline{\phi}^{xx}_{yy}=\overline{(Q\overline{V}^{xy}-\overline{K}^{y}_{x})_x}^{y}-\overline{(Q\overline{U}^{xy}+\overline{K}^{x}_{y})_y}^{x}. \tag{33}$$

We solve eq (33) for ϕ by relaxation with the following condition along the walls parallel to the y axis:

$$\phi_x=\overline{f}^{x}\overline{V}^{x}. \tag{34}$$

A similar condition is used along the two other walls.

The stream function used for this model is not produced from an analysis of the wind observations. It is determined by applying an initialization scheme to an analysis of the geopotential. This conversion is performed on a routine basis for the filtered baroclinic model used in Canada. Unfortunately, our operational initialization scheme does not use the same finite-difference approximations as those appearing in eq (33). We use eq (33) in the present set of experiments simply to obtain fields balanced under the same conditions as those used in the model.

With these modifications, the winds are perfectly nondivergent initially and at the end of the first time step. This property virtually eliminates the gravity waves from the integration. On the other hand, the geopotential used by the model differs slightly from the originally analyzed geopotential. These discrepancies will vanish when we adapt our operational initialization scheme to the model described here.

5. THE MODEL EXPERIMENTS

The semi-implicit numerical integration scheme was tested by comparison with an explicit model. A model developed by Shuman and Stackpole (1969) was used for this purpose. By using the operators defined earlier, we can write this model as

$$\overline{U}^{t}_{t}=\overline{-\overline{\phi}^{y}_{x}+Q\overline{V}^{xy}-\overline{K}^{y}_{x}}^{xy}, \tag{35}$$

$$\overline{V}^{t}_{t}=\overline{-\overline{\phi}^{x}_{y}-Q\overline{U}^{xy}-\overline{K}^{x}_{y}}^{xy}, \tag{36}$$

and

$$\overline{\phi}^{t}_{t}=\overline{-\overline{m^2}^{xy}[\overline{\phi}^{xy}(\overline{U}^{y}_{x}+\overline{V}^{x}_{y})+\overline{U}^{xy}\overline{\phi}^{y}_{x}+\overline{V}^{xy}\overline{\phi}^{x}_{y}]}^{xy}. \tag{37}$$

The numerical integration of these equations is straightforward.

For the implicit model, we will use equations

$$\overline{U^*}^{t}_{t}=-\overline{\phi}^{y2t}_{x}+Q\overline{V}^{xy}-\overline{K}^{y}_{x}, \tag{38}$$

$$\overline{V^*}^{t}_{t}=-\overline{\phi}^{x2t}_{y}-Q\overline{U}^{xy}-\overline{K}^{x}_{y}, \tag{39}$$

and

$$\overline{\phi}^{t}_{t}=-m^2\phi_0\overline{(\overline{U^*}^{y}_{x}+\overline{V^*}^{x}_{y})}^{2t} - \overline{\overline{m^2}^{xy}[\overline{(\phi-\phi_0)}^{xy}(\overline{U}^{y}_{x}+\overline{V}^{x}_{y})+\overline{U}^{xy}\overline{\phi}^{y}_{x}+\overline{V}^{xy}\overline{\phi}^{x}_{y}]}^{xy}. \tag{40}$$

In these equations, ϕ_0 is a constant and is assigned a value of 56000 $m^2\ s^{-2}$ for predictions of the 500-mb flow. The actual value of this constant is not very important as long as it is near the mean value of ϕ over the grid.

The implicit model operates in terms of the variables U^* and V^* internally. The wind components U and V are obtained from eq (27) and (28) when required. Both models use the boundary conditions defined by eq (29), (30), and (34).

The numerical procedure used to perform the time step in the implicit model has already been described by Robert (1969). It will be repeated here for convenience:

$$\overline{U^*}^{t}_{t}+\overline{\phi}^{y2t}_{x}=a, \tag{41}$$

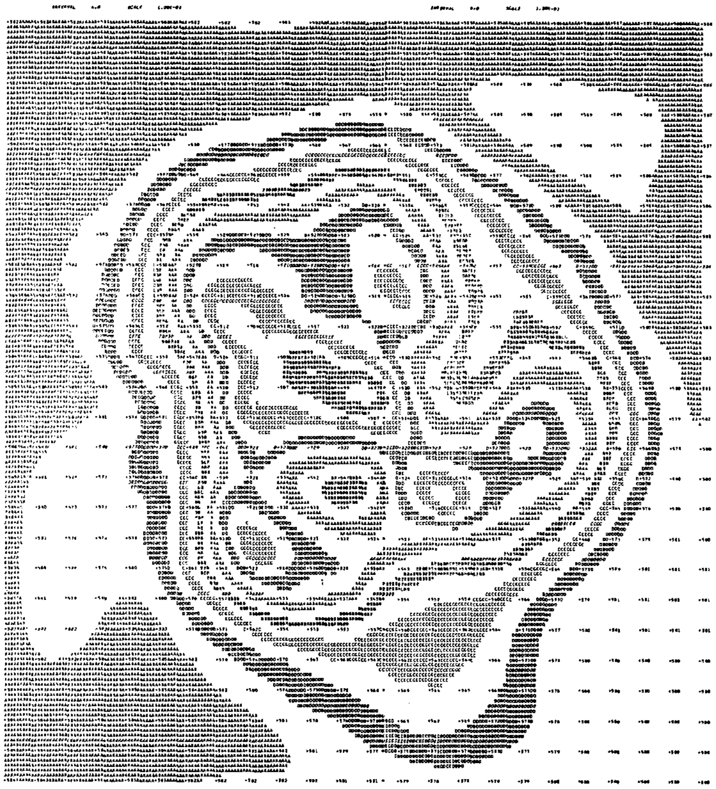

FIGURE 2.—The 5-day forecast of the 500-mb geopotential produced by the explicit model.

