# MAGNETIC TORNADOES: THREE-DIMENSIONAL AFFINE MOTIONS IN IDEAL MAGNETOHYDRODYNAMICS 

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#### Abstract

From Hamilton's principle and a factorization Ansatz we derive a class of exact solutions for three-dimensional motion in ideal, compressible MHD. These exact nonlinear solutions are motions generated by time-dependent affine transformations, under which the fluid rotates, circulates and deforms. They reduce to three-dimensional self-similar solutions when rotation is absent. Continuous symmetries of Hamilton's principle for the affine MHD motions generate various constants of motion. Discrete symmetries establish duality relations among classes of solutions. In a special case, rotational and circulatory MHD motion is expressed as classical mechanical motion upon its own symmetry group, the Lie group $\mathrm{O}(4)$, in the well-known Arnold-Lax-Euler commutator form, $\dot{M}=[\omega, M]$.


## 1. Introduction

Three-dimensional plasma dynamics can be very complex when considered in all its microscopic detail. However, the gross features of plasma equilibrium and dynamics on time scales intermediate between particle collision times and transport times are describable by the relatively simple equations of ideal magnetohydrodynamics (MHD).

In the ideal MHD model, electrically neutral plasma convects like an adiabatic fluid that carries an embedded magnetic field. During convection, induced electrical currents flow instantaneously to oppose change of magnetic flux through every co-moving surface. The resultant magnetic stresses alter the convective motion of the plasma by opposing bending of magnetic field lines. Thus MHD flow is anisotropic and essentially three dimensional.

We shall seek motions in three-dimensional MHD whose time dependence factorizes in the Lagrange representation. The result is a class of MHD motions of affine type which reduce to self-similar motions in a special case.

We derive affine MHD motions from Hamilton's principle, for arbitrary initial distributions of material and magnetic field. The affine motion is expressed in a special case as torque-free rotational motion on the Lie group $O(4)$, coupled to dilational motion along principal axes of the initial moment of inertia tensor.

Affine motions for MHD generalize earlier work by F. J. Dyson [1] on isothermal expansion of an ideal fluid whose initial density profile is of Gaussian shape. Dyson's work on affine isothermal fluid expansion bears on traditional analysis of ellipsoidal figures of equilibrium for self-gravitating, incompressible fluids. That traditional work is summarized by Chandrasekar [2]. Before Dyson's work was published, L. V. Ovsjannikov [3] had also studied affine motion for ideal fluids. Subsequently S. I. Anisimov and Yu I. Lysikov [4] have found special solutions to Dyson's equations, that involve
elliptic integrals for $\gamma=5 / 3$ ideal gas. Affine motions are also derived in [5] as a type of groupinvariant solution for ideal, compressible fluids in one dimension.

Use of the Lie group $\mathrm{O}(4)$ as the configuration space for affine MHD motion follows recent mathematical formulations of classical mechanics and fluid dynamics on Lie groups [6] to [10]. A similar use of groups as configuration spaces has recently been applied to homothetic motions in Einstein spaces by O.I. Bogoyavlenskii and S.P. Novikov [11]. The obverse of the affine motion problem, application of the equations of a classical rigid body to study hydrodynamics has recently been discussed by Dolzhanskii [12] and by Visik and Dolzhanskii [13].

In the next section we explain how time dependence factorizes out for affine motions in the Lagrange representation of ideal MHD. We then derive the equations for affine motion from Hamilton's principle, and use Noether's theorem to find constants of motion that generate the symmetry group $O(3) \times O(3)$, isomorphic to $O(4)$. Following that result, we use $O(4)$ as a configuration space for the rotational part of the motion, which becomes torque-free rotational motion in four dimensions (six degrees of freedom) with a time-dependent inertia tensor. In the last section we study the coupling between the free rotational motion and the dilational motion along the three principal axes of the initial mass distribution.

## 2. Lagrange representation of three-dimensional MHD

In the Lagrange representation the particle paths are fundamental objects, and partial derivatives of the particle paths are basic dependent variables. The paths of fluid particles through fixed Eulerian space are given by vector functions $x\left(x_{0}, t\right)$ with initial conditions $\boldsymbol{x}\left(x_{0}, 0\right)=x_{0}$, the Lagrange coordinate. The partial derivatives of the particle paths specify the components of velocity

$$
v^{i}\left(x_{0}, t\right)=\partial x^{i} /\left.\partial t\right|_{x_{0}} \stackrel{\text { def }}{=} \dot{x}^{i}
$$

and they specify displacement gradients

$$
F_{j}^{i}\left(x_{0}, t\right)=\partial x^{i} /\left.\partial x\right|_{t},
$$

with subscripts $t, x_{0}$ that label the variables held constant in the partial derivatives.
In the Lagrange representation the equation of motion for ideal MHD is

$$
\rho \frac{\partial^{2} x_{i}}{\partial t^{2}}=-\frac{\partial \operatorname{det} F}{\partial F_{j}^{i}} \frac{\partial}{\partial x \hat{j}}\left(p+\frac{B^{2}}{8 \pi}\right)+\frac{1}{4 \pi} \mathrm{~B}^{\mathrm{k}} \nabla^{0}{ }_{\mathrm{k}} \mathrm{~B}_{\mathrm{i}} .
$$

In the motion equation $\nabla^{0}{ }_{k}$ is the covariant derivative in Lagrange curvilinear coordinates. MHD motion in Lagrange coordinates requires the following subsidiary relations:

$$
\begin{aligned}
& \rho \operatorname{det} F=\rho_{0} \stackrel{\operatorname{def}}{=} \rho\left(x_{0}, 0\right), \\
& B^{i}=F_{i}^{i} B \dot{d} / \operatorname{det} F, \\
& s(\rho, p)=s_{0} \stackrel{\operatorname{def}}{=} s\left(\rho_{0}, p_{0}\right), \\
& e(\rho, s)=e\left(\rho, s_{0}\right) .
\end{aligned}
$$

These subsidiary conditions represent, respectively, conservation of ion mass, Faraday's Law of magnetic induction, and the equations of state for adiabatic convection and specific internal energy. In Faraday's Law and the motion equation one uses Ampere's Law, curl B=4 J/c, and Ohm's Law for the case of infinite conductivity, $\boldsymbol{E}+\boldsymbol{v} \times \boldsymbol{B} / \boldsymbol{c}=0$, in order to eliminate current density, $\boldsymbol{J}$, and electric field, $\boldsymbol{E}$, in favor of magnetic field, $\boldsymbol{B}$, and particle velocity, $\boldsymbol{v}$.

Faraday's Law implies preservation of the divergence equation $\operatorname{div} \boldsymbol{B}=0$, which thus may be taken as an initial condition.

## 3. Hamilton's principle for three-dimensional MHD

The equation of motion for ideal MHD follows from Hamilton's principle:

$$
0=\delta S=\delta \int \mathrm{d} t \mathrm{~d}^{3} x_{0} \sqrt{\operatorname{det} g_{0}} \rho_{0}\left\{g_{i j}(x) \frac{\dot{x}^{\dot{i}} \dot{x}^{j}}{2}-e\left(\rho, s_{0}\right)-g_{i j}(x) \frac{B^{i} B^{j}}{8 \pi \rho}\right\} .
$$

Hamilton's principle states that the action $S$ is stationary for variations of particle paths $\delta x^{k}$ that vanish on the boundaries of the Lagrange domain of integration. Variations of the particle paths result in variations of the density, specific energy, and magnetic field through the MHD subsidiary relations stated in the previous section. In the action integrand $g_{i j}(x)$ is the Euclidean metric tensor in whatever Eulerian coordinate system we choose and det $g_{0}$ is its determinant in Lagrange coordinates. The particle interpretation of MHD motion is quite clear in this form of Hamilton's principle: the action integrand is the difference between kinetic and potential energy densities on the particle trajectories.

## 4. Factorization Ansatz

Time dependence factorizes in all of the variables, $e, \rho, p, v^{i}, B^{i}$, of the Lagrange representation of MHD, provided the displacement gradient $F_{j}^{i}$ is a function of time only:

$$
F_{j}^{i}=F_{j}^{i}(t) .
$$

The MHD subsidiary relations confirm this factorization of time dependence. Once factorized, ideal MHD motion reduces to classical particle dynamics, i.e. the problem reduces to solution of Newton's law of motion for the nine components of $F_{j}^{i}(t)$.
When the displacement gradient $F_{j}^{i}$ is a function of time only, the particle paths are determined by the affine transformation

$$
x^{i}\left(x_{0}, t\right)=F_{j}^{j}(t) x_{b}^{i} .
$$

This affine transformation stretches the initial configuration of particles, and rotates it relative to both Euler and Lagrange coordinate frames.

## 5. Hamilton's principle for affine motions in MHD

Hamilton's principle is expressed in terms of the affine displacement gradient $\boldsymbol{F}_{\mathrm{j}}{ }^{i}(t)$ in Cartesian coordinates as

$$
0=\delta S=\delta \int \mathrm{d} t\left\{\frac{1}{2} \delta_{i j} \dot{F}_{k}^{i} \dot{F}_{l}^{j} I_{0}^{k l}-E(\operatorname{det} F) \Pi_{0}-\delta_{i j} \frac{F_{k}^{i} F_{l}^{j}}{\operatorname{det} F} S^{k l}\right\}
$$

where $I_{0}^{k l}, \Pi_{0}$, and $S^{k l}$ are constants defined by integrals over the initial distributions of matter and magnetic fields,

$$
\begin{aligned}
& I_{0}^{k l}=\int \mathrm{d}^{3} x_{0} \rho_{0} x_{0}^{k} x_{0}^{l} \quad \text { (moment of inertia) }, \\
& \Pi_{0}=\int \mathrm{d}^{3} x_{0} p_{0} \quad \text { (integrated pressure), } \\
& S^{k l}=\int \mathrm{d}^{3} x_{0} \frac{B_{0}^{k} B_{0}^{\prime}}{8 \pi} \quad \text { (magnetic stress). }
\end{aligned}
$$