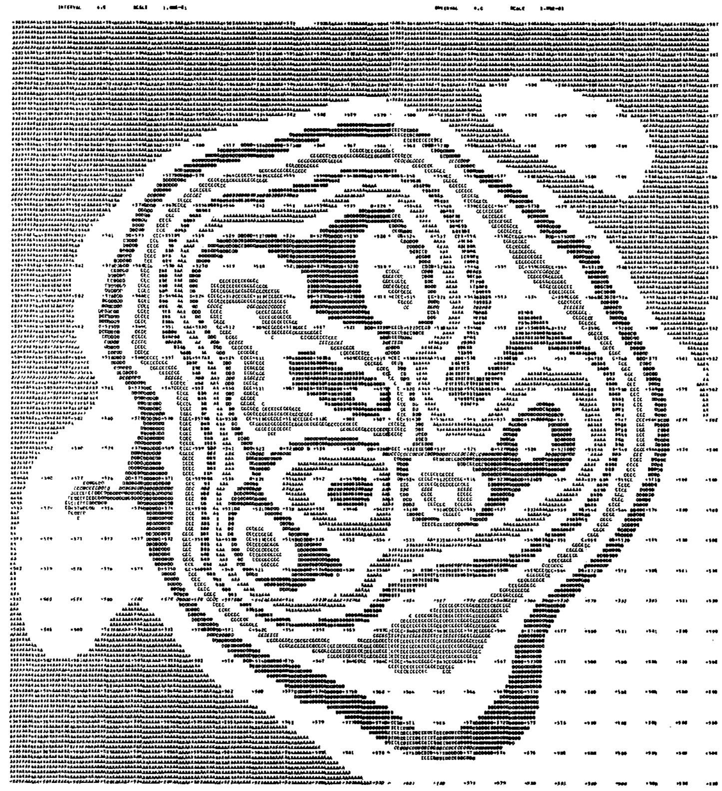

FIGURE 3.—The 5-day forecast of the 500-mb geopotential produced by the semi-implicit model.

$$\overline{V^*}^t_t+\overline{\phi}^{x2t}_y=b, \quad (42)$$

and

$$\overline{\phi}^t_t+m^2\phi_0\overline{(\overline{U^*}^y_x+\overline{V^*}^x_y)}^{2t}=c. \quad (43)$$

We first calculate a, b, and c and then perform the transformations

$$\overline{U^*}^{2t}+\Delta t\overline{\phi}^{y2t}_x=U^*(t-\Delta t)+\Delta ta=A, \quad (44)$$

$$\overline{V^*}^{2t}+\Delta t\overline{\phi}^{x2t}_y=V^*(t-\Delta t)+\Delta tb=B, \quad (45)$$

and

$$\overline{\phi}^{2t}+\Delta tm^2\phi_0\overline{(\overline{U^*}^y_x+\overline{V^*}^x_y)}^{2t}=\phi(t-\Delta t)+\Delta tc=C. \quad (46)$$

Eliminating the divergence from the above equations gives

$$m^2\Delta t^2\phi_0\overline{(\overline{\phi}^{yy}_{xx}+\overline{\phi}^{xx}_{yy})}^{2t}-\overline{\phi}^{2t}=-C+m^2\Delta t\phi_0(\overline{A}^y_x+\overline{B}^x_y). \quad (47)$$

We solve this equation for $\overline{\phi}^{2t}$ and then we solve eq (44) and (45) for $\overline{U^*}^{2t}$ and $\overline{V^*}^{2t}$. Finally, we produce the predictions

$$U^*(t+\Delta t)=2\overline{U^*}^{2t}-U^*(t-\Delta t), \quad (48)$$

$$V^*(t+\Delta t)=2\overline{V^*}^{2t}-V^*(t-\Delta t), \quad (49)$$

and

$$\phi(t+\Delta t)=2\overline{\phi}^{2t}-\phi(t-\Delta t). \quad (50)$$

A time step of 10 min is used for the explicit mode compared to 60 min for the implicit version. The numerical integration proceeds about four times faster with the implicit scheme.

6. RESULTS

A 5-day integration is performed from the 500-mb chart of 00 GMT on Feb. 21, 1969. The prediction produced by the explicit model appears in figure 2. A similar prediction made with the implicit model is given in figure 3. The two predictions are almost identical. There are no differences exceeding 20 m.

The result is quite interesting. It shows that the implicit model duplicates the explicit model very well. It also shows that the errors arising from time truncation remain quite small even after an integration of the order of 5 days.

Both models were integrated to 20 days with calculations of total mass, total energy, and mean square potential vorticity. Only slight variations of these quantities were detected during the integration. A slight amount of decoupling between the even and odd time steps appears after about 10 days in both models, but this is only a minor problem that could be eliminated with a very weak filter.

7. CONCLUSION

The semi-implicit time integration scheme has been tested successfully with a spectral barotropic model of the primitive meteorological equations, a grid point barotropic model, and a three-level spectral model (Robert 1970). In all cases, the truncation errors associated with the 60-min time step remain an order of magnitude smaller than the errors associated with space truncation.

Implicit integration schemes are well worth considering for atmospheric models because of the substantial economy in computing that arises. In Canada, it appears that this numerical procedure will become an essential part of our models, enabling us to integrate the primitive equations in a period of time short enough for all operational applications.

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