Hamilton's principle says that the action should be stationary for all variations of the particle paths in configuration space. The configuration space for affine flows is the space of trajectories whose coordinates are given by

$$
x^{i}=F_{j}^{i}(t) x_{0}^{j}
$$

By definition, variation of these particle trajectories does not change the identity of the particles, labeled by $x \dot{b}$; and the variation is performed at fixed time, $t$. Consequently the variation of particle trajectories is given by

$$
\delta x^{i}=\left[\delta F_{j}^{i}(t)\right] x_{b}^{j} .
$$

Thus it is the displacement gradient now regarded as a generalized coordinate which is being varied, subject to vanishing endpoint conditions. Hamilton's principle then provides the correct equations of motion for affine MHD.

Variation of the action with respect to generalized coordinates $F_{j}^{i}(t)$ produces the following motion equation, for homogeneous boundary conditions;

$$
F_{k}^{i} \ddot{F}_{l}^{i} I_{0}^{k l}=\delta^{i j}\left[\frac{\Pi_{0}}{(\operatorname{det} F)^{\gamma-1}}+\frac{\operatorname{Tr}\left(F S F^{\mathrm{T}}\right)}{\operatorname{det} F}\right]-2 \frac{F_{k}^{i} F_{[ }^{i} S^{k l}}{\operatorname{det} F},
$$

where for convenience we have taken a polytropic adiabat, $p / \rho^{\gamma}=p_{0} / \rho_{\gamma}^{\gamma}$. In curvilinear coordinates, metric tensor terms also contribute curvature forces, which are not considered in the present work. In addition, stresses and forces at the boundary have been neglected by imposition of homogeneous boundary conditions. In the case that the magnetic stress tensor $S$ is absent and the initial moment of inertia is transformed to the identity, one recovers Dyson's equation [1] for the affine motion of a spinning gas cloud.

## 6. Tensor virial equation

The symmetric part of the previous affine motion equation is equivalent to the following relation, called the tensor virial equation [14], which holds for arbitrary MHD motion:

$$
\frac{1}{2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} t^{2}} I^{i k}=\int_{V} \mathrm{~d}^{3} x T^{i k}+\frac{1}{2} \int_{S} \mathrm{~d} S_{i}\left\{-p\left(\delta^{i j} x^{k}+\delta^{i k} x^{j}\right)+M^{i j} x^{k}+M^{i k} x^{j}\right\} .
$$

Here $T^{i j}$ is the total momentum-flux-density tensor, $M^{i j}$ is the magnetic stress tensor, and $I^{i k}$ is the moment of inertia of the fluid,

$$
T^{i j}=\rho v^{i} v^{i}+\delta^{i j} p-M^{i j}, \quad M^{i j}=-\delta^{i j} B^{2} / 8 \pi+B^{i} B^{i} / 4 \pi, \quad I^{i k}=\int_{V} d^{3} x \rho x^{i} x^{k} .
$$

For homogeneous boundary conditions the surface integral vanishes in the virial equation. In that case one considers an isolated magnetic system for which pressures and magnetic fields vanish on the enclosing surface, perhaps located at infinity.

For an isolated system, the trace of the tensor virial equation indicates that fluid motion, gas pressure, and magnetic forces all contribute toward expansion.

$$
\frac{1}{2} \frac{d^{2}}{d t^{2}}(\operatorname{Tr} I)=\int_{V} d^{3} x\left[\rho v^{2}+3 p+B^{2} / 8 \pi\right]>0 .
$$

Thus, the gross motion is expansive: the sum of principal moments of inertia increases with time. However, details of the MHD fluid motion can still be quite intricate even for an isolated magnetic system, because of anisotropy of the magnetic forces. In a later section of this paper, temporal evolution of the principal moments of inertia is described in terms of Newtonian dynamics.

## 7. Continuous symmetries and constants of motion

Before discussion of the motion of the fluid in detail, consider the symmetries of Hamilton's principle and the associated constants of motion. The Lagrangian in Hamilton's principle is, in matrix notation

$$
L=\frac{1}{2} \operatorname{Tr}\left(\dot{F}_{0} \dot{F}^{\mathrm{T}}\right)-\frac{\operatorname{Tr}\left(F S F^{\mathrm{T}}\right)}{\operatorname{det} F}
$$

This Lagrangian admits translations in the time variable $t \rightarrow t^{\prime}=t+\delta t$. Consequently there exists an energy integral

$$
H=\frac{1}{2} \operatorname{Tr}\left(\dot{\mathrm{~F}} \dot{I}_{0} \dot{F}^{\mathrm{T}}\right)+E(\operatorname{det} F) \Pi_{0}+\frac{\operatorname{Tr}\left(F S F^{\mathrm{T}}\right)}{\operatorname{det} F}
$$

which is the Hamiltonian when expressed in terms of the following canonically conjugate variables:

$$
\Pi_{\mathrm{j}}^{i}=\delta_{\mathrm{jk}} \dot{F}_{I}^{\mathrm{k}} I_{0}^{i i}, \quad Q_{\mathrm{j}}^{i}=F_{j}^{i} .
$$

The Lagrangian $L$ also admits the transformation

$$
F \rightarrow F^{\prime}=O_{1} F,
$$

where $O_{1}$ is an orthogonal $3 \times 3$ matrix that reorients the Euler coordinate frame,

$$
x \rightarrow x^{\prime}=O_{1} x .
$$

Such rotational invariance of the Lagrangian leads by Noether's theorem to a matrix constant of motion,

$$
J=Q \Pi^{\mathrm{T}}-\Pi Q^{\mathrm{T}} .
$$

The skew-symmetric, matrix constant of motion, $J$, is the material angular momentum. Here the field angular momentum does not contribute: it vanishes when the displacement current is neglected in the MHD approximation.
With the canonical Poisson bracket relations

$$
\begin{aligned}
& \left\{Q^{i}, Q^{k}{ }_{l}\right\}=0=\left\{\Pi_{j}^{i}, \Pi_{l}^{k}\right\}, \\
& \left\{Q^{i}, \Pi_{l}^{k}\right\}=\delta^{i k} \delta_{j l},
\end{aligned}
$$

one quickly verifies that the components of $J$ commute with the Hamiltonian and generate the $\mathrm{O}(3)$ Lie algebra.

## 8. Discrete symmetries and duality relations

Discrete symmetries also exist for affine flows. For example, the Lagrangian $L$ is invariant under the substitution $F \rightarrow F^{\prime}=F^{\mathrm{T}}$. This discrete symmetry reverses the roles of Euler and Lagrange coordinates, and is related to the symmetry between space-fixed and body-fixed coordinates in the dynamics of a rigid body (cf. [1], p. 96).

Invariance of the Lagrangian under the adjoint transformation $F \leftrightarrow F^{\mathrm{T}}$ implies that $F^{\mathrm{T}}$ is a solution of the equations of motion whenever $F$ is a solution. For affine motions of incompressible fluids, duality under transposition of $F$ is called the Dedekind duality principle by Chandrasekar [2].

Another example of discrete symmetry for affine MHD is time-reversal invariance. Invariance of the Lagrangian under time reversal $t \leftrightarrow-t$-means that expanding solutions can be time-reversed to describe three-dimensional implosions.

## 9. Topological linkage numbers

Finally, there are preserved topological linkage numbers, called "helicities" in [14]. For example, every motion of ideal MHD - affine motions included-must preserve the number, $N$, of linkages among lines of magnetic flux. By definition of $N$,

$$
N=\int A_{k} B^{k} \mathrm{~d}^{3} x=\int A_{0_{k}} B_{0}^{k} \mathrm{~d}^{3} x_{0}=N_{0}
$$

This preservation of magnetic flux linkage, $N$, follows by substitution into the definition of $N$ directly from the Lagrange representations of magnetic field, $B^{k}$, and vector potential $A_{k}$,

$$
\begin{aligned}
& A_{k}\left(x_{0}, t\right)=A_{j}\left(x_{0}, 0\right) F^{-i j}\left(x_{0}, t\right), \\
& B^{i}=\epsilon^{i j k} \nabla_{i} A_{k} .
\end{aligned}
$$

So ideal MHD motion preserves $N$, the number of linkages of the magnetic field with itself. In other words, stretching and rotation by affine motions cannot unlink lines of magnetic flux.

Another topological quantity for ideal MHD is the number, $N^{\prime}$, of linkages between lines of magnetic flux and lines of vorticity. The definition of $N^{\prime}$ is

$$
N^{\prime}=\int \mathrm{d}^{3} x v \cdot B=\int \mathrm{d}^{3} x \delta_{i j} v^{i} B^{j}
$$

which is sometimes called "cross helicity". For cross helicity to be preserved requires there be no hydrodynamic source of vorticity; so gradients of density and pressure must remain everywhere colinear, i.e. $\nabla \rho \times \nabla p=0$. Under affine flow with polytropic adiabats $p / \rho^{\gamma}=p_{0} / \rho_{0}^{\gamma}$ one finds, for example,

$$
\epsilon^{i j k} \frac{\partial p}{\partial x^{j}} \frac{\partial \rho}{\partial x^{k}}=\frac{F_{1}^{i}(t)}{(\operatorname{det} F)^{\gamma}} \epsilon^{\operatorname{lm} n} \frac{\partial p_{0}}{\partial x_{0}^{m}} \frac{\partial \rho_{0}}{\partial x_{0}^{n}} .
$$

Both sides of this equation will vanish so that cross helicity will be preserved under affine motion, provided the pressure and density are related functionally at the initial time. For example, the initial conditions may be isothermal or isentropic; or the density or pressure may be initially uniform. In these situations, cross helicity is preserved under affine MHD flow.

## 10. Three-dimensional description of affine motion

At this point one may discuss affine MHD motion in detail, in terms of the transformation matrix $F_{j}{ }_{j}(t)$ whose temporal evolution stretches the initial configuration of particles, and rotates the particle configuration relative to both Euler and Lagrange coordinate frames. Accordingly, the displacement gradient $F_{j}^{i}(t)$ may be decomposed into a matrix product

$$
F=R_{1} D R_{2},
$$

where $R_{1}$ and $R_{2}$ are orthogonal and $D$ is diagonal. Each matrix depends on time: the diagonal matrix $D$ characterizes changes of shape of the spatial distributions of mass and magnetic field of the fluid; while the orthogonal matrices $R_{1}$ and $R_{2}$ specify respectively the changes of orientation of fluid relative to initial Euler and Lagrange coordinate frames, cf. [1].

In terms of matrix coordinates $F=R_{1} D R_{2}$ with nine degrees of freedom, the matrix equations of affine MHD motion are, for a polytropic equation of state,

$$
\ddot{F} I_{0} F^{\mathrm{T}}=1\left[\frac{\Pi_{0}}{(\operatorname{det} F)^{\gamma-1}}+\frac{\operatorname{Tr}\left(F S F^{\mathrm{T}}\right)}{\operatorname{det} F}\right]-2 \frac{F S F^{\mathrm{T}}}{\operatorname{det} F} .
$$

Upon substitution of the triple product $F(t)=R_{1} D R_{2}$ into this equation, one obtains the following separated equations [16]

$$
\begin{aligned}
& \dot{J}=0, \\
& \dot{K}=\frac{2}{\operatorname{det} F}\left[S, F^{\mathrm{T}} F\right], \\
& \ddot{D}_{i}=-\frac{\partial}{\partial D_{i}} U,
\end{aligned}
$$

where the skew-symmetric matrices $J, K$ represent fluid angular momentum and circulation, respectively, with $I_{0}=1$.

$$
\begin{aligned}
& J \stackrel{\text { def }}{=} \int \mathrm{d}^{3} x \rho\left(x_{i} v_{i}-x_{i} v_{i}\right)=F \dot{F}^{\mathrm{T}}-\dot{F} F^{\mathrm{T}}, \\
& K \stackrel{\text { def }}{=}\left(v_{l, k}-v_{k, l}\right) F_{k i} F_{l j}=F^{\mathrm{T}} \dot{F}-\dot{F}^{\mathrm{T}} F .
\end{aligned}
$$

The bracket in the $\dot{K}$ equation is the matrix commutator, and the potential function $U$ in the equation for the dilation matrix $D$ is given by

$$
U=\frac{1}{4} \operatorname{Tr}\left(\omega_{1} L+\omega_{2} N\right)+\frac{\operatorname{Tr} S^{*} D^{2}}{\operatorname{det} D}-\frac{\Pi_{0}}{\gamma-2}(\operatorname{det} D)^{-\gamma+2}
$$

with dynamical matrix quantities $\omega_{1}, \omega_{2}, L, N, S^{*}$ defined by

$$
\begin{aligned}
& \omega_{1}=-R_{1}^{-1} \dot{R}_{1}, \quad \omega=\dot{R}_{2} R_{2}^{-1}, \\
& L=R_{1}^{-1} J R_{1}=D^{2} \omega_{1}+\omega_{1} D^{2}-2 D \omega_{2} D, \\
& N=R_{2}^{-1} K R_{2}=D^{2} \omega_{2}+\omega_{2} D^{2}-2 D \omega_{1} D, \\
& S^{*}=R_{2} S R_{2}^{-1} .
\end{aligned}
$$

The quantities $\omega_{1}, \omega_{2}$ are angular velocities of rotation and circulation, respectively. The quantities $L, N$ represent the angular momentum and circulation expressed in fixed, Eulerian coordinates. Finally $S^{*}=R_{2} S R_{2}^{-1}$ is the magnetic stress tensor referred to the fixed Eulerian frame.

The equations of motion for $J, K$, and $D$ first of all express conservation of fluid angular momentum, $J$. The circulation $K$ is also conserved, provided the magnetic stress tensor $S$ can be simultaneously diagonalized with the initial mass distribution $I_{0}$. However, when the commutator [ $S, F^{\top} F$ ] does not vanish, the circulation experiences a restoring torque due to magnetic stresses, which are developed as the lines of magnetic field wind around themselves during fluid circulation. Finally, the last equation for the dilation matrix $D$ expresses the coupling between expansion of the fluid and its circulation and rotation. In the potential $U$, the centrifugal, hydrodynamic, and magnetic forces each are represented in conservative form, so energetic trade offs among them are clear.

## 11. Angular MHD motion in the group $\mathbf{O}(4)$

Suppose that by suitable linear transformations and rescaling of initial conditions the Lagrangian for affine MHD motions can be brought into a form where the kinetic and magnetic energies are sums of squares,

$$
L=\frac{1}{2} \operatorname{Tr}\left(\dot{F} \dot{F}^{\mathrm{T}}\right)-\Pi_{0} E(\operatorname{det} F)-\frac{\operatorname{Tr}\left(F F^{\mathrm{T}}\right)}{\operatorname{det} F} .
$$

Then both $J$ and $K$ of the previous section are preserved and the nondiagonal parts of the affine MHD equation of motion can be expressed as angular motion in the group $O$ (4). In particular, one can obtain an Arnold-Lax-Euler (ALE) representation of the angular motion as a commutator relation.

$$
\dot{M}=[\omega, M],
$$

with angular velocity $\omega$ in the Lie algebra $O$ (4), and angular momentum $M$ linearly related to $\omega$ by the (time-dependent) inertia tensor. The time dependence of the inertia tensor is determined from the dilatational motion, which itself is coupled to the angular motion by inertial forces due to rotation in both Euler and Lagrange spaces. The dilatational motion is discussed in the next section.

In order to regard the group $\mathrm{O}(4)$ as configuration space for the angular part of affine MHD motion one recalls the isomorphism between $[O(3) \times O(3)]$ and $O(4)$, which is familiar to physicists in connection with the Kepler problem of planetary motion and its quantum-mechanical analog, the hydrogen atom [17].

The ( 1,1 ) representation of $\mathrm{O}(4)$ establishes an isomorphism between the real $3 \times 3$ matrix $F_{i}^{i}$, without symmetry; and a symmetric, traceless $4 \times 4$ matrix $\Phi_{\beta}^{\alpha}$, where $\alpha, \beta=0,1,2,3$. Namely,

In canonical form $\Phi$ is expressed as

$$
\Phi=\Omega^{-1} \Delta \Omega ; \quad \Omega^{-1}=\Omega^{\mathrm{T}}
$$

where $\Delta$ is the diagonal matrix $\operatorname{diag}\left(\Delta_{0}, \Delta_{1}, \Delta_{2}, \Delta_{3}\right)$ with elements

$$
\Delta_{0}=\frac{1}{2} \operatorname{Tr} D ; \quad \Delta_{i}=D_{i}-\frac{1}{2} \operatorname{Tr} D,
$$

and $\Omega$ is a $4 \times 4$ orthogonal matrix, i.e. $\Omega$ is an element of the group $O(4)$.
The time derivative of the orthogonal matrix $\Omega$, when expressed as

$$
\dot{\Omega}=\omega \Omega,
$$

defines an angular velocity $\omega$ in the Lie algebra $\mathrm{O}(4)$, whose elements are antisymmetric. In matrix form the angular velocity $\omega$ is given by

$$
\omega_{\beta}^{\alpha}=\left[\begin{array}{cccc}
0 & -\beta_{1} & -\beta_{2} & -\beta_{3} \\
\beta_{1} & 0 & \alpha_{3} & -\alpha_{2} \\
\beta_{2} & -\alpha_{3} & 0 & \alpha_{1} \\
\beta_{3} & \alpha_{3} & -\alpha_{1} & 0
\end{array}\right],
$$

where $\alpha_{i}$ and $\beta_{i}, i=1,2,3$, are related to angular velocities $\dot{\theta}_{i}$ and $\dot{\phi}_{i}$ in Euler and Lagrange space by

$$
\alpha_{i}=\frac{1}{2}\left(\dot{\theta}_{i}+\dot{\varphi}_{i}\right), \quad \beta_{i}=\frac{1}{2}\left(\dot{\theta}_{i}-\dot{\varphi}_{i}\right), \quad \dot{\theta}_{i}=\frac{1}{2} \epsilon_{i j k} \delta^{i l} \omega_{1}{ }^{k}{ }_{1}, \quad \dot{\varphi}_{i}=\frac{1}{2} \epsilon_{i j k} \delta^{j} \omega_{2}{ }_{1}{ }^{k} .
$$

In terms of $\omega$ and $\Delta$ the time derivative of the matrix $\Phi=\Omega^{-1} \Delta \Omega$ may be expressed as

$$
\dot{\Phi}=\Omega^{-1}(\dot{\Delta}+[\Delta, \omega]) \Omega .
$$

Here the components of the commutator matrix $[\Delta, \omega]$ are

$$
[\Delta, \omega]_{\beta}^{\alpha}=\Gamma^{\alpha}{ }_{\beta} \omega^{\alpha}{ }_{\beta} \quad \text { (no sum), }
$$

where the antisymmetric matrix $\Gamma$ is defined as

$$
\Gamma_{\beta}^{\alpha}=\Delta_{\alpha}-\Delta_{\beta} .
$$

The four-dimensional angular momentum $\Psi$ is given by the commutator relation

$$
\Psi=[\Phi, \dot{\Phi}]=\Phi \dot{\Phi}-\dot{\Phi} \Phi
$$

whose components belong to a $(1,1) \oplus(1,1)$ representation of $O(3) \times O(3)$,

$$
\Psi_{\beta}^{\alpha}=\frac{1}{2}\left[\begin{array}{cccc}
0 & K^{2}{ }_{3}-J^{2}{ }_{3} & K^{3}{ }_{1}-J_{1}^{3} & K^{1}{ }_{2}-J^{1}{ }_{2} \\
J^{2}-K_{3}^{2} & 0 & J^{1}{ }_{2}+K^{1} & J^{1}{ }_{3}+K^{1}{ }_{3} \\
J^{3}-K_{1}{ }_{1} & J^{2}+K^{2}{ }_{1} & 0 & J^{2}+K^{2} \\
J^{1}{ }_{2}-K_{2}^{1} & J^{3}+K_{1}{ }_{1} & J^{3}+K^{3} & 0
\end{array}\right],
$$

The six independent components of the $4 \times 4$ antisymmetric matrix $\Psi$ are six constants of motion. One may also write $\Psi$ in its canonical form

$$
\Psi=\Omega^{-1} M \Omega .
$$

From the commutator definition of $\Psi$ the matrix $M$ is expressible as

$$
M=[\Delta,[\Delta, \omega]],
$$

which is also an element of the Lie algebra $O(4)$, an antisymmetric matrix with components

$$
M_{\beta}^{\alpha}=\left(\Gamma_{\beta}^{\alpha}\right)^{2} \omega_{\beta}^{\alpha} \quad \text { (no sum). }
$$

Since $J$ and $K$ are constants of motion, $\Psi=0$, and one finds (cf. [1], eq. (48))

$$
\dot{M}+[M, \omega]=0,
$$

which is the ALE equation we have sought, for conservation of angular momentum $J$ and relative vorticity $K$ in four-dimensional notation.

The ALE equation means that motion in $\mathrm{O}(4)$ generated by $\omega$ preserves the eigenvalues of $M$. This is also clear from the expressions for $\Psi$ and $\dot{\Psi}$. The motion in $\mathrm{O}(4)$ is deformable-body, rotational motion with $\Psi$ the angular momentum in $O(4)$ "space coordinates", and $M$ the angular momentum in $\mathrm{O}(4)$ "body coordinates". If the moment of inertia operator defined by $\left(\Gamma^{\alpha}{ }_{\beta}\right)^{2}=\left(\Delta_{\alpha}-\Delta_{\beta}\right)^{2}$ were not time dependent, the ALE equation would express rigid-body motion on $O(4)$, which is completely integrable [17].

Thus the ALE equation on $O(4)$ expresses how the time dependence of the moment of inertia tensor affects rotational motion. In turn, we seek to express how rotational motion affects dilatational motion.

## 12. Dilatational motion

When the Lagrangian for affine MHD motions can be brought into a form where the kinetic and magnetic energies are sums of squares,

$$
L=\frac{1}{2} \operatorname{Tr} \dot{F} \dot{F}^{\mathrm{T}}-U(\operatorname{det} F)-\frac{\operatorname{Tr} F F^{\mathrm{T}}}{\operatorname{det} F}
$$

by suitable linear transformations and rescaling of the initial conditions, one may take advantage of the following trace identity for the isomorphism between $F$ and $\Phi$,

$$
\operatorname{Tr} D^{2}=\operatorname{Tr} F F^{\mathrm{T}}=\operatorname{Tr} \Phi^{2}=\operatorname{Tr} \Delta^{2}
$$

in order to translate the Lagrangian in four-dimensional notation into the form

$$
L=\frac{1}{2} \operatorname{Tr} \dot{\Phi}^{2}-U(\operatorname{det} F)-\frac{\operatorname{Tr} D^{2}}{\operatorname{det} F}
$$

where $D_{i}$ and $\dot{D}_{i}$ are now to be used as generalized coordinates and velocities. By direct calculation one finds

$$
\operatorname{Tr} \dot{\Phi}^{2}=\operatorname{Tr}(\dot{\Delta}+[\Delta, \omega])^{2}=\operatorname{Tr} \dot{\Delta}^{2}+\operatorname{Tr}[\Delta, \omega]^{2}=\operatorname{Tr} \dot{\Delta}^{2}-\operatorname{Tr} M \omega=\operatorname{Tr} \dot{D}^{2}-\operatorname{Tr} M \omega
$$

So the kinetic energy separates into a sum of dilational and rotational parts in the four-dimensional notation. In terms of matrix components the kinetic energy is

$$
\begin{aligned}
\operatorname{Tr} \dot{\Phi}^{2} & =\sum_{\alpha} \dot{\Delta}_{\alpha}^{2}-\sum_{\alpha, \beta}\left(\Gamma_{\beta}^{\alpha}\right)^{2}\left(\omega_{\beta}^{\alpha}\right)^{2} \\
& =\sum_{i} \dot{D}_{i}^{2}+2\left[\sum_{i}\left(D_{i}-\operatorname{Tr} D\right)^{2} \beta_{i}^{2}+\left(D_{1}-D_{2}\right)^{2} \alpha_{3}^{2}+\left(D_{2}-D_{3}\right)^{2} \alpha_{1}^{2}+\left(D_{3}-D_{1}^{2}\right) \alpha_{2}^{2}\right]
\end{aligned}
$$

The Lagrangian is then expressible as

$$
\begin{aligned}
L(D, \dot{D})= & \frac{1}{2} \sum_{i} \dot{D}_{i}^{2}-U(\operatorname{det} F)-\frac{1}{\operatorname{det} F} \sum_{i} D_{i}^{2} \\
& +\frac{1}{2} \sum_{j, k, 1}\left\{\left[\epsilon^{j k l}\left(D_{i}+D_{k}\right) \beta_{l}\right]^{2}+\left[\epsilon^{j k l}\left(D_{j}-D_{k}\right) \alpha_{l}\right]^{2}\right\}
\end{aligned}
$$

and the Euler-Lagrange equations of motion for this Lagrangian become

$$
\begin{aligned}
\ddot{D}_{i}= & \frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial L}{\partial \dot{D}_{i}}=\frac{\partial L}{\partial D_{i}}=-\frac{\partial U}{\partial D_{i}}+\frac{\operatorname{Tr} D^{2}}{\operatorname{det} F^{2}} \frac{\partial \operatorname{det} F}{\partial D_{i}}-\frac{2 D_{i}}{\operatorname{det} F} \\
& +\sum_{j, k, 1}\left\{\left(D_{j}+D_{k}\right)\left(\epsilon^{j k} \beta_{l}\right)^{2}\left(\delta_{i j}+\delta_{i k}\right)+\left(D_{i}-D_{k}\right)\left(\epsilon^{j k l} \alpha_{l}\right)^{2}\left(\delta_{i j}-\delta_{j k}\right)\right],
\end{aligned}
$$

where $\operatorname{det} F=D_{1} D_{2} D_{3}$, and the last term is the $i$ th diagonal element of the matrix of centrifugal force, $\left[A^{2} D+D A^{2}-2 A D B-2 B D A+D B^{2}+B^{2} D\right]$, with skew-symmetric matrices $A$ and $B$ from the earlier discussion. When this set of equations is rewritten as

$$
\ddot{D}_{i}=-\frac{\partial}{\partial D_{i}}\left(U+\frac{\operatorname{Tr} D^{2}}{\operatorname{det} F}+\operatorname{Tr} \frac{1}{2} M \omega\right)
$$

one sees two interesting features. First, the magnetic, rotational, and hydrodynamic forces all enter as potential forces, which makes the energetic trade offs among them easy to study. However, more importantly, the magnetic force contributes a restoring force which opposes the expansion caused by thermodynamic pressure forces. If the initial magnetic stress tensor is large enough, the magnetic restoring force may actually overcome thermodynamic expansion and cause oscillatory dilations along the principal axes. In particular, solutions are possible which show multiple oscillations followed by collapse to a current sheet and then dramatic, explosive, expansion.
The first question about such dilational oscillations concerns their stability. Certainly dilatational motion can be baroclinically unstable, for example. Also, instability can arise for deformable rotations of magnetic fluids, just as rotational instability occurs for rigid bodies. So stability is a major question for these affine MHD motions. However, a detailed analysis of stability is outside the scope of the present paper.

## 13. Discussion

The magnetic fluid analysis discussed here provides an exact, nonlinear description of plasma motion in three dimensions. Besides its conceivable utility in astrophysical models, there are several possible applications of this description and its associated Lie-group formalism for study of threedimensional asymmetries in magnetic fusion research. For example, direct numerical integrations of the affine MHD equations in specific cases offer benchmark tests of the accuracy of more sophisticated computer simulation codes for design of magnetic fusion reactors.

A one-dimensional example further illustrates the behavior of the affine solutions. Consider a cylindrical Z-pinch with homogeneous boundary conditions at the outside of the plasma. In that case a closed-form solution exists [19] for an isothermal plasma with Gaussian density profile, which exhibits radial, unstable oscillations; which may also be seen experimentally. In this case the restoration force is a geometrical effect due to cylindrical, $1 / r$ divergence.
Affine MHD analysis also contributes toward better understanding of three-dimensional effects in magnetic plasmas by providing a mathematical framework for study of bifurcation points [20] of the equations of motion, and their nonlinear instability upon expansion or implosion. Detailed study of
stability, and consideration of special cases of affine MHD motion obtained by numerical integration remain to be done.

In fact, much remains to be done with the magnetic tornado system. In addition to stability analysis and specific numerical integrations, there remain the open problems of non-homogeneous boundary conditions, curvilinear coordinates, and dissipative effects. Also there are several other fluid dynamics models to which the affine factorization Ansatz applies, such as the Boussinesq model. Study of the nonlinear dynamical systems which result would be an interesting line of future research, especially in regard to their chaotic properties, such as strange attractors which seem to exist in the dissipative cases.

